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Research Article

Direct and Inverse Approximation Theorems for Baskakov Operators with the Jacobi-Type Weight

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We introduce a new norm and a new K-functional $K_{\varphi^\lambda}(f;t)_{w,\lambda}$. Using this K-functional, direct and inverse approximation theorems for the Baskakov operators with the Jacobi-type weight are obtained in this paper.

1. Introduction and Main Results

Let f be a function defined on the interval $[0,\infty)$. The operators $V_n(f;x)$ are defined as follows:

$$V_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) v_{n,k}(x), \tag{1.1}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \tag{1.2}$$

which were introduced by Baskakov in 1957 [1]. Becker [2] and Ditzian [3] had studied these operators and obtained direct and converse theorems. In [4, 5] Totik gave a result: if $f \in C_B[0,+\infty)$, $0 < \alpha < 1$, then $\|V_n(f;x) - f(x)\|_\infty = O(n^\alpha)$ if and only if $x^\alpha(1+x)^\alpha|\Delta_h^2(f;x)| \le kh^{2\alpha}$, where $h > 0$ and k is a positive constant. We may formulate the following question: do the Baskakov operators have similar property in the case of weighted approximation with the Jacobi weights? It is well known that the weighted approximation is not a simple extension, because the Baskakov operators are unbounded for the usual weighted norm $\|f\|_w = \|wf\|_\infty$. Xun and Zhou [6] introduced the norm

$$\|f\|_w = \|wf\|_\infty + |f(0)|, \quad f \in C_B[0,\infty) \tag{1.3}$$

and have discussed the rate of convergence for the Baskakov operators with the Jacobi weights and obtained

$$w(x)|V_n(f;x) - f(x)| = O(n^{-\alpha}) \Longleftrightarrow K(f;t)_w = O(t^\alpha), \tag{1.4}$$

where $w(x) = x^a(1+x)^{-b}$, $0 < a < 1$, $b > 0$, $0 < \alpha < 1$, and $C_B[0,\infty)$ is the set of bounded continuous functions on $[0,\infty)$.

In this paper, we introduce a new norm and a new K-functional, using the K-functional, and we get direct and inverse approximation theorems for the Baskakov operators with the Jacobi-type weight.

First, we introduce some useful definitions and notations.

Definition 1.1. Let $C_B[0,\infty)$ denote the set of bounded continuous functions on the interval $[0,\infty)$, and let

$$\begin{aligned} C_{a,b,\lambda} &= \left\{f \mid f \in C_B[0,\infty), \varphi^{2(2-\lambda)} wf \in C_B[0,\infty)\right\}, \\ C^0_{a,b,\lambda} &= \{f \mid f \in C_{a,b,\lambda}, f(0) = 0\}, \end{aligned} \tag{1.5}$$

where $\varphi(x) = \sqrt{x(1+x)}, w(x) = x^a(1+x)^{-b}$, $x \in [0,\infty)$, $0 \le a < \lambda \le 1$, and $b \ge 0$.

Moreover, the K-functional is given by

$$K_{\varphi^\lambda}(f;t)_{w,\lambda} = \inf_{g\in D}\left\{\left\|\varphi^{2(1-\lambda)}(f-g)\right\|_w + t\left\|\varphi^{2(2-\lambda)}g''\right\|_w\right\}, \tag{1.6}$$

where $D = \{g \mid g \in C^0_{a,b,\lambda}, g' \in A.C._{loc}[0,\infty), \|\varphi^{2(2-\lambda)}g''\|_w < \infty\}$.

We are now in a position to state our main results.

Theorem 1.2. *If $f \in C^0_{a,b,\lambda}$, then*

$$\left\|\varphi^{2(1-\lambda)}(V_n(f) - f)\right\|_w \le MK_{\varphi^\lambda}\left(f;n^{-1}\right)_{w,\lambda}. \tag{1.7}$$

Theorem 1.3. *Suppose $f \in C^0_{a,b,\lambda}$, $0 < \alpha < 1$. Then the following statements are equivalent:*

$$\begin{aligned} &(1) \quad \varphi^{2(1-\lambda)}(x)w(x)|(V_n(f(x)) - f(x))| = O(n^{-\alpha}), \quad n \ge 2; \\ &(2) \quad K_{\varphi^\lambda}(f;t)_{w,\lambda} = O(t^\alpha), \quad 0 < t < 1. \end{aligned} \tag{1.8}$$

Throughout this paper, M denotes a positive constant independent of x, n, and f which may be different in different places. It is worth mentioning that for $\lambda = 1$, we recover the results of [6].

2. Auxiliary Lemmas

To prove the theorems, we need some lemmas. By simple computation, we have

$$V_n''(f;x) = n(n+1)\sum_{k=0}^{\infty} v_{n+2,k}(x)\left(f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right) \tag{2.1}$$

or

$$V_n''(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right)\left(\frac{k(k-1)}{x^2} - \frac{2k(n+k)}{x(1+x)} + \frac{(n+k)(n+k+1)}{(1+x)^2}\right). \tag{2.2}$$

Lemma 2.1. *Let $c \geq 0,\ d \in \mathbb{R}$. Then*

$$\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}\right)^{-c}\left(1+\frac{k}{n}\right)^{-d} \leq Mx^{-c}(1+x)^{-d}, \quad \text{for } x > 0. \tag{2.3}$$

Proof. We notice [7]

$$\begin{aligned} \sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{n}{k}\right)^{l} &\leq Mx^{-l}, \quad \text{for } l \in \mathbb{N}, \\ \sum_{k=0}^{\infty} v_{n,k}(x)\left(1+\frac{k}{n}\right)^{m} &\leq M(1+x)^{m}, \quad \text{for } m \in \mathbb{Z}. \end{aligned} \tag{2.4}$$

For $c = 0,\ d = 0$, the result of (2.3) is obvious. For $c > 0,\ d \neq 0$, there exists $m \in \mathbb{Z}$, such that $0 < -2d/m < 1$. Using Hölder's inequality, we have

$$\begin{aligned} \sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}\right)^{-c}\left(1+\frac{k}{n}\right)^{-d} &\leq \left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}\right)^{-2c}\right)^{1/2}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(1+\frac{k}{n}\right)^{-2d}\right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{n}{k}\right)^{[2c]+1}\right)^{c/([2c]+1)}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(1+\frac{k}{n}\right)^{m}\right)^{-d/m} \\ &\leq M\left(x^{-([2c]+1)}\right)^{c/([2c]+1)}\left((1+x)^{m}\right)^{-d/m} \\ &\leq Mx^{-c}(1+x)^{-d}. \end{aligned} \tag{2.5}$$

For $c > 0$, $d = 0$ or $c = 0$, $d \neq 0$, the proof is similar to that of (2.5). Thus, this proof is completed. □

Lemma 2.2. *Let $f \in C^0_{a,b,\lambda}$, $n \in \mathbb{N}$. Then*

$$\left| w(x)\varphi^{2(1-\lambda)}(x)V_n(f;x) \right| \le M\left\| \varphi^{2(1-\lambda)} f \right\|_w. \tag{2.6}$$

Proof. By Lemma 2.1, we get

$$\begin{aligned}\left| w(x)\varphi^{2(1-\lambda)}(x)V_n(f;x) \right| &= \left| w(x)\varphi^{2(1-\lambda)}(x)\sum_{k=1}^{\infty} f\left(\frac{k}{n}\right) v_{n,k}(x) \right| \\ &\le \left\| \varphi^{2(1-\lambda)} f \right\|_w w(x)\varphi^{2(1-\lambda)}(x)\sum_{k=1}^{\infty} v_{n,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{2(\lambda-1)}\left(\frac{k}{n}\right) \\ &\le M\left\| \varphi^{2(1-\lambda)} f \right\|_w.\end{aligned} \tag{2.7}$$

□

Lemma 2.3. *Let $f \in C^0_{a,b,\lambda}$, $n \in \mathbb{N}$. Then*

$$\left\| \varphi^{2(2-\lambda)} V_n''(f) \right\|_w \le Mn\left\| \varphi^{2(1-\lambda)} f \right\|_w. \tag{2.8}$$

Proof. For $x \in E_n^c = [0, 1/n]$, $x \neq 0$, $(n+1)x(x+1) \le 2n \cdot 2x \le 4$; using (2.1) and Lemma 2.1, we have

$$\begin{aligned}&\left| w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x) \right| \\ &\quad\le w(x)\varphi^{2(1-\lambda)}(x)n(n+1)x(1+x) \\ &\qquad\times \left(\sum_{k=0}^{\infty} v_{n+2,k}(x) w^{-1}\left(\frac{k+2}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k+2}{n}\right) + 2\sum_{k=0}^{\infty} v_{n+2,k}(x) w^{-1}\left(\frac{k+1}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k+1}{n}\right) \right. \\ &\qquad\quad \left. + \sum_{k=1}^{\infty} v_{n+2,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right) \right) \left\| \varphi^{2(1-\lambda)} f \right\|_w \\ &\quad\le Mnw(x)\varphi^{2(1-\lambda)}(x)w^{-1}(x)\varphi^{-2(1-\lambda)}(x)\left\| \varphi^{2(1-\lambda)} f \right\|_w \\ &\quad\le Mn\left\| \varphi^{2(1-\lambda)} f \right\|_w.\end{aligned} \tag{2.9}$$

For $x \in E_n = (1/n, \infty)$, by (2.2), we get

$$\begin{aligned}
&\left| w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x) \right| \\
&\quad = \left| n^2 w(x)\varphi^{-2\lambda}(x)\sum_{k=1}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right)\left(\left(\frac{k}{n}-x\right)^2 - \frac{1+2x}{n}\left(\frac{k}{n}-x\right) - \frac{x(1+x)}{n}\right) \right| \\
&\quad \le n^2 w(x)\varphi^{-2\lambda}(x)\left\| \varphi^{2(1-\lambda)} f \right\|_w \sum_{k=1}^{\infty} v_{n,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right) \\
&\qquad \cdot \left(\left(\frac{k}{n}-x\right)^2 + \frac{1+2x}{n}\left|\frac{k}{n}-x\right| + \frac{x(1+x)}{n}\right) \\
&\quad := n^2 w(x)\varphi^{-2\lambda}(x)\left\| \varphi^{2(1-\lambda)} f \right\|_w (I_1(n,x) + I_2(n,x) + I_3(n,x)).
\end{aligned} \tag{2.10}$$

Note that for $x \in E_n$, one has the following inequality [7]

$$n^{2m} V_n\left((t-x)^{2m}; x\right) \le M n^m (\varphi(x))^{2m}, \quad m \in \mathbb{N}. \tag{2.11}$$

Applying Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
I_1(n,x) &= \sum_{k=1}^{\infty} v_{n,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)^2 \\
&\le \left(\sum_{k=1}^{\infty} v_{n,k}(x) w^{-2}\left(\frac{k}{n}\right)\varphi^{-4(1-\lambda)}\left(\frac{k}{n}\right)\right)^{1/2}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}-x\right)^4\right)^{1/2} \\
&\le M x^{-a-1+\lambda}(1+x)^{b+\lambda-1}\frac{x(1+x)}{n} \\
&\le M n^{-1} w^{-1}(x)\varphi^{2\lambda}(x),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
I_2(n,x) &= \sum_{k=1}^{\infty} v_{n,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right)\left|\frac{k}{n}-x\right|\frac{1+2x}{n} \\
&\le \frac{1+2x}{n}\left(\sum_{k=1}^{\infty} v_{n,k}(x) w^{-2}\left(\frac{k}{n}\right)\varphi^{-4(1-\lambda)}\left(\frac{k}{n}\right)\right)^{1/2}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}-x\right)^2\right)^{1/2} \\
&\le M w^{-1}(x)\varphi^{2\lambda}(x) n^{-3/2}\left(1+\frac{1}{x}\right)^{1/2}.
\end{aligned} \tag{2.13}$$

Note that for $x > 1/n$, one has $1 + 1/x < 2n$. Hence,

$$I_2(n,x) \le Mn^{-1}w^{-1}(x)\varphi^{2\lambda}(x), \tag{2.14}$$

$$\begin{aligned} I_3(n,x) &= \sum_{k=1}^{\infty} v_{n,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right)\frac{x(1+x)}{n} \\ &\le Mn^{-1}x(1+x)x^{-a-1+\lambda}(1+x)^{b-1+\lambda} \\ &= Mn^{-1}w^{-1}(x)\varphi^{2\lambda}(x). \end{aligned} \tag{2.15}$$

Combining (2.9)–(2.14), we get

$$\left| w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)\right|_w \le Mn\left\|\varphi^{2(1-\lambda)}f\right\|_w. \tag{2.16}$$

Thus,

$$\left\|\varphi^{2(2-\lambda)}V_n''(f)\right\|_w \le Mn\left\|\varphi^{2(1-\lambda)}f\right\|_w. \tag{2.17}$$

The proof is completed. □

Lemma 2.4. *Let $f \in D$, $n \in \mathbb{N}$, and $n \ge 2$. Then*

$$\left\|\varphi^{2(2-\lambda)}V_n''(f)\right\|_w \le M\left\|\varphi^{2(2-\lambda)}f''\right\|_w. \tag{2.18}$$

Proof. (1) For the case $\lambda \ne 1$ or $a \ne 0$, if $\lambda - 2 + b \ge 0$, using (2.1) and Lemma 2.1, we have

$$\begin{aligned} &\left| w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)\right| \\ &\quad = \left| w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=0}^{\infty} v_{n+2,k}(x)\int_0^{1/n}\int_0^{1/n} f''\left(\frac{k}{n}+u+v\right)du\,dv\right| \\ &\quad \le \left\|\varphi^{2(2-\lambda)}f''\right\|_w \left| w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=0}^{\infty} v_{n+2,k}(x) \right. \\ &\qquad\qquad \left. \cdot\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}+u+v\right)^{\lambda-a-2}\left(1+\frac{k}{n}+u+v\right)^{\lambda+b-2} du\,dv\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\varphi^{2(2-\lambda)} f''\right\|_w \left| w(x)\varphi^{2(2-\lambda)}(x) n(n+1)\sum_{k=1}^{\infty} v_{n+2,k}(x)\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k+2}{n}\right)^{\lambda+b-2} du dv\right| \\
&\quad + \left\|\varphi^{2(2-\lambda)} f''\right\|_w \left| w(x)\varphi^{2(2-\lambda)}(x) n(n+1) v_{n+2,0}(x)\int_0^{1/n}\int_0^{1/n}(u+v)^{\lambda-a-2}(1+u+v)^{\lambda+b-2} du dv\right| \\
&\leq \left\|\varphi^{2(2-\lambda)} f''\right\|_w w(x)\varphi^{2(2-\lambda)}(x) n(n+1)\sum_{k=1}^{\infty} v_{n+2,k}(x) n^{-2}\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k+2}{n}\right)^{\lambda+b-2} \\
&\quad + 3^{\lambda+b-2}\left\|\varphi^{2(2-\lambda)} f''\right\|_w w(x)\varphi^{2(2-\lambda)}(x) n(n+1) v_{n+2,0}(x)\int_0^{1/n}\frac{1}{1+a-\lambda}u^{\lambda-a-1}du \\
&\leq 2\cdot 3^{\lambda+b-2}\left\|\varphi^{2(2-\lambda)} f''\right\|_w w(x)\varphi^{2(2-\lambda)}(x)\sum_{k=1}^{\infty} v_{n+2,k}(x)\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k}{n}\right)^{\lambda+b-2} \\
&\quad + 2\cdot 3^{\lambda+b-2}\left\|\varphi^{2(2-\lambda)} f''\right\|_w w(x)\varphi^{2(2-\lambda)}(x) n(n+1) v_{n+2,0}(x)\frac{1}{(1+a-\lambda)(\lambda-a)}\left(\frac{1}{n}\right)^{\lambda-a} \\
&\leq 2\cdot 3^{\lambda+b-2}\left\|\varphi^{2(2-\lambda)} f''\right\|_w \left(1+\frac{x^{2+a-\lambda} n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a}\right).
\end{aligned}
\tag{2.19}
$$

(i) If $x \in E_n^c$,

$$
\frac{x^{2+a-\lambda} n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a} \leq \frac{n(n+1)}{(1+a-\lambda)(\lambda-a\ )}\left(\frac{1}{n}\right)^{2} \leq \frac{2}{(1+a-\lambda)(\lambda-a)}. \tag{2.20}
$$

(ii) If $x \in E_n,\ n \geq 2$,

$$
\begin{aligned}
\frac{x^{2+a-\lambda} n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a} &\leq \frac{n^{\lambda-a}x^2 n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a} \\
&\leq \frac{x^2 n(n+1)}{(1+a-\lambda)(\lambda-a)n(n-1)x^2} \\
&\leq \frac{3}{(1+a-\lambda)(\lambda-a)}.
\end{aligned}
\tag{2.21}
$$

Combining (2.19)–(2.21), we have

$$
\left| w(x)\varphi^{2(2-\lambda)}(x) V_n''(f;x)\right| \leq M\left\|\varphi^{2(2-\lambda)} f''\right\|_w. \tag{2.22}
$$

Thus,

$$
\left\|\varphi^{2(2-\lambda)}(x) V_n''(f)\right\|_w \leq M\left\|\varphi^{2(2-\lambda)} f''\right\|_w. \tag{2.23}
$$

If $\lambda - 2 + b < 0$, we have

$$\begin{aligned}
&\left|w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)\right| \\
&\quad\leq \left\|\varphi^{2(2-\lambda)}f''\right\|_w \left|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=1}^{\infty}v_{n+2,k}(x)\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k}{n}\right)^{\lambda+b-2}dudv\right| \\
&\qquad + \left\|\varphi^{2(2-\lambda)}f''\right\|_w \left|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)v_{n+2,0}(x)\int_0^{1/n}\int_0^{1/n}(u+v)^{\lambda-a-2}dudv\right|.
\end{aligned} \tag{2.24}$$

By using the method similar to that of (2.19)–(2.23), it is not difficult to obtain the same inequality as (2.23).

(2) For the case $\lambda = 1, a = 0$, the proof is similar to that of case (1) and even simpler. Therefore the proof is completed. □

Lemma 2.5 (see [8, page 200]). *Let $\Omega(t)$ be an increasing positive function on $(0, a)$, the inequality $(r > \alpha)$*

$$\Omega(h) \leq M\left[t^{\alpha} + \left(\frac{h}{t}\right)^{r}\Omega(t)\right] \tag{2.25}$$

holds true for $h, t \in (0, a)$. Then one has

$$\Omega(t) = O(t^{\alpha}). \tag{2.26}$$

3. Proofs of Theorems

3.1. Proof of Theorem 1.2

Proof. First, we prove it as follows.

(i) If $x \in E_n^c$, then

$$w(x)\varphi^{2(1-\lambda)}(x)\sum_{k=0}^{\infty}v_{n,k}(x)\left|\int_{k/n}^{x}\left|\frac{k}{n}-u\right|w^{-1}(u)\varphi^{-2(2-\lambda)}(u)du\right| \leq Mn^{-1}. \tag{3.1}$$

(ii) If $x \in E_n$, then

$$V_n\left((t-x)^2(1+t)^{b-2+\lambda};x\right) \leq Mn^{-1}\varphi^2(x)(1+x)^{b-2+\lambda}. \tag{3.2}$$

The Proof of (3.1)

In fact, (i) for $k = 0$, since $x \in E_n^c$, we have

$$\begin{aligned} &w(x)\varphi^{2(1-\lambda)}(x)v_{n,0}(x)\int_0^x uw^{-1}(u)\varphi^{-2(2-\lambda)}(u)du \\ &\quad = w(x)\varphi^{2(1-\lambda)}(x)(1+x)^{-n}\int_0^x u^{\lambda-a-1}(1+u)^{b-2+\lambda}du. \end{aligned} \tag{3.3}$$

If $b-2+\lambda \le 0$, we get

$$\begin{aligned} &w(x)\varphi^{2(1-\lambda)}(x)(1+x)^{-n}\int_0^x u^{\lambda-a-1}(1+u)^{b-2+\lambda}du \\ &\quad \le Mw(x)\varphi^{2(1-\lambda)}(x)(1+x)^{-n}x^{\lambda-a} \\ &\quad \le Mn^{-1}. \end{aligned} \tag{3.4}$$

If $b-2+\lambda > 0$, we have

$$\begin{aligned} &w(x)\varphi^{2(1-\lambda)}(x)(1+x)^{-n}\int_0^x u^{\lambda-a-1}(1+u)^{b-2+\lambda}du \\ &\quad \le Mw(x)\varphi^{2(1-\lambda)}(x)(1+x)^{-n+b-2+\lambda}\int_0^x u^{\lambda-a-1}du \\ &\quad \le Mn^{-1}. \end{aligned} \tag{3.5}$$

(ii) If $k \ge 1$, since $x \in E_n^c$, we have

$$\begin{aligned} &w(x)\varphi^{2(1-\lambda)}(x)\sum_{k=1}^{\infty}v_{n,k}(x)\left|\int_{k/n}^x\left|\frac{k}{n}-u\right|w^{-1}(u)\varphi^{-2(2-\lambda)}(u)du\right| \\ &\quad \le Mw(x)\varphi^{2(1-\lambda)}(x)\sum_{k=1}^{\infty}v_{n,k}(x)\left(\frac{k}{n}-x\right)\varphi^{-2(2-\lambda)}(x)\left(1+\frac{k}{n}\right)^b\left|\int_{k/n}^x u^{-a}du\right| \\ &\quad \le Mw(x)\varphi^{-2}(x)\sum_{k=1}^{\infty}v_{n,k}(x)\left(\frac{k}{n}-x\right)\left(1+\frac{k}{n}\right)^b\left(\left(\frac{k}{n}\right)^{1-a}-x^{1-a}\right) \\ &\quad \le Mw(x)\varphi^{-2}(x)\sum_{k=1}^{\infty}v_{n,k}(x)\left(\frac{k}{n}-x\right)^{2-a}\left(1+\frac{k}{n}\right)^b \le Mn^{-1}. \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we obtain (3.1).

The proof of (3.2)

If $b-2+\lambda \le 0$, by (9.5.10) and (9.6.3) of [7], using the Cauchy-Schwarz inequality and the Hölder inequality, we obtain

$$\begin{aligned} V_n\left((t-x)^2(1+t)^{b-2+\lambda};x\right) &\le \left(V_n\left((t-x)^4;x\right)\right)^{1/2}\left(V_n\left((1+t)^{2(b-2+\lambda)};x\right)\right)^{1/2} \\ &\le \left(V_n\left((t-x)^4;x\right)\right)^{1/2}\left(V_n\left((1+t)^{-2};x\right)\right)^{(2-b-\lambda)/2} \\ &\le Mn^{-1}\varphi^2(x)(1+x)^{b-2+\lambda}. \end{aligned} \tag{3.7}$$

If $b-2+\lambda > 0$, by (2.3), we get $V_n((1+t)^{b-2+\lambda};x) \le M(1+x)^{b-2+\lambda}$, and using the Cauchy-Schwarz inequality and the Hölder inequality, we have

$$\begin{aligned} V_n\left((t-x)^2(1+t)^{b-2+\lambda};x\right) &\le \left(V_n\left((t-x)^4;x\right)\right)^{1/2}\left(V_n\left((1+t)^{2(b-2+\lambda)};x\right)\right)^{1/2} \\ &\le Mn^{-1}\varphi^2(x)(1+x)^{b-2+\lambda}. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain (3.2).

Next, we prove Theorem 1.2. For $g \in D$, if $x \in E_n^c$, by (3.1), we have

$$\begin{aligned} &\left|w(x)\varphi^{2(1-\lambda)}(x)\left(V_n(g;x)-g(x)\right)\right| \\ &\quad= \left|w(x)\varphi^{2(1-\lambda)}(x)V_n\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\quad\le w(x)\varphi^{2(1-\lambda)}(x)\left\|\varphi^{2(2-\lambda)}g''\right\|_w V_n\left(\left|\int_x^t |t-u|w^{-1}(u)\varphi^{-2(2-\lambda)}(u)du\right|;x\right) \\ &\quad\le M\left\|\varphi^{2(2-\lambda)}g''\right\|_w w(x)\varphi^{2(1-\lambda)}(x)\sum_{k=0}^{\infty}v_{n,k}(x)\left|\int_{k/n}^x \left|\frac{k}{n}-u\right|w^{-1}(u)\varphi^{-2(2-\lambda)}(u)du\right| \\ &\quad\le Mn^{-1}\left\|\varphi^{2(2-\lambda)}g''\right\|_w. \end{aligned} \tag{3.9}$$

If $x \in E_n$, by (3.2), we get

$$\begin{aligned}
&\left|w(x)\varphi^{2(1-\lambda)}(x)(V_n(g;x)-g(x))\right| \\
&\quad= \left|w(x)\varphi^{2(1-\lambda)}(x)V_n\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\
&\quad\le M\left\|\varphi^{2(2-\lambda)}g''\right\|_w w(x)\varphi^{2(1-\lambda)}(x)V_n\left(\left|\int_x^t |t-u|w^{-1}(u)\varphi^{-2(2-\lambda)}(u)du\right|;x\right) \\
&\quad\le M\left\|\varphi^{2(2-\lambda)}g''\right\|_w \Big|\varphi^{-2}(x)V_n\left((t-x)^2;x\right) \\
&\qquad\qquad + x^{-2-a+\lambda}w(x)\varphi^{2(1-\lambda)}(x)V_n\left((t-x)^2(1+t)^{b-2+\lambda};x\right)\Big| \\
&\quad\le Mn^{-1}\left\|\varphi^{2(2-\lambda)}g''\right\|_w.
\end{aligned} \tag{3.10}$$

Therefore, for $f \in C^0_{a,b,\lambda}$, $g \in D$, by Lemma 2.2 and (3.9), (3.10), and the definition of $K_{\varphi^\lambda}(f;n^{-1})_{w,\lambda}$, we obtain

$$\begin{aligned}
&\left|w(x)\varphi^{2(1-\lambda)}(x)(V_n(f;x)-f(x))\right| \\
&\quad\le \left|w(x)\varphi^{2(1-\lambda)}(x)(V_n(f-g;x))\right| + \left|w(x)\varphi^{2(1-\lambda)}(x)(f(x)-g(x))\right| \\
&\qquad + \left|w(x)\varphi^{2(1-\lambda)}(x)(V_n(g;x)-g(x))\right| \\
&\quad\le M\left\|\varphi^{2(1-\lambda)}(f-g)\right\|_w + \left|w(x)\varphi^{2(1-\lambda)}(x)(f(x)-g(x))\right| \\
&\qquad + \left|w(x)\varphi^{2(1-\lambda)}(x)(V_n(g;x)-g(x))\right| \\
&\quad\le M\left\{\left\|\varphi^{2(1-\lambda)}(f-g)\right\|_w + n^{-1}\left\|\varphi^{2(2-\lambda)}g''\right\|_w\right\}.
\end{aligned} \tag{3.11}$$

Taking the infimum on the right-hand side over all $g \in D$, we get

$$\left|w(x)\varphi^{2(1-\lambda)}(x)(V_n(f;x)-f(x))\right| \le MK_{\varphi^\lambda}\left(f;n^{-1}\right)_{w,\lambda}. \tag{3.12}$$

This completes the proof of Theorem 1.2. □

3.2. Proof of Theorem 1.3

Proof. By Theorem 1.2, we know (2) ⇒ (1). Now,we will prove (1) ⇒ (2). In view of (1), we get

$$\left\|\varphi^{2(1-\lambda)}(V_n(f)-f)\right\|_w \le Mn^{-\alpha}. \tag{3.13}$$

By the definition of K-functional, we may choose $g \in D$ to satisfy

$$\left\|\varphi^{2(1-\lambda)}(f-g)\right\|_w + n^{-1}\left\|\varphi^{2(2-\lambda)}g''\right\|_w \le 2K_{\varphi^\lambda}\left(f;n^{-1}\right)_{w,\lambda}. \tag{3.14}$$

Using Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
K_{\varphi^\lambda}(f;t)_{w,\lambda} &\le \left\|\varphi^{2(1-\lambda)}(V_n(f)-f)\right\|_w + t\left\|\varphi^{2(2-\lambda)}V_n''(f)\right\|_w \\
&\le Mn^{-\alpha} + t\left(\left\|\varphi^{2(2-\lambda)}V_n''(f-g)\right\|_w + \left\|\varphi^{2(2-\lambda)}V_n''(g)\right\|_w\right) \\
&\le Mn^{-\alpha} + t\left(nM\left\|\varphi^{2(1-\lambda)}(f-g)\right\|_w + M\left\|\varphi^{2(2-\lambda)}g''\right\|_w\right) \\
&\le Mn^{-\alpha} + tnM\left(\left\|\varphi^{2(1-\lambda)}(f-g)\right\|_w + n^{-1}\left\|\varphi^{2(2-\lambda)}g''\right\|_w\right).
\end{aligned} \tag{3.15}$$

Taking the infimum on the right-hand side over all $g \in D$, we get

$$K_{\varphi^\lambda}(f;t)_{w,\lambda} \le M\left(n^{-\alpha} + \frac{t}{n^{-1}}K_{\varphi^\lambda}\left(f;n^{-1}\right)_{w,\lambda}\right). \tag{3.16}$$

By Lemma 2.4, we get

$$K_{\varphi^\lambda}\left(f;n^{-1}\right)_{w,\lambda} \le M(n^{-\alpha}). \tag{3.17}$$

Leting $(n+1)^{-1} < t \le n^{-1}$, we get

$$K_{\varphi^\lambda}(f;t)_{w,\lambda} \le MK_{\varphi^\lambda}\left(f;n^{-1}\right)_{w,\lambda} \le M\left(\frac{n}{n+1}\right)^{-\alpha}(n+1)^{-\alpha} \le M(n+1)^{-\alpha} \le Mt^{\alpha}. \tag{3.18}$$

This completes the proof of Theorem 1.3. □

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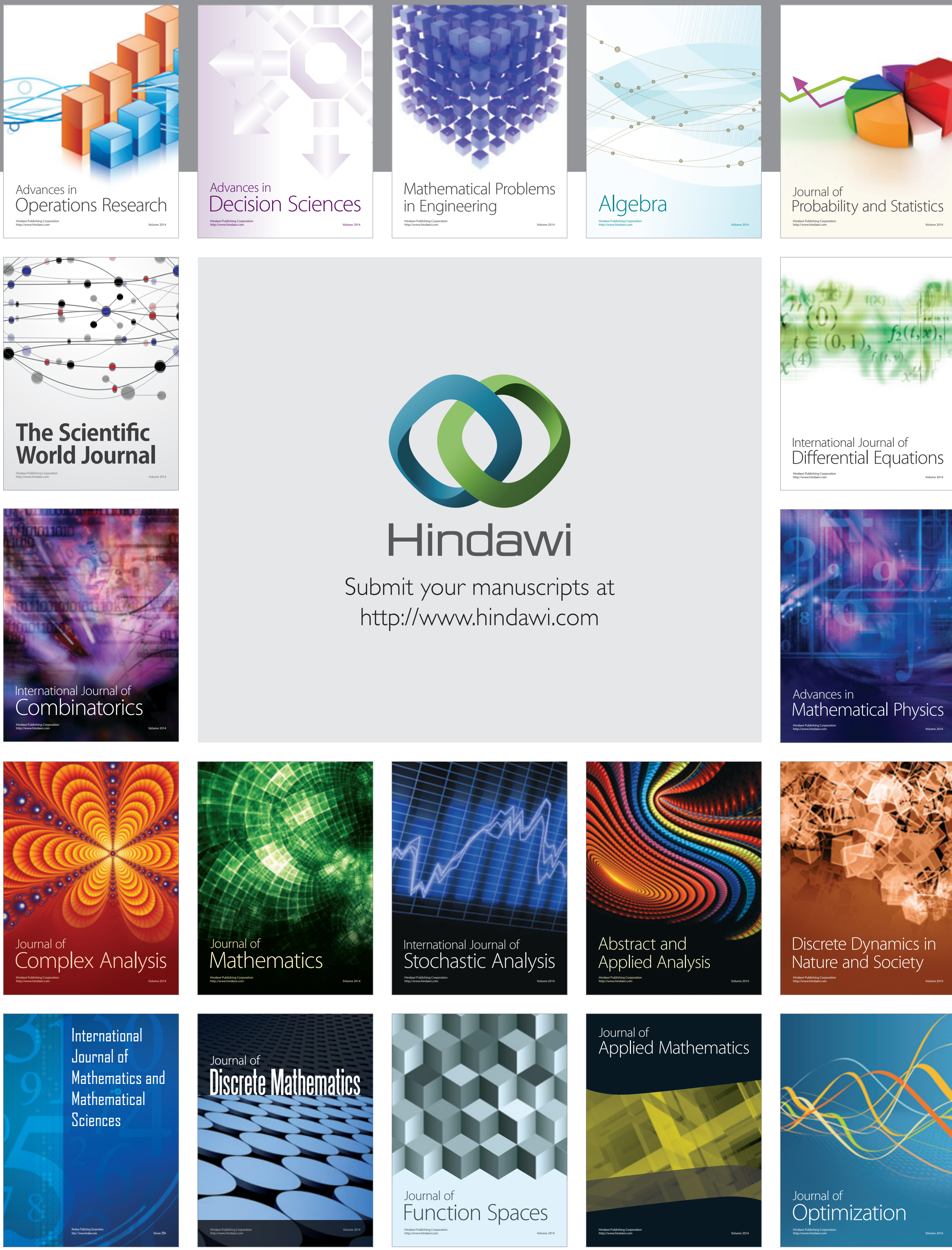
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