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Research Article

Some Embeddings into the Morrey and Modified Morrey Spaces Associated with the Dunkl Operator

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We consider the generalized shift operator, associated with the Dunkl operator $\Lambda_\alpha(f)(x) = (d/dx)f(x)+((2\alpha+1)/x)((f(x)-f(-x))/2)$, $\alpha > -1/2$. We study some embeddings into the Morrey space (D-Morrey space) $L_{p,\lambda,\alpha}$, $0 \le \lambda < 2\alpha+2$ and modified Morrey space (modified D-Morrey space) $\widetilde{L}_{p,\lambda,\alpha}$ associated with the Dunkl operator on $\mathbb{R}$. As applications we get boundedness of the fractional maximal operator M_β, $0 \le \beta < 2\alpha+2$, associated with the Dunkl operator (fractional D-maximal operator) from the spaces $L_{p,\lambda,\alpha}$ to $L_\infty(\mathbb{R})$ for $p = (2\alpha+2-\lambda)/\beta$ and from the spaces $\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ to $L_\infty(\mathbb{R})$ for $(2\alpha+2-\lambda)/\beta \le p \le (2\alpha+2)/\beta$.

1. Introduction

On the real line, the Dunkl operators are differential-difference operators introduced in 1989 by Dunkl [1] and are denoted by Λ_α, where α is a real parameter $> -1/2$. These operators are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Rösler in [2] shows that the Dunkl kernel verifies a product formula. This allows us to define the Dunkl translations τ_x, $x \in \mathbb{R}$.

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [3]). Later, Morrey spaces found important applications to Navier-Stokes [4, 5] and Schrödinger [6–8] equations, elliptic problems with discontinuous coefficients [9, 10], and potential theory [11–13]. An exposition of the Morrey spaces can be found in the book [14].

In the present work, we study some embeddings into the D-Morrey and modified D-Morrey spaces. As applications we give boundedness of the fractional D-maximal operator in the D-Morrey and modified D-Morrey spaces.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give some embeddings into the D-Morrey and modified D-Morrey spaces. In Section 4, we prove the boundedness of the fractional D-maximal operator M_β from the spaces $L_{p,\lambda,\alpha}$ to $L_\infty(\mathbb{R})$ for $p = (2\alpha + 2 - \lambda)/\beta$ and from the spaces $\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ to $L_\infty(\mathbb{R})$ for $(2\alpha + 2 - \lambda)/\beta \le p \le (2\alpha + 2)/\beta$.

2. Preliminaries

On the real line, we consider the first-order differential-difference operator defined by

$$\Lambda_\alpha(f)(x) := \frac{d}{dx} f(x) + \frac{2\alpha + 1}{x}\left(\frac{f(x) - f(-x)}{2}\right), \quad \alpha > -\frac{1}{2}, \tag{2.1}$$

which is called the Dunkl operator. For $\lambda \in \mathbb{C}$, the Dunkl kernel $E_\alpha(\lambda\cdot)$ on $\mathbb{R}$ was introduced by Dunkl in [1] (see also [15–17]) and is given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha + 1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R}, \tag{2.2}$$

where j_α is the normalized Bessel function of the first kind and order α [18], defined by

$$j_\alpha(z) := 2^\alpha \Gamma(\alpha + 1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1)\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n + \alpha + 1)}, \quad z \in \mathbb{C}. \tag{2.3}$$

The Dunkl kernel $E_\alpha(\lambda\cdot)$ is the unique analytic solution on $\mathbb{R}$ of the initial problem for the Dunkl operator (see [1]).

Let μ_α be the weighted Lebesgue measure on $\mathbb{R}$ given by

$$d\mu_\alpha(x) := \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha + 1)} dx. \tag{2.4}$$

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\begin{gathered} \|f\|_{L_{p,\alpha}} := \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if } p \in [1, \infty), \\ \|f\|_{L_{\infty,\alpha}} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| \quad \text{if } p = \infty. \end{gathered} \tag{2.5}$$

For $1 \le p < \infty$, we denote by $WL_{p,\alpha}(\mathbb{R})$ the weak $L_{p,\alpha}$ spaces defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} := \sup_{r>0} r\left(\mu_\alpha\{x \in \mathbb{R} : |f(x)| > r\}\right)^{1/p}. \tag{2.6}$$

Note that

$$L_{p,\alpha}(\mathbb{R}) \subset WL_{p,\alpha}(\mathbb{R}), \quad \|f\|_{WL_{p,\alpha}} \le \|f\|_{L_{p,\alpha}} \quad \forall f \in L_{p,\alpha}. \tag{2.7}$$

For all $x, y, z \in \mathbb{R}$, we put

$$W_\alpha(x, y, z) := (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x, y, z), \tag{2.8}$$

where

$$\sigma_{x,y,z} := \begin{cases} \dfrac{x^2 + y^2 - z^2}{2xy} & \text{if } x, y \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x, y, z) := \begin{cases} d_\alpha \dfrac{\left(\left[(|x| + |y|)^2 - z^2\right]\left[z^2 - (|x| - |y|)^2\right]\right)^{\alpha - 1/2}}{|xyz|^{2\alpha}} & \text{if } |z| \in A_{x,y}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.10}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+1/2))$ and $A_{x,y} = [||x| - |y||, |x| + |y|]$.

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} := \begin{cases} W_\alpha(x, y, z)d\mu_\alpha(z) & \text{if } x, y \in \mathbb{R} \setminus \{0\}, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases} \tag{2.11}$$

For $x, y \in \mathbb{R}$ and f being a continuous function on $\mathbb{R}$, the Dunkl translation operator τ_x is given by

$$\tau_x f(y) := \int_{\mathbb{R}} f(z) d\nu_{x,y}(z). \tag{2.12}$$

Using the change of variable $z = \Psi(x, y, \theta) = \sqrt{x^2 + y^2 - 2xy\cos\theta}$, we have also (see [2])

$$\tau_x f(y) = C_\alpha \int_0^\pi \left[f(\Psi) + f(-\Psi) + \frac{x + y}{\Psi}(f(\Psi) - f(-\Psi))\right] d\nu_\alpha(\theta), \tag{2.13}$$

where $d\nu_\alpha(\theta) = (1 - \cos\theta)\sin^{2\alpha}\theta d\theta$ and $C_\alpha = \Gamma(\alpha+1)/2\sqrt{\pi}\Gamma(\alpha+1/2)$.

Proposition 2.1 (see Soltani [16]). *For all $x \in \mathbb{R}$ the operator τ_x extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,*

$$\|\tau_x f\|_{L_{p,\alpha}} \leq 4\|f\|_{L_{p,\alpha}}. \tag{2.14}$$

Let $B(0,t) =]-t,t[$, $t > 0$ and $\mu_\alpha(]-t,t[) = b_\alpha t^{2\alpha+2}$, where $b_\alpha = 2^{-\alpha-1}((\alpha+1)\Gamma(\alpha+1))^{-1}$. For $L_{1,\alpha}^{\mathrm{loc}}(\mathbb{R})$ (the space of locally integrable functions on $\mathbb{R}$), we consider

$$Mf(x) := \sup_{r>0} (\mu_\alpha B(0,r))^{-1} \int_{B(0,r)} \tau_x |f|(y)\, d\mu_\alpha(y). \tag{2.15}$$

Theorem 2.2 (see [19]). (1) *If $f \in L_{1,\alpha}(\mathbb{R})$, then for every $\beta > 0$,*

$$\mu_\alpha\{x \in \mathbb{R} : Mf(x) > \beta\} \leq \frac{C}{\beta}\|f\|_{L_{1,\alpha}}, \tag{2.16}$$

where $C > 0$ is independent of f.

(2) *If $f \in L_{p,\alpha}(\mathbb{R})$, $1 < p \leq \infty$, then $Mf \in L_{p,\alpha}(\mathbb{R})$ and*

$$\|Mf\|_{L_{p,\alpha}} \leq C_p \|f\|_{L_{p,\alpha}}, \tag{2.17}$$

where $C_p > 0$ is independent of f.

Corollary 2.3. *If $f \in L_{1,\alpha}^{\mathrm{loc}}(\mathbb{R})$, then*

$$\lim_{r\to 0} (\mu_\alpha B(0,r))^{-1} \int_{B(0,r)} \tau_x f(y)\, d\mu_\alpha(y) = f(x) \tag{2.18}$$

for a.e. $x \in \mathbb{R}$.

3. Some Embeddings into the D-Morrey and Modified D-Morrey Spaces

Definition 3.1 (see [20]). Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha + 2$, and $[t]_1 = \min\{1,t\}$, $t > 0$. We denote by $L_{p,\lambda,\alpha}(\mathbb{R})$ Morrey space ($\equiv$ D-Morrey space) and by $\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ the modified Morrey space ($\equiv$ modified D-Morrey space), associated with the Dunkl operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norms

$$\begin{gathered}\|f\|_{L_{p,\lambda,\alpha}} := \sup_{x\in\mathbb{R},t>0} \left(t^{-\lambda} \int_{B(0,t)} \tau_x |f|^p(y)\, d\mu_\alpha(y) \right)^{1/p}, \\ \|f\|_{\widetilde{L}_{p,\lambda,\alpha}} := \sup_{x\in\mathbb{R},t>0} \left([t]_1^{-\lambda} \int_{B(0,t)} \tau_x |f|^p(y)\, d\mu_\alpha(y) \right)^{1/p},\end{gathered} \tag{3.1}$$

respectively.

If $\lambda < 0$ or $\lambda > 2\alpha + 2$, then $\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}$.

Note that

$$L_{p,\alpha}(\mathbb{R}) \hookrightarrow \widetilde{L}_{p,0,\alpha}(\mathbb{R}) = L_{p,0,\alpha}(\mathbb{R}),$$
$$\|f\|_{\widetilde{L}_{p,0,\alpha}} = \|f\|_{L_{p,0,\alpha}} \leq 4\|f\|_{L_{p,\alpha}}, \tag{3.2}$$

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \hookrightarrow L_{p,\alpha}(\mathbb{R}), \qquad \|f\|_{L_{p,\alpha}} \leq \|f\|_{\widetilde{L}_{p,\lambda,\alpha}},$$
$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \hookrightarrow L_{p,\lambda,\alpha}(\mathbb{R}), \qquad \|f\|_{L_{p,\lambda,\alpha}} \leq \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}. \tag{3.3}$$

Definition 3.2 (see [19]). Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha + 2$. We denote by $WL_{p,\lambda,\alpha}(\mathbb{R})$ the weak D-Morrey space and by $W\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ the modified weak D-Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with finite norms

$$\|f\|_{WL_{p,\lambda,\alpha}} := \sup_{r>0} r \sup_{x\in\mathbb{R},t>0} \left(t^{-\lambda}\mu_\alpha\{y \in B(0,t) : \tau_x|f|(y) > r\}\right)^{1/p},$$
$$\|f\|_{W\widetilde{L}_{p,\lambda,\alpha}} := \sup_{r>0} r \sup_{x\in\mathbb{R},t>0} \left([t]_1^{-\lambda}\mu_\alpha\{y \in B(0,t) : \tau_x|f|(y) > r\}\right)^{1/p}, \tag{3.4}$$

respectively.

We note that

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset WL_{p,\lambda,\alpha}(\mathbb{R}), \qquad \|f\|_{WL_{p,\lambda,\alpha}} \leq \|f\|_{L_{p,\lambda,\alpha}},$$
$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \subset W\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}), \qquad \|f\|_{W\widetilde{L}_{p,\lambda,\alpha}} \leq \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}. \tag{3.5}$$

Lemma 3.3 (see [20]). *Let $1 \leq p < \infty$. Then*

$$L_{p,2\alpha+2,\alpha}(\mathbb{R}) = L_\infty(\mathbb{R}),$$
$$\|f\|_{L_{p,2\alpha+2,\alpha}} = 4b_\alpha^{1/p}\|f\|_{L_\infty}. \tag{3.6}$$

Lemma 3.4. *Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha + 2$. Then*

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) = L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R}),$$
$$\max\left\{\|f\|_{L_{p,\lambda,\alpha}}, \|f\|_{L_{p,\alpha}}\right\} \leq \|f\|_{\widetilde{L}_{p,\lambda,\alpha}} \leq \max\left\{\|f\|_{L_{p,\lambda,\alpha}}, 4\|f\|_{L_{p,\alpha}}\right\}. \tag{3.7}$$

Proof. Let $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$. Then by (3.3) we have

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \hookrightarrow L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R}),$$
$$\max\left\{ \|f\|_{L_{p,\lambda,\alpha}}, \|f\|_{L_{p,\alpha}} \right\} \le \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}. \tag{3.8}$$

Let $f \in L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R})$. Then

$$\begin{aligned}\|f\|_{\widetilde{L}_{p,\lambda,\alpha}} &= \sup_{x\in\mathbb{R},t>0}\left([t]_1^{-\lambda}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p} \\ &= \max\left\{\sup_{x\in\mathbb{R},0<t\le1}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p},\right. \\ &\qquad \left.\sup_{x\in\mathbb{R},t>1}\left(\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p}\right\} \le \max\left\{\|f\|_{L_{p,\lambda,\alpha}}, 4\|f\|_{L_{p,\alpha}}\right\}.\end{aligned} \tag{3.9}$$

Therefore, $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ and the embedding $L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R}) \hookrightarrow \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ is valid.

Thus $\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) = L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R})$. □

From Lemmas 3.3 and 3.4 for $1 \le p < \infty$, we have

$$\widetilde{L}_{p,2\alpha+2,\alpha}(\mathbb{R}) = L_\infty(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R}). \tag{3.10}$$

Lemma 3.5. *Let $0 \le \lambda \le 2\alpha + 2$. Then*

$$L_{(2\alpha+2)/(2\alpha+2-\lambda),\alpha}(\mathbb{R}) \hookrightarrow L_{1,\lambda,\alpha}(\mathbb{R}), \qquad \|f\|_{L_{1,\lambda,\alpha}} \le 4b_\alpha^{\lambda/(2\alpha+2)}\|f\|_{L_{(2\alpha+2)/(2\alpha+2-\lambda),\alpha}}. \tag{3.11}$$

Proof. The embedding is a consequence of Hölder's inequality and Proposition 2.1. Indeed,

$$\begin{aligned}\|f\|_{L_{1,\lambda,\alpha}} &= \sup_{x\in\mathbb{R},t>0} t^{-\lambda}\int_{B(0,t)}\tau_x|f|(y)d\mu_\alpha(y) \\ &\le \sup_{x\in\mathbb{R},t>0} t^{-\lambda}(\mu_\alpha B(0,t))^{\lambda/(2\alpha+2)}\left(\int_{B(0,t)}\tau_x|f|(y)^{(2\alpha+2)/(2\alpha+2-\lambda)}d\mu_\alpha(y)\right)^{(2\alpha+2-\lambda)/(2\alpha+2)} \\ &\le 4b_\alpha^{\lambda/(2\alpha+2)}\|f\|_{L_{(2\alpha+2)/(2\alpha+2-\lambda),\alpha}}.\end{aligned} \tag{3.12}$$

□

On the D-Morrey spaces, the following embedding is valid.

Lemma 3.6 (see [20]). *Let* $0 \leq \lambda < 2\alpha+2$ *and* $0 \leq \beta < 2\alpha+2-\lambda$. *Then for* $p = (2\alpha+2-\lambda)/\beta$,

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset L_{1,2\alpha+2-\beta,\alpha}(\mathbb{R}), \qquad \|f\|_{L_{1,2\alpha+2-\beta,\alpha}} \leq b_\alpha^{1/p'} \|f\|_{L_{p,\lambda,\alpha}}, \tag{3.13}$$

where $1/p + 1/p' = 1$.

On the modified D-Morrey spaces, the following embedding is valid.

Lemma 3.7. *Let* $0 \leq \lambda < 2\alpha+2$ *and* $0 \leq \beta < 2\alpha+2-\lambda$. *Then for* $(2\alpha+2-\lambda)/\beta \leq p \leq (2\alpha+2)/\beta$,

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \subset_{>} L_{1,2\alpha+2-\beta,\alpha}(\mathbb{R}), \qquad \|f\|_{L_{1,2\alpha+2-\beta,\alpha}} \leq b_\alpha^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}. \tag{3.14}$$

Proof. Let $0 < \lambda < 2\alpha+2$, $0 < \beta < 2\alpha+2-\lambda$, $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$, and $(2\alpha+2-\lambda)/\beta \leq p \leq (2\alpha+2)/\beta$. By the Hölder's inequality, we have

$$\begin{aligned}
\|f\|_{L_{1,2\alpha+2-\beta,\alpha}} &= \sup_{x\in\mathbb{R},t>0} [t]_1^{\beta-2\alpha-2} \int_{B(0,t)} \tau_x |f|(y)\, d\mu_\alpha(y) \\
&\leq b_\alpha^{1/p'} \sup_{x\in\mathbb{R},t>0} \left([t]_1 t^{-1}\right)^{-(2\alpha+2)/p'} [t]_1^{\beta-(2\alpha+2-\lambda)/p} \\
&\quad \times \left([t]_1^{-\lambda} \int_{B(0,t)} \tau_x |f|^p(y)\, d\mu_\alpha(y) \right)^{1/p} \\
&= b_\alpha^{1/p'} \sup_{x\in\mathbb{R},t>0} \left([t]_1 t^{-1}\right)^{2\alpha+2-\beta} \left([t]_1 t^{-1}\right)^{-(2\alpha+2)/p'} [t]_1^{\beta-(2\alpha+2-\lambda)/p} \\
&\quad \times \left([t]_1^{-\lambda} \int_{B(0,t)} \tau_x |f|^p(y)\, d\mu_\alpha(y) \right)^{1/p} \\
&\leq b_\alpha^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\alpha}} \sup_{t>0} \left([t]_1 t^{-1}\right)^{(2\alpha+2)/p-\beta} [t]_1^{\beta-(2\alpha+2-\lambda)/p}.
\end{aligned} \tag{3.15}$$

Note that

$$\begin{aligned}
\sup_{t>0} \left([t]_1 t^{-1}\right)^{(2\alpha+2)/p-\beta} [t]_1^{\beta-(2\alpha+2-\lambda)/p} &= \max\left\{ \sup_{0<t\leq 1} t^{\beta-(2\alpha+2-\lambda)/p}, \sup_{t>1} t^{\beta-(2\alpha+2)/p} \right\} < \infty \\
&\qquad \text{iff } \frac{2\alpha+2-\lambda}{\beta} \leq p \leq \frac{2\alpha+2}{\beta}.
\end{aligned} \tag{3.16}$$

Therefore, $f \in L_{1,2\alpha+2-\beta,\alpha}(\mathbb{R})$ and

$$\|f\|_{L_{1,2\alpha+2-\beta,\alpha}} \le b_\alpha^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}. \tag{3.17}$$

□

4. Some Applications

In this section, using the results of Section 3, we get the boundedness of the fractional D-maximal operator in the D-Morrey and modified D-Morrey spaces.

For $0 \le \beta < 2\alpha + 2$, we define the fractional maximal functions

$$\begin{aligned} M_\beta f(x) &:= \sup_{t>0} (\mu_\alpha B(0,t))^{-1+\beta/(2\alpha+2)} \int_{B(0,t)} \tau_x |f|(y)\, d\mu_\alpha(y), \\ M_{p,\beta} f(x) &:= \left(M_\beta |f|^p\right)^{(1/p)}(x). \end{aligned} \tag{4.1}$$

In the case $\beta = 0$, we denote $M_{p,0}f$ by $M_p f$. Note that $M_1 f = Mf$.

Lemma 4.1. *Let $1 \le p < \infty$, $0 \le \beta < 2\alpha + 2$, and $f \in L_{p,2\alpha+2-\beta,\alpha}(\mathbb{R})$. Then $M_{p,\beta} f \in L_\infty(\mathbb{R})$ and the following equality*

$$\|M_{p,\beta} f\|_{L_\infty} = b_\alpha^{(\beta/(2\alpha+2)-1)(1/p)} \|f\|_{L_{p,2\alpha+2-\beta,\alpha}} \tag{4.2}$$

is valid.

Proof.

$$\begin{aligned} \|M_{p,\beta} f\|_{L_\infty} &= b_\alpha^{(\beta/(2\alpha+2)-1)(1/p)} \sup_{x\in\mathbb{R}, t>0} \left(t^{\beta-2\alpha-2} \int_{B(0,t)} \tau_x |f|^p(y)\, d\mu_\alpha(y) \right)^{1/p} \\ &= b_\alpha^{(\beta/(2\alpha+2)-1)(1/p)} \|f\|_{L_{p,2\alpha+2-\beta,\alpha}}. \end{aligned} \tag{4.3}$$

□

Taking $\beta = 0$ in Lemma 4.1 and using Lemma 3.3, we get for $M_p f$ the following result.

Corollary 4.2. *Let $1 \le p < \infty$. Then*

$$\|M_p f\|_{L_\infty} = 4\|f\|_{L_\infty}. \tag{4.4}$$

Lemma 4.3. *Let $1 \le p < \infty$, $0 \le \beta < 2\alpha + 2$, and $f \in \widetilde{L}_{p,2\alpha+2-\beta,\alpha}(\mathbb{R})$. Then $M_{p,\beta} f \in L_\infty(\mathbb{R})$ and the following equality*

$$\|M_{p,\beta} f\|_{L_\infty} = b_\alpha^{(\beta/(2\alpha+2)-1)(1/p)} \|f\|_{\widetilde{L}_{p,2\alpha+2-\beta,\alpha}} \tag{4.5}$$

is valid.

Corollary 4.4. *Let* $0 \leq \lambda < 2\alpha + 2$ *and* $0 \leq \beta < 2\alpha + 2 - \lambda$. *Then the operator* M_β *is bounded from* $L_{p,\lambda,\alpha}$ *to* L_∞ *for* $p = (2\alpha + 2 - \lambda)/\beta$. *Moreover,*

$$\|M_\beta f\|_{L_\infty} = b_\alpha^{\beta/(2\alpha+2)-1} \|f\|_{L_{1,2\alpha+2-\beta,\alpha}} \leq b_\alpha^{\beta/(2\alpha+2)-1/p} \|f\|_{L_{p,\lambda,\alpha}}. \tag{4.6}$$

Corollary 4.5. $1 \leq p < \infty, 0 \leq \lambda < 2\alpha + 2, 0 \leq \beta < 2\alpha + 2 - \lambda$. *Then the operator* M_β *is bounded from* $\widetilde{L}_{p,\lambda,\alpha}$ *to* L_∞ *for* $(2\alpha + 2 - \lambda)/\beta \leq p \leq (2\alpha + 2)/\beta$. *Moreover,*

$$\|M_\beta f\|_{L_\infty} = b_\alpha^{\beta/(2\alpha+2)-1} \|f\|_{\widetilde{L}_{1,2\alpha+2-\beta,\alpha}} \leq b_\alpha^{\beta/(2\alpha+2)-1/p} \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}. \tag{4.7}$$

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