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Research Article

Asymptotically Linear Solutions for Some Linear Fractional Differential Equations

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We establish here that under some simple restrictions on the functional coefficient $a(t)$ the fractional differential equation ${}_0D_t^{\alpha}[tx' - x + x(0)] + a(t)x = 0$, $t > 0$, has a solution expressible as $ct + d + o(1)$ for $t \to +\infty$, where ${}_0D_t^{\alpha}$ designates the Riemann-Liouville derivative of order $\alpha \in (0,1)$ and $c, d \in \mathbb{R}$.

1. Introduction

Consider the ordinary differential equation

$$x'' + f(t,x) = 0, \quad t \geq 1, \tag{1.1}$$

where the function $f : [1,+\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous such that

$$\left|f(t,x)\right| \leq h(t) \cdot g\left(\frac{|x|}{t}\right), \quad t \geq 1, \ x \in \mathbb{R}. \tag{1.2}$$

Here, the functions $h : [1,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to [0,+\infty)$ are continuous, and there exists $\varepsilon \in [0,1]$ with

$$\int_1^{+\infty} t^{\varepsilon} h(t)dt < +\infty. \tag{1.3}$$

Then, given $c, d \in \mathbb{R}$, (1.1) has a solution $x(t)$, defined in a neighborhood of $+\infty$, which is expressible as $ct+o(t)$ for $\varepsilon = 0$, as $ct+o(t^{1-\varepsilon})$ for $\varepsilon \in (0,1)$ and, finally, as $ct+d+o(1)$ for $\varepsilon = 1$ when $t \to +\infty$. Such a solution is called *asymptotically linear* in the literature. In particular, these developments apply to the homogeneous linear differential equation $x'' + a(t)x = 0$.

A unifying technique of proof for such estimates can be read in [1] and is based on the next reformulation of the differential equation (1.1)

$$y(t) = t^{-\varepsilon}\left[d - \int_t^{+\infty} \tau^{\varepsilon} f(\tau, x(\tau))d\tau\right],$$
$$x(t) = \left[c - d(\operatorname{sgn}\varepsilon - 1)\right]t + \varepsilon t\int_t^{+\infty} \frac{y(\tau)}{\tau}d\tau - (1-\varepsilon)\int_{t_0}^{t} y(\tau)d\tau \tag{1.4}$$

for some $t_0 \geq 1$ large enough. For a different approach, the so-called Riccatian method, in the case of *intermediate* asymptotic ($\varepsilon \in (0,1)$, $c = 0$), see the technique from [2, 3].

The study of asymptotically linear solutions to linear and nonlinear ordinary differential equations is of importance in fluid mechanics, differential geometry (Jacobi fields, e.g., [4, page 239]), bidimensional gravity (the geodesics of the Euclidean planar spray $x'' = 0$ being the asymptotically linear solutions $x(t) = ct + d$), and others.

In this note, we are interested in the existence of a *fractional variant* for the problem of asymptotically linear solutions which can be formulated as follows: *are there any nontrivial fractional differential equations which have only asymptotically linear solutions and also their solution sets contain solutions (asymptotically linear) for all the prescribed values of numbers c, d, and ε?* To the best of our knowledge, this is an open problem in the theory of fractional differential equations.

Fractional differential equations have been of great interest during the last few years. This follows from the intensive development of the theory of fractional calculus [5, 6] followed by the applications of its methods in various sciences and engineering [7]. We can mention that the fractional differential equations are playing an important role in fluid dynamics, traffic model with fractional derivative, measurement of viscoelastic material properties, modeling of viscoplasticity, control theory, economy, nuclear magnetic resonance, mechanics, optics, signal processing, and so on. Basically, the fractional differential equations are used to investigate the dynamics of the complex systems; the models based on these derivatives have given superior results as those based on the classical derivatives, see [8, page 305], [9–11].

To introduce a fractional differential operator of order $1 + \alpha$, there are *three options*. The first two consist of a mixed ordinary differential-Caputo fractional differential operator, namely, $({}_0^C D_t^{\alpha} x)'(t) = {}_0D_t^{\alpha}(x')(t)$, and, respectively, a Riemann-Liouville fractional differential operator $({}_0D_t^{\alpha} x)'(t) = ({}_0D_t^{1+\alpha} x)(t)$.

We recall that $({}_0D_t^{\alpha} f)(t) = (1/\Gamma(1-\alpha)) \cdot (d/dt)[\int_0^t (f(s)/(t-s)^{\alpha})ds]$ represents the Riemann-Liouville derivative of order α of some function f, cf. [8, page 68], and Γ stands for Euler's function Gamma. Remark as well that, in general, ${}_0D_t^{\alpha+\beta} x \neq {}_0D_t^{\alpha}({}_0D_t^{\beta} x)$ for $\alpha, \beta \in (0,1)$, see [8, page 74]. To deal with iterations, Miller and Ross [12] coined the term *sequential fractional* differential operator of order $\alpha+\beta$ for the quantity ${}_0D_t^{\alpha}({}_0D_t^{\beta} x)$, cf. [8, pages 108, 122].

Also, the quantity $({}_0^C D_t^{\alpha} f)(t) = (1/\Gamma(1-\alpha)) \int_0^t (f'(s)/(t-s)^{\alpha})ds$ has been called the *Caputo derivative* in physics, see [8, page 79], and it is often preferred due to its sound explanation of what the initial data signify.

The first variant of differential operator was used in [13] to study the existence of solutions $x(t)$ of nonlinear fractional differential equations that obey the restrictions

$$x(t) \longrightarrow 1 \quad \text{when } t \longrightarrow +\infty, \quad x' \in \left(L^1 \cap L^\infty\right)((0,+\infty),\mathbb{R}). \tag{1.5}$$

The second variant of differential operator, see [14], was employed to prove that, for any real numbers x_0, x_1, the linear fractional differential equation

$${}_0D_t^{1+\alpha}x + a(t)x = 0, \quad t > 0, \tag{1.6}$$

possesses a solution $x(t)$ with the asymptotic development

$$x(t) = [x_0 + O(1)]t^{\alpha-1} + x_1 t^\alpha \quad \text{when } t \longrightarrow +\infty. \tag{1.7}$$

A recent application of the Caputo derivative can be found in [15].

All of these fractional differential operators are based upon the natural splitting of the second-order operator d^2/dt^2, namely, $x'' = (x')'$. Here, we shall introduce a different fractionalizing of x'' which is based on the identities

$$tx'' = \left(tx' - x\right)' = \left[tx' - x + x(0)\right]', \quad t > 0, \tag{1.8}$$

stemming from the integration technique in the Lie algebra L_2, cf. [16, page 23].

In the following section, we give a positive (partial) answer to the preceding open question. In fact, we produce some simple conditions regarding the continuous function $a : [0,+\infty) \to \mathbb{R}$ such that, given $c \in \mathbb{R} - \{0\}$, the fractional differential equation (FDE) below

$${}_0D_t^{\alpha}\left[tx' - x + x(0)\right] + a(t)x = 0, \quad t > 0, \tag{1.9}$$

possesses a solution with the asymptotic development $x(t) = ct + x(0) + o(1)$ when $t \to +\infty$.

2. Asymptotically Linear Solutions

Let us start with a result regarding the case of intermediate asymptotic.

Proposition 2.1. *Set the numbers $\varepsilon \in (0,1)$, $c \neq 0$, and $c_1 \in (0,1)$, $A > 0$, such that*

$$\max\left\{|c|, \frac{1}{1-\varepsilon}\right\} \cdot \Gamma(1-\alpha)A \le c_1. \tag{2.1}$$

Assume also that $a \in C([0,+\infty),\mathbb{R})$ is confined to

$$\left(1 + t^{1-\varepsilon}\right)|a(t)| \le \frac{A}{t^\alpha}, \quad t > 0. \tag{2.2}$$

Then, the FDE

$$_0D_t^{\alpha}(tx' - x) + a(t)x = 0, \quad t > 0, \tag{2.3}$$

has a solution $x \in C([0,+\infty),\mathbb{R}) \cap C^1((0,+\infty),\mathbb{R})$, *with* $\lim_{t\searrow 0}[t^{2-\alpha}x'(t)] = 0$, *which verifies the asymptotic formula* $x(t) = ct + O(t^{\varepsilon})$ *when* $t \to +\infty$.

Proof. Introduce the complete metric space $\mathcal{M} = (D,\delta)$, where $D = \{y \in C((0,+\infty),\mathbb{R}) : \sup_{t>0}[t^{-\varepsilon}|y(t)|] \le c_1,\ t > 0\}$ and the metric δ is given by the usual formula

$$\delta(y_1,y_2) = \sup_{t>0}\frac{|y_1(t) - y_2(t)|}{t^{\varepsilon}}, \quad y_1,y_2 \in D. \tag{2.4}$$

In particular, $\lim_{t\searrow 0}y(t) = 0$ for all $y \in D$.

Introduce the function $x : (0,+\infty) \to \mathbb{R}$ via the formulas

$$y = tx' - x, \quad x(t) = ct - t\int_t^{+\infty}\frac{y(s)}{s^2}ds, \quad t > 0. \tag{2.5}$$

Since $\lim_{t\searrow 0}x(t) = 0$, we deduce that x can be continued backward to 0; so, its extension x belongs to $C([0,+\infty),\mathbb{R}) \cap C^1((0,+\infty),\mathbb{R})$. Also, $\lim_{t\searrow 0}[t^{1-\alpha}y(t)] = \lim_{t\searrow 0}[t^{2-\alpha}x'(t)] = 0$.

Define further the integral operator $T : \mathcal{M} \to \mathcal{M}$ by the formula

$$(T)(y)(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t\frac{a(s)}{(t-s)^{1-\alpha}}\left[cs - s\int_s^{+\infty}\frac{y(\tau)}{\tau^2}d\tau\right]ds, \quad t > 0. \tag{2.6}$$

The estimate

$$\begin{aligned}
|(T)(y)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t\frac{|a(s)|}{(t-s)^{1-\alpha}}\left(|c|s + \frac{c_1}{1-\varepsilon}s^{\varepsilon}\right)ds \\
&\le t^{\varepsilon}\frac{1}{\Gamma(\alpha)}\int_0^t\frac{|a(s)|}{(t-s)^{1-\alpha}}\left(\frac{c_1}{1-\varepsilon} + |c|s^{1-\varepsilon}\right)ds \\
&\le t^{\varepsilon}\frac{1}{\Gamma(\alpha)}\int_0^t\frac{|a(s)|}{(t-s)^{1-\alpha}}\left(1 + s^{1-\varepsilon}\right)ds \cdot \max\left\{|c|, \frac{c_1}{1-\varepsilon}\right\} \\
&\le t^{\varepsilon}\frac{1}{\Gamma(\alpha)}\int_0^t\frac{ds}{(t-s)^{1-\alpha}s^{\alpha}} \cdot A\max\left\{|c|, \frac{c_1}{1-\varepsilon}\right\} \\
&\le \Gamma(1-\alpha) \cdot A\max\left\{|c|, \frac{1}{1-\varepsilon}\right\} \cdot t^{\varepsilon} \\
&\le c_1t^{\varepsilon}, \quad t > 0,
\end{aligned} \tag{2.7}$$

shows that T is well defined by taking into account (2.1), (2.2).

Now, given $y_1, y_2 \in D$, we have

$$
\begin{aligned}
&\left|(T)(y_1)(t) - (T)(y_2)(t)\right| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}} \cdot s \int_s^{+\infty} \frac{d\tau}{\tau^{2-\varepsilon}} ds \cdot \delta(y_1, y_2) \\
&\quad \leq \frac{1}{\Gamma(\alpha)(1-\varepsilon)} \int_0^t \frac{s^{\varepsilon}|a(s)|}{(t-s)^{1-\alpha}} ds \cdot \delta(y_1, y_2) \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \cdot \max\left\{|c|, \frac{1}{1-\varepsilon}\right\} \cdot \int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}} ds \cdot t^{\varepsilon} \cdot \delta(y_1, y_2) \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \max\left\{|c|, \frac{1}{1-\varepsilon}\right\} \cdot \int_0^t \frac{(1+s^{1-\varepsilon})|a(s)|}{(t-s)^{1-\alpha}} ds \cdot t^{\varepsilon}\delta(y_1, y_2) \\
&\quad \leq \Gamma(1-\alpha) \cdot A \max\left\{|c|, \frac{1}{1-\varepsilon}\right\} \cdot t^{\varepsilon}\delta(y_1, y_2) \\
&\quad \leq t^{\varepsilon} \cdot c_1 \delta(y_1, y_2), \quad t > 0,
\end{aligned} \tag{2.8}
$$

and so $\delta(T(y_1), T(y_2)) \leq c_1 \delta(y_1, y_2)$.

The operator T being a contraction, it has a unique fixed point $y_0 \in D$. Since $t\int_t^{+\infty}(y_0(s)/s^2)ds = O(t^{\varepsilon})$ when $t \to +\infty$, the proof is complete. □

Theorem 2.2. *Assume that* (2.1) *holds true and* $a \in C([0, +\infty), \mathbb{R})$ *verifies the sharper restriction*

$$
\left(1 + t^{1-\varepsilon}\right)|a(t)| \leq A \min\left\{\frac{1}{t^{\alpha}}, \frac{1}{t^{\beta}}\right\}, \quad t > 0, \tag{2.9}
$$

where $1 > \beta > \alpha + \varepsilon$. *Then, the solution* x *of FDE* (2.3) *from Proposition 2.1 has the asymptotic development* $x(t) = ct + o(1)$ *when* $t \to +\infty$.

Proof. Notice that

$$
\int_0^t \frac{ds}{(t-s)^{1-\alpha} s^{\beta}} = t^{\alpha-\beta} \int_0^1 \frac{du}{(1-u)^{1-\alpha} u^{\beta}} = t^{\alpha-\beta} B(\alpha, 1-\beta), \tag{2.10}
$$

where B is the Beta function, cf. [8, page 6].

Via (2.9), we have the estimate

$$\begin{aligned}
\left|y_0(t)\right| &= \left|T(y_0)(t)\right| \\
&\le t^{\varepsilon}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(1+s^{1-\varepsilon})|a(s)|}{(t-s)^{1-\alpha}}ds\cdot\max\left\{|c|,\frac{c_1}{1-\varepsilon}\right\} \\
&\le t^{\varepsilon}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}s^{\beta}}\cdot A\max\left\{|c|,\frac{c_1}{1-\varepsilon}\right\} \\
&\le t^{\varepsilon+\alpha-\beta}\cdot\frac{\Gamma(1-\beta)}{\Gamma(\alpha+1-\beta)}A\max\left\{|c|,\frac{1}{1-\varepsilon}\right\} \\
&= o(1) \quad \text{when } t\longrightarrow+\infty.
\end{aligned} \tag{2.11}$$

By means of L'Hôpital's rule, we conclude that (recall (2.5))

$$\lim_{t\to+\infty}\left[-t\int_t^{+\infty}\frac{y_0(s)}{s^2}ds\right]=\lim_{t\to+\infty}y_0(t)=0. \tag{2.12}$$

The proof is complete. □

Our main contribution is given next.

Theorem 2.3. *Set the numbers $\varepsilon\in(0,1-\alpha)$, $\beta\in(\alpha+\varepsilon,1)$, c, d with $c^2+d^2>0$, and $c_1\in(0,1)$, $A>0$, such that*

$$\max\left\{|c|,|d|,\frac{1}{1-\varepsilon}\right\}\cdot\Gamma(1-\alpha)A\le c_1. \tag{2.13}$$

Assume also that $a\in C([0,+\infty),\mathbb{R})$ satisfies the inequality

$$\left(\frac{1}{t^{\varepsilon}}+1+t^{1-\varepsilon}\right)|a(t)|\le A\min\left\{\frac{1}{t^{\alpha}},\frac{1}{t^{\beta}}\right\},\quad t>0. \tag{2.14}$$

Then the FDE (1.9) has a solution $x\in C([0,+\infty),\mathbb{R})\cap C^1((0,+\infty),\mathbb{R})$, with $x(0)=d$ and $\lim_{t\searrow 0}[t^{2-\alpha}x'(t)]=0$, which has the asymptotic development

$$x(t)=ct+d+o(1)\quad \text{when } t\longrightarrow+\infty. \tag{2.15}$$

Proof. Keeping the notations from Proposition 2.1, introduce the change of variables

$$y=tx'-x+d,\quad x(t)=ct+d-t\int_t^{+\infty}\frac{y(s)}{s^2}ds,\quad t>0,\ y\in D, \tag{2.16}$$

and the integral operator $T : \mathcal{M} \longrightarrow \mathcal{M}$ with the formula

$$(T)(y)(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t \frac{a(s)}{(t-s)^{1-\alpha}} \left[cs + d - s \int_s^{+\infty} \frac{y(\tau)}{\tau^2} d\tau \right] ds, \quad t > 0. \tag{2.17}$$

As before, we have the estimates

$$\begin{aligned}
\left| T(y)(t) \right| &\leq t^{\varepsilon} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}} \left(\frac{1}{s^{\varepsilon}} + 1 + s^{1-\varepsilon} \right) ds \cdot \max\left\{ |c|, |d|, \frac{c_1}{1-\varepsilon} \right\} \\
&\leq c_1 t^{\varepsilon}, \quad t > 0, \\
\left| (T)(y_1)(t) - (T)(y_2)(t) \right| &\leq \frac{t^{\varepsilon}}{\Gamma(\alpha)} \max\left\{ |c|, |d|, \frac{1}{1-\varepsilon} \right\} \int_0^t \frac{\left(1/s^{\varepsilon} + 1 + s^{1-\varepsilon}\right)|a(s)|}{(t-s)^{1-\alpha}} ds \cdot \delta(y_1, y_2) \\
&\leq t^{\varepsilon} \cdot c_1 \delta(y_1, y_2), \quad t > 0,
\end{aligned} \tag{2.18}$$

for all $y, y_1, y_2 \in D$.

Finally, for the fixed point y_0 of the operator T, we have that

$$\begin{aligned}
\left| y_0(t) \right| &= \left| T(y_0)(t) \right| \\
&\leq t^{\varepsilon+\alpha-\beta} \cdot \frac{\Gamma(1-\beta)}{\Gamma(\alpha+1-\beta)} A \max\left\{ |c|, |d|, \frac{1}{1-\varepsilon} \right\} \\
&= o(1) \quad \text{when } t \longrightarrow +\infty.
\end{aligned} \tag{2.19}$$

The proof is complete. □

3. Conclusion

A particular case of Theorem 2.3 is when $c = 0$, $d = 1$, that is, when the solution of (1.9) reads as $x(t) = 1 + o(1)$ for $t \rightarrow +\infty$. Notice from (2.14) that the behavior of the functional coefficient $a(t)$ is confined to $\lim_{t \to +\infty} a(t) = 0$. However, there is no restriction with respect to the (eventual) zeros of $a(t)$. On the other hand, in the recent contribution [13, Section 3], we were forced to request that the functional coefficient of the FDE has *a unique zero* in $(0, +\infty)$. In conclusion, the fractional differential operators proposed in (1.9), (2.3) allow more freedom for the functional coefficient.

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