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Research Article

Unital Compact Homomorphisms between Extended Analytic Lipschitz Algebras

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Let X and K be compact plane sets with $K \subseteq X$. We define $A(X,K) = \{f \in C(X) : f|_K \in A(K)\}$, where $A(K) = \{g \in C(X) : g$ is analytic on $\text{int}(K)\}$. For $\alpha \in (0,1]$, we define $\text{Lip}(X,K,\alpha) = \{f \in C(X) : p_{\alpha,K}(f) = \sup\{|f(z) - f(w)|/|z-w|^\alpha : z,w \in K, z \neq w\} < \infty\}$ and $\text{Lip}_A(X,K,\alpha) = A(X,K) \cap \text{Lip}(X,K,\alpha)$. It is known that $\text{Lip}_A(X,K,\alpha)$ is a natural Banach function algebra on X under the norm $\|f\|_{\text{Lip}(X,K,\alpha)} = \|f\|_X + p_{\alpha,K}(f)$, where $\|f\|_X = \sup\{|f(x)| : x \in X\}$. These algebras are called extended analytic Lipschitz algebras. In this paper we study unital homomorphisms from natural Banach function subalgebras of $\text{Lip}_A(X_1,K_1,\alpha_1)$ to natural Banach function subalgebras of $\text{Lip}_A(X_2,K_2,\alpha_2)$ and investigate necessary and sufficient conditions for which these homomorphisms are compact. We also determine the spectrum of unital compact endomorphisms of $\text{Lip}_A(X,K,\alpha)$.

1. Introduction and Preliminaries

We let $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, $\mathbb{D}(\lambda,r) = \{z \in \mathbb{C} : |z-\lambda| < r\}$, and $\overline{\mathbb{D}(\lambda,r)} = \{z \in \mathbb{C} : |z-\lambda| \leq r\}$ denote the field of complex numbers, the open unit disc, the closed unit disc, and the open and closed discs with center at λ and radius r, respectively. We also denote $\mathbb{D}(0,r)$ by $\mathbb{D}_r$.

Let A and B be unital commutative semisimple Banach algebras with maximal ideal spaces $\mathfrak{M}(A)$ and $\mathfrak{M}(B)$. A homomorphism $T : A \to B$ is called *unital* if $T1_A = 1_B$. If T is a unital homomorphism from A into B, then T is continuous and there exists a norm-continuous map $\varphi : \mathfrak{M}(B) \to \mathfrak{M}(A)$ such that $\widehat{Tf} = \widehat{f} \circ \varphi$ for all $f \in A$, where $\widehat{g}$ is the Gelfand transform g. In fact, φ is equal the adjoint of $T^* : B^* \to A^*$ restricted to $\mathfrak{M}(B)$. Note that T^* is a weak*-weak* continuous map from B^* into A^*. Thus φ is a continuous map from $\mathfrak{M}(B)$ with the Gelfand topology into $\mathfrak{M}(A)$ with the Gelfand topology.

Let A be a unital commutative semisimple Banach algebra, and let T be an endomorphism of A, a homomorphism from A into A. We denote the spectrum of T by $\sigma(T)$ and define

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}. \tag{1.1}$$

For a compact Hausdorff space X, we denote by $C(X)$ the Banach algebra of all continuous complex-valued functions on X.

Definition 1.1. Let X be a compact Hausdorff space. A *Banach function algebra* on X is a subalgebra A of $C(X)$ which contains 1_X, the constant function 1 on X, separates the points of X, and is a unital Banach algebra with an algebra norm $\|\cdot\|$. If the norm of a Banach function algebra on X is $\|\cdot\|_X$, the uniform norm on X, it is called a *uniform algebra* on X.

Let A and B be Banach function algebras on X and Y, respectively. If $\varphi : Y \to X$ is a continuous mapping such that $f \circ \varphi \in B$ for all $f \in A$ and if $T : A \to B$ is defined by $Tf = f \circ \varphi$, then T is a unital homomorphism, which is called the *induced homomorphism* from A into B by φ. In particular, if $Y = X$ and $B = A$, then T is called the *induced endomorphism* of A by the self-map φ of X.

Let A be a Banach function algebra on a compact Hausdorff space X. For $x \in X$, the map $e_x : A \to \mathbb{C}$, defined by $e_x(f) = f(x)$, is an element of $\mathfrak{M}(A)$ and is called the *evaluation homomorphism* on A at x. This fact implies that A is semisimple and $\|f\|_X \le \|\widehat{f}\|_{\mathfrak{M}(A)}$ for all $f \in A$. Note that the map $x \mapsto e_x : X \to \mathfrak{M}(A)$ is a continuous one-to-one mapping. If this map is onto, we say that A is *natural*.

Proposition 1.2. *Let X and Y be compact Hausdorff spaces, and let A and B be natural Banach function algebras on X and Y, respectively. Then every unital homomorphism $T : A \to B$ is induced by a unique continuous map $\varphi : Y \to X$. In particular, if X is a compact plane set and the coordinate function Z belongs to A, then $\varphi = TZ$ and so $\varphi \in B$.*

Proof. Let $T : A \to B$ be a unital homomorphism. Since A and B are unital commutative semisimple Banach algebras, there exists a continuous map $\psi : \mathfrak{M}(B) \to \mathfrak{M}(A)$ such that $\widehat{Tf} = \widehat{f} \circ \psi$ for all $f \in A$. The naturality of the Banach function algebra A on X implies that the map $J_A : X \to \mathfrak{M}(A)$, defined by $J_A(x) = e_x$, is a homeomorphism and so $J_A^{-1} : \mathfrak{M}(A) \to X$ is continuous. Since B is a Banach function algebra on Y, the map $J_B : Y \to \mathfrak{M}(B)$, defined by $J_B(y) = e_y$, is continuous. We now define the map $\varphi : Y \to X$ by $\varphi = J_A^{-1} \circ \psi \circ J_B$. Clearly, φ is continuous. Let $f \in A$. Since

$$\begin{aligned}(Tf)(y) &= \widehat{Tf}(e_y) = \left(\widehat{f} \circ \psi\right)(J_B(y)) = \left(\widehat{f} \circ J_A\right)(\varphi(y)) \\ &= \widehat{f}\left(e_{\varphi(y)}\right) = e_{\varphi(y)}(f) = f(\varphi(y)) \\ &= (f \circ \varphi)(y),\end{aligned} \tag{1.2}$$

for all $y \in Y$, we have $Tf = f \circ \varphi$. Therefore, T is induced by φ.

Now, let X be a compact plane set, and let $Z \in A$. Then $\varphi = Z \circ \varphi = TZ$, and so $\varphi \in B$. □

Corollary 1.3. *Let X be a compact Hausdorff space, and let A be a natural Banach function algebra on X. Then every unital endomorphism T of A is induced by a unique continuous self-map φ of X. In particular, if X is a compact plane set and A contains the coordinate function Z, then $\varphi = TZ$ and so $\varphi \in A$.*

Definition 1.4. Let X be a compact plane set which is connected by rectifiable arcs, and let $\delta(z,w)$ be the geodesic metric on X, the infimum of the length of the arcs joining z and w. X is called *uniformly regular* if there exists a constant C such that, for all $z,w \in X$, $\delta(z,w) \leq C|z-w|$.

The following lemma occurs in [1] but it is important and we will be using it in the sequel.

Lemma 1.5 (see [1, Lemma 1.5]). *Let H and K be two compact plane sets with $H \subsetneq \mathrm{int}(K)$. Then there exists a finite union of uniformly regular sets in $\mathrm{int}(K)$ containing H, namely Y, and then a positive constant C such that for every analytic complex-valued function f on $\mathrm{int}(K)$ and any $z,w \in H$,*

$$\left|f(z) - f(w)\right| \leq C|z-w|\left(\|f\|_Y + \|f'\|_Y\right). \tag{1.3}$$

Let X be a compact plane set. We denote by $A(X)$ the algebra of all continuous complex-valued functions on X which are analytic on $\mathrm{int}(X)$, the interior of X, and call it the *analytic uniform algebra* on X. It is known that $A(X)$ is a natural uniform algebra on X.

Let X and K be compact plane sets such that $K \subseteq X$. We define $A(X,K) = \{f \in C(X) : f|_K \in A(K)\}$. Clearly, $A(X,K) = A(X)$ if $K = X$, and $A(X,K) = C(X)$ if $\mathrm{int}(K)$ is empty. We know that $A(X,K)$ is a natural uniform algebra on X (see [2]) and call it the *extended analytic uniform algebra* on X with respect to K.

Let (X,d) be a compact metric space. For $\alpha \in (0,1]$, we denote by $\mathrm{Lip}(X,\alpha)$ the algebra of all complex-valued functions f for which $p_{\alpha,X}(f) = \sup\{|f(z) - f(w)|/d^{\alpha}(z,w) : z,w \in X, z \neq w\} < \infty$. For $f \in \mathrm{Lip}(X,\alpha)$, we define the *$\alpha$-Lipschitz* norm f by $\|f\|_{\mathrm{Lip}(X,\alpha)} = \|f\|_X + p_{\alpha,X}(f)$. Then $(\mathrm{Lip}(X,\alpha), \|\cdot\|_{\mathrm{Lip}(X,\alpha)})$ is a unital commutative Banach algebra. For $\alpha \in (0,1)$, we denote by $\mathrm{lip}(X,\alpha)$ the algebra of all complex-valued functions f on X for which $|f(z) - f(w)|/d^{\alpha}(z,w) \to 0$ as $d(z,w) \to 0$. Then $\mathrm{lip}(X,\alpha)$ is a unital closed subalgebra of $\mathrm{Lip}(X,\alpha)$. These algebras are called *Lipschitz algebras* of order α and were first studied by Sherbert in [3, 4]. We know that the Lipschitz algebras $\mathrm{Lip}(X,\alpha)$ and $\mathrm{lip}(X,\alpha)$ are natural Banach function algebras on X.

Let (X,d) be a compact metric space, and let K be a compact subset of X. For $\alpha \in (0,1]$, we denote by $\mathrm{Lip}(X,K,\alpha)$ the algebra of all complex-valued functions f on X for which $p_{\alpha,K}(f) = \sup\{|f(z) - f(w)|/d^{\alpha}(z,w) : z,w \in K, z \neq w\} < \infty$. In fact, $\mathrm{Lip}(X,K,\alpha) = \{f \in C(X) : f|_K \in \mathrm{Lip}(K,\alpha)\}$. For $f \in \mathrm{Lip}(X,K,\alpha)$, we define $\|f\|_{\mathrm{Lip}(X,K,\alpha)} = \|f\|_X + p_{\alpha,K}(f)$. Then $\mathrm{Lip}(X,K,\alpha)$ under the algebra norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$ is a unital commutative Banach algebra. Moreover, $\mathrm{Lip}(X,\alpha)$ is a subalgebra of $\mathrm{Lip}(X,K,\alpha)$; $\mathrm{Lip}(X,K,\alpha) = \mathrm{Lip}(X,\alpha)$ if $X \setminus K$ is finite, and $\mathrm{Lip}(X,K,\alpha) = C(X)$ if K is finite. For $\alpha \in (0,1)$, we denote by $\mathrm{lip}(X,K,\alpha)$ the algebra of all complex-valued functions f on X for which $|f(z) - f(w)|/d^{\alpha}(z,w) \to 0$ as $d(z,w) \to 0$ with $z,w \in K$. In fact, $\mathrm{lip}(X,K,\alpha) = \{f \in C(X) : f|_K \in \mathrm{lip}(K,\alpha)\}$. Clearly, $\mathrm{lip}(X,K,\alpha)$ is a closed unital subalgebra of $\mathrm{Lip}(X,K,\alpha)$. Moreover, $\mathrm{lip}(X,\alpha)$ is a subalgebra of $\mathrm{lip}(X,K,\alpha)$; $\mathrm{lip}(X,K,\alpha) = \mathrm{lip}(X,\alpha)$ if $X \setminus K$ is finite, and $\mathrm{lip}(X,K,\alpha) = C(X)$ if K is finite. The Banach algebras $\mathrm{Lip}(X,K,\alpha)$ and $\mathrm{lip}(X,K,\alpha)$ are Banach function algebras on X and were first introduced by Honary and Moradi in [5].

Let X be a compact plane set. We define $\mathrm{Lip}_A(X,\alpha) = \mathrm{Lip}(X,\alpha) \cap A(X)$ for $\alpha \in (0,1]$ and $\mathrm{lip}_A(X,\alpha) = \mathrm{Lip}(X,\alpha) \cap A(X)$ for $\alpha \in (0,1)$. These algebras are called *analytic Lipschitz algebras*. We know that analytic Lipschitz algebras $\mathrm{Lip}_A(X,\alpha)$ and $\mathrm{lip}_A(X,\alpha)$ under the norm $\|\cdot\|_{\mathrm{Lip}(X,\alpha)}$ are natural Banach function algebras on X (see [6]).

Let X and K be compact plane sets with $K \subseteq X$. We define $\mathrm{Lip}_A(X,K,\alpha) = \mathrm{Lip}(X,K,\alpha) \cap A(X,K)$ for $\alpha \in (0,1]$ and $\mathrm{lip}_A(X,K,\alpha) = \mathrm{lip}(X,K,\alpha) \cap A(X,K)$ for $\alpha \in (0,1)$. Then $\mathrm{Lip}_A(X,K,\alpha)$ and $\mathrm{lip}(X,K,\alpha)$ are closed unital subalgebras of $\mathrm{Lip}(X,K,\alpha)$ and $\mathrm{lip}(X,K,\alpha)$ under the norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$, respectively. Moreover, $\mathrm{Lip}_A(X,K,\alpha) = \mathrm{Lip}_A(X,\alpha)[\mathrm{lip}_A(X,K,\alpha) = \mathrm{Lip}_A(X,\alpha)]$ if $K = X$, and $\mathrm{Lip}_A(X,K,\alpha) = \mathrm{Lip}(X,K,\alpha)[\mathrm{lip}_A(X,K,\alpha) = \mathrm{lip}(X,K,\alpha)]$ if $\mathrm{int}(K)$ is empty.

The algebras $\mathrm{Lip}_A(X,K,\alpha)$ and $\mathrm{lip}(X,K,\alpha)$ are called *extended analytic Lipschitz algebras* and were first studied by Honary and Moradi in [5]. They showed that the extended analytic Lipschitz algebras $\mathrm{Lip}_A(X,K,\alpha)$ and $\mathrm{lip}_A(X,K,\alpha)$ under the norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$ are natural Banach function algebras on X [5, Theorem 2.4].

Behrouzi and Mahyar in [1] studied endomorphisms of some uniform subalgebras of $A(X)$ and some Banach function subalgebras of $\mathrm{Lip}_A(X,\alpha)$ and investigated some necessary and sufficient conditions for these endomorphisms to be compact, where X is a compact plane set and $\alpha \in (0,1]$.

In Section 2, we study unital homomorphisms from natural Banach function subalgebras of $\mathrm{Lip}_A(X_1,K_1,\alpha_1)$ to natural Banach function subalgebras of $\mathrm{Lip}_A(X_2,K_2,\alpha_2)$ and investigate necessary and sufficient conditions for which these homomorphisms are compact. In Section 3, we determine the spectrum of unital compact endomorphisms of $\mathrm{Lip}_A(X,K,\alpha)$.

2. Unital Compact Homomorphisms

We first give a sufficient condition for which a continuous map $\varphi : X_2 \to X_1$ induces a unital homomorphism T from a subalgebra B_1 of $A(X_1,K_1)$ into a subalgebra B_2 of $A(X_2,K_2)$.

Proposition 2.1. *Let X_j and K_j be compact plane sets with $\mathrm{int}(K_j) \neq \emptyset$ and $K_j \subseteq X_j$, and let B_j be a subalgebra of $A(X_j,K_j)$ which is a natural Banach function algebra on X_j under an algebra norm $\|\cdot\|_j$, where $j \in \{1,2\}$. If $\varphi \in B_2$ with $\varphi(X_2) \subseteq \mathrm{int}(K_1)$, then φ induces a unital homomorphism $T : B_1 \to B_2$. Moreover, if $Z \in B_1$, then $\varphi = TZ$.*

Proof. The naturality of Banach function algebra B_2 on X_2 implies that $\sigma_{B_2}(h) = h(X_2)$, where $\sigma_A(h)$ is the spectrum of $h \in A$ in the Banach algebra A. Let $f \in B_1$. Since $\varphi \in B_2$, $\varphi(X_2) \subseteq \mathrm{int}(K_1)$, and f is analytic on $\mathrm{int}(K_1)$, we conclude that f is analytic on an open neighborhood of $\sigma_{B_2}(\varphi)$. By using the Functional Calculus Theorem [2, Theorem 5.1 in Chapter I], there exists $g \in B_2$ such that $\widehat{g} = f \circ \widehat{\varphi}$ on $\mathcal{M}(B_2)$. It follows that

$$\begin{aligned} g(z) &= e_z(g) = \widehat{g}(e_z) = f(\widehat{\varphi}(e_z)) \\ &= f(e_z(\varphi)) = f(\varphi(z)) = (f \circ \varphi)(z), \end{aligned} \tag{2.1}$$

for all $z \in X_2$ and so $g = f \circ \varphi$. Therefore, $f \circ \varphi \in B_2$. This implies that the map $T : B_1 \to B_2$ defined by $Tf = f \circ \varphi$ is a unital homomorphism from B_1 into B_2, which is induced by φ. Now let $Z \in B_1$. Then $\varphi = TZ$ by Proposition 1.2. □

Corollary 2.2. *Let X and K be compact plane sets with* $\mathrm{int}(K) \neq \emptyset$ *and $K \subseteq X$. Let B be a subalgebra of $A(X,K)$ which is a natural Banach function algebra on X under an algebra norm $\|\cdot\|_B$. If $\varphi \in B$ with $\varphi(X) \subseteq \mathrm{int}(K)$, then φ induces a unital endomorphism T of B. Moreover, if $Z \in B$, then $\varphi = TZ$.*

Proposition 2.3. *Suppose that $\alpha_j \in (0,1]$, $z_j \in \mathbb{C}$, $0 < r_j < R_j$, $G_j = \mathbb{D}(z_j, R_j)$, $\Omega_j = \mathbb{D}(z_j, r_j)$, $X_j = \overline{G_j}$, and $K_j = \overline{\Omega_j}$, where $j \in \{1,2\}$. Then for each $\rho \in (r_1, R_1]$ there exists a continuous map $\varphi_\rho : X_2 \to X_1$ with $\varphi_\rho(X_2) = \overline{\mathbb{D}(z_1,\rho)}$ such that $\varphi_\rho \in Lip_A(X_2, K_2, \alpha_2)$ and φ_ρ does not induce any homomorphism from $Lip_A(X_1, K_1, \alpha_1)$ to $Lip_A(X_2, K_2, \alpha_2)$.*

Proof. Let $\rho \in (r_1, R_1]$. We define the map $\varphi_\rho : X_2 \to X_1$ by

$$\varphi_\rho(z) = \begin{cases} z_1 + \dfrac{\rho(z-z_2)}{r_2} & |z-z_2| \leq r_2, \\ z_1 + \dfrac{\rho(z-z_1)}{|z-z_2|} & r_2 < |z-z_2| \leq R_2. \end{cases} \tag{2.2}$$

Clearly, φ_ρ is a continuous mapping, $\varphi_\rho(X_2) = \overline{\mathbb{D}(z_1,\rho)}$, and $\varphi_\rho \in \mathrm{Lip}_A(X_2, K_2, \alpha_2)$. We now define the function $f_\rho : X_1 \to \mathbb{C}$ by

$$f_\rho(z) = \begin{cases} \dfrac{\rho(z-z_1)}{r_1} & |z-z_1| \leq r_1, \\ \dfrac{\rho(z-z_1)}{|z-z_1|} & r_1 < |z-z_1| \leq R_1. \end{cases} \tag{2.3}$$

Then, $f_\rho \in \mathrm{Lip}_A(X_1, K_1, \alpha_1)$. Since $0 < r_1 r_2/\rho < r_2$ and

$$(f_\rho \circ \varphi_\rho)(z) = \begin{cases} \dfrac{\rho^2}{r_1 r_2}(z-z_2) & |z-z_2| \leq \dfrac{r_1 r_2}{\rho}, \\ \dfrac{\rho(z-z_2)}{|z-z_2|} & \dfrac{r_1 r_2}{\rho} < |z-z_2| \leq R_2, \end{cases} \tag{2.4}$$

we conclude that $f_\rho \circ \varphi_\rho \notin \mathrm{Lip}_A(X_2, K_2, \alpha_2)$. Therefore, φ_ρ does not induce any homomorphism from $\mathrm{Lip}_A(X_1, K_1, \alpha_1)$ to $\mathrm{Lip}_A(X_2, K_2, \alpha_2)$. Hence, the proof is complete. □

Corollary 2.4. *Suppose that $\alpha \in (0,1]$, $\lambda \in \mathbb{C}$, $0 < r < R$, $G = \mathbb{D}(\lambda, R)$, $\Omega = \mathbb{D}(\lambda, r)$, $X = \overline{G}$, and $K = \overline{\Omega}$. Then for each $\rho \in (r, R]$, there exists a continuous self-map φ_ρ of X with $\varphi_\rho(X) = \overline{\mathbb{D}(\lambda,\rho)}$ such that $\varphi_\rho \in Lip_A(X, K, \alpha)$ and φ_ρ does not induce any endomorphism of $Lip_A(X, K, \alpha)$.*

We now give a sufficient condition for a unital homomorphism from a subalgebra B_1 of $\mathrm{Lip}_A(X_1, K_1, \alpha_1)$ into a subalgebra B_2 of $\mathrm{Lip}_A(X_2, K_2, \alpha_2)$ to be compact.

Theorem 2.5. *Suppose that $\alpha_j \in (0,1]$, X_j and K_j are compact plane sets with* $\mathrm{int}(K_j) \neq \emptyset$ *and $K_j \subseteq X_j$, and B_j is a subalgebra of $Lip_A(X_j, K_j, \alpha_j)$ which is a natural Banach function algebra on X_j under the norm $\|\cdot\|_{Lip(X_j,K_j,\alpha_j)}$, where $j \in \{1,2\}$. Let $\varphi : X_2 \to X_1$ be a continuous mapping. If φ is constant or $\varphi \in B_2$ with $\varphi(X_2) \subseteq \mathrm{int}(K_1)$, then φ induces a unital compact homomorphism $T : B_1 \to B_2$.*

Proof. If $\varphi : X_2 \to X_1$ is constant, then the map $T : B_1 \to B_2$ defined by $Tf = f \circ \varphi$ is a unital homomorphism from B_1 into B_2 with dim $T(B_1) \leq 1$, and so it is compact.

Let $\varphi : X_2 \to X_1$ be a nonconstant mapping with $\varphi \in B_2$ and $\varphi(X_2) \subseteq \Omega_1$. Then the map $T : B_1 \to B_2$ defined by $Tf = f \circ \varphi$ is a unital homomorphism from B_1 to B_2 by Proposition 2.1. To prove the compactness of T, let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in B_1 with $\|f_n\|_{\mathrm{Lip}(X_1,K_1,\alpha_1)} \leq 1$ for all $n \in \mathbb{N}$. This implies that $\{f_n|_{K_1}\}_{n=1}^{\infty}$ is a bounded sequence in $C(K_1)$ which is equicontinuous on $(K_1, d_1^{\alpha_1})$. By Arzela-Ascoli's theorem, $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that $\{f_{n_j}|_{K_1}\}_{j=1}^{\infty}$ is convergent in $C(K_1)$. Since $f_{n_j}|_{K_1} \in A(K_1)$ for all $j \in \mathbb{N}$, $\{f_{n_j}|_{K_1}\}_{j=1}^{\infty}$ is convergent in $A(K_1)$. By Montel's theorem, the sequences $\{f_{n_j}\}_{j=1}^{\infty}$ and $\{f'_{n_j}\}_{j=1}^{\infty}$ are uniformly convergent on the compact subsets of $\mathrm{int}(K_1)$. Since $\varphi(X_2)$ and K_1 are compact sets in the complex plane and $\varphi(X_2) \subseteq \mathrm{int}(K_1)$, by using Lemma 1.5, we deduce that there exists a finite union of uniformly regular sets in $\mathrm{int}(K_1)$ containing $\varphi(X_2)$, namely Y, and then a positive constant C such that for every analytic complex-valued function f on $\mathrm{int}(K_1)$ and any $z, w \in \varphi(X_2)$

$$\left|f(z) - f(w)\right| \leq C|z - w|\left(\|f\|_Y + \|f'\|_Y\right). \tag{2.5}$$

Therefore, there exists a positive constant C such that

$$\left|f_{n_j}(\varphi(z)) - f_{n_j}(\varphi(w))\right| \leq C\left|\varphi(z) - \varphi(w)\right|\left(\left\|f_{n_j}\right\|_Y + \left\|f'_{n_j}\right\|_Y\right), \tag{2.6}$$

for all $j \in \mathbb{N}$ and any $z, w \in X_2$. Let $j, k \in \mathbb{N}$. Then, for all $z, w \in K_2$ with $\varphi(z) \neq \varphi(w)$, we have

$$\begin{aligned} &\frac{\left|\left(\left(f_{n_j} \circ \varphi\right) - \left(f_{n_k} \circ \varphi\right)\right)(z) - \left(\left(f_{n_j} \circ \varphi\right) - \left(f_{n_k} \circ \varphi\right)\right)(w)\right|}{|z - w|^{\alpha_2}} \\ &\quad = \frac{\left|\left(f_{n_j} - f_{n_k}\right)(\varphi(z)) - \left(f_{n_j} - f_{n_k}\right)(\varphi(w))\right|}{\left|\varphi(z) - \varphi(w)\right|} \cdot \frac{\left|\varphi(z) - \varphi(w)\right|}{|z - w|^{\alpha_2}} \\ &\quad \leq C p_{\alpha_2,K_2}(\varphi)\left(\left\|f_{n_j} - f_{n_k}\right\|_Y + \left\|f'_{n_j} - f'_{n_k}\right\|_Y\right). \end{aligned} \tag{2.7}$$

The above inequality is certainly true for all $z, w \in K_2$ with $z \neq w$ and $\varphi(z) = \varphi(w)$. Therefore,

$$p_{\alpha_2,K_2}\left(\left(f_{n_j} \circ \varphi\right) - \left(f_{n_k} \circ \varphi\right)\right) \leq C p_{\alpha_2,K_2}(\varphi)\left(\left\|f_{n_i} - f_{n_j}\right\|_Y + \left\|f'_{n_i} - f'_{n_j}\right\|_Y\right), \tag{2.8}$$

and so

$$\left\|\left(f_{n_j} \circ \varphi\right) - \left(f_{n_k} \circ \varphi\right)\right\|_{\mathrm{Lip}(X_2,K_2,\alpha_2)} \leq \left(1 + C p_{\alpha_2,K_2}(\varphi)\right)\left(\left\|f_{n_j} - f_{n_k}\right\|_Y + \left\|f'_{n_j} - f'_{n_k}\right\|_Y\right). \tag{2.9}$$

Since Y is a compact subset of $\mathrm{int}(K_1)$, we deduce that the sequences $\{f_{n_j}\}_{j=1}^{\infty}$ and $\{f'_{n_j}\}_{j=1}^{\infty}$ are convergent uniformly on Y. Therefore, $\{f_{n_j} \circ \varphi\}_{j=1}^{\infty}$ is a Cauchy sequence on $\mathrm{Lip}(X_2, K_2, \alpha_2)$, that is $\{Tf_{n_j}\}_{j=1}^{\infty}$ is convergent in $\mathrm{Lip}(X_2, K_2, \alpha_2)$. Hence, T is compact. □

Corollary 2.6. *Suppose that $\alpha \in (0,1]$, X and K are compact plane sets with $\operatorname{int}(K) \neq \emptyset$, and $K \subseteq X$. Let B be a subalgebra of $Lip_A(X, K, \alpha)$ which is a natural Banach function algebra on X with the norm $\|\cdot\|_{Lip(X_2,K_2,\alpha_2)}$, and let φ be a self-map of X. If φ is constant or $\varphi \in B$ with $\varphi(X) \subseteq \operatorname{int}(K)$, then φ induces a unital compact endomorphism of B.*

Definition 2.7.

(a) A *sector* in $\mathbb{D}(z_0, r)$ at a point $\omega \in \partial\mathbb{D}(z_0, r)$ is the region between two straight lines in $\mathbb{D}(z_0, r)$ that meet at ω and are symmetric about the radius to ω.

(b) If f is a complex-valued function on $\mathbb{D}(z_0, r)$ and $\omega \in \partial\mathbb{D}(z_0, r)$, then $\angle\lim_{z\to\omega} f(z) = L$ means that $f(z) \to L$ as $z \to \omega$ through any sector at ω. When this happens, we say that L is *angular* (or *non-tangential*) *limit* of f at ω.

(c) An analytic map $\varphi : \mathbb{D}(z_0, r) \to \mathbb{D}_\rho$ has an *angular derivation* at a point $\omega \in \partial\mathbb{D}_r(z_0, r)$ if for some $\eta \in \partial\mathbb{D}_\rho$

$$\angle\lim_{z\to\omega} \frac{\eta - f(z)}{\omega - z} \tag{2.10}$$

exists (finitely). We call the limit the *angular derivative* of φ at ω and denote it by $\angle\varphi'(\omega)$.

Lemma 2.8. *Let $0 < r \leq 1$, and let $\varphi : \mathbb{D}(z_0, r) \to \mathbb{D}_\rho$ be an analytic function and $\psi : \mathbb{D} \to \mathbb{D}$ defined by $\psi(z) = (1/\rho)\varphi(z_0 + rz)$. Then φ has angular derivation at $\omega \in \partial\mathbb{D}(z_0, r)$ if and only if ψ has angular derivation at $(\omega - z_0)/r \in \partial\mathbb{D}$. Moreover,*

$$\angle\varphi'(\omega) = \frac{r}{\rho}\angle\psi'\left(\frac{\omega - z_0}{r}\right). \tag{2.11}$$

The following result is a modification of Julia-Caratheodory's theorem. For further details and proof of Julia-Caratheodory's theorem, see [7, pages 295–300].

Theorem 2.9. *Take $0 < r \leq 1$. Let $\varphi : \mathbb{D}(z_0, r) \to \mathbb{D}$ be a nonconstant analytic function and $\omega \in \partial\mathbb{D}(z_0, r)$. Then the following are equivalent:*

(i) $\liminf_{z\to\omega}(\|\varphi\|_{\mathbb{D}_r} - |\varphi(z)|)/(r - |z|) = \delta < \infty$,

(ii) $\angle\lim_{z\to\omega}(\eta - \varphi(z))/(\omega - z)$ *exists for some* $\eta \in \partial\mathbb{D}$,

(iii) $\angle\lim_{z\to\omega}\varphi'(z)$ *exists and* $\angle\lim_{z\to\omega}\varphi(z) = \eta \in \partial\mathbb{D}$.

The boundary point η in (ii) and (iii) is the same, and $\delta > 0$ in (i). Also the limit of the difference quotients in (ii) coincides with the limit of the derivative in (iii), and both are equal to $\omega\overline{\eta}\delta$.

Note that the existence of the angular derivative φ at $\omega \in \partial\mathbb{D}(z_0, r)$, according to Theorem 2.9, is equivalent to $\liminf_{z\to\omega}(\|\varphi\|_{\mathbb{D}(z_0,r)} - |\varphi(z)|)/(r - |z - z_0|) < \infty$. In this case the angular derivative of φ at ω is nonzero.

Proposition 2.10. *Let X be a compact plane set, and let $\mathbb{D}(z_0, r) \subseteq X$ and $K = \overline{\mathbb{D}(z_0, r)}$. Suppose that $c \in \partial\mathbb{D}(z_0, r)$ and $\varphi \in Lip_A(X, K, 1)$ is a nonconstant function such that $|\varphi(c)| = \|\varphi\|_{\overline{\mathbb{D}(z_0,r)}}$. Then the angular derivative of φ at c exists and is nonzero.*

Proof. Let $\Gamma = \{z \in \mathbb{D}(z_0, r) : |z - c|/(r - |z - z_0|) < 2\}$. For every $z \in \Gamma$ we have

$$\frac{\|\varphi\|_{\mathbb{D}(z_0,r)} - |\varphi(z)|}{r - |z - z_0|} = \frac{|\varphi(c)| - |\varphi(z)|}{r - |z - z_0|} \leq \frac{|z - c|}{r - |z - z_0|} \frac{|\varphi(z) - \varphi(c)|}{|z - c|} < 2p_{1,K}(\varphi). \tag{2.12}$$

Therefore, $\liminf_{z \to \omega}(\|\varphi\|_{\mathbb{D}(z_0,r)} - |\varphi(z)|)/(r - |z - z_0|) < \infty$, and, by Theorem 2.9, the proof is complete. □

Definition 2.11.

(a) A plane set X at $c \in \partial X$ has an *internal circular tangent* if there exists a disc D in the complex plane such that $c \in \partial D$ and $\overline{D} \setminus \{c\} \subseteq \operatorname{int}(X)$.

(b) A plane set X is called *strongly accessible from the interior* if it has an internal circular tangent at each point of its boundary. Such sets include the closed unit disc $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}(z_0, r)} \setminus \bigcup_{k=1}^{n} \mathbb{D}(z_k, r_k)$, where closed discs $\overline{\mathbb{D}(z_k, r_k)}$ are mutually disjoint in $\mathbb{D}(z_0, r)$.

(c) A compact plane set X has *peak boundary* with respect to $B \subseteq C(X)$ if for each $c \in \partial X$ there exists a nonconstant function $h \in B$ such that $\|h\|_X = h(c) = 1$.

Example 2.12. The closed unit disc $\overline{\mathbb{D}}$ has peak boundary with respect to $A(\overline{\mathbb{D}})$ because, if $c \in \partial\overline{\mathbb{D}}$, then the function $h : \overline{\mathbb{D}} \to C$ defined by $h(z) = (1/2)(1 + \overline{c}z)$ belongs to $A(\overline{\mathbb{D}})$ and satisfies $\|h\|_{\overline{\mathbb{D}}} = h(c) = 1$.

Let X be a compact plane set. The algebra $R(X)$ consists of all functions in $C(X)$ which can be approximated by rational functions with poles off X. It is known that $R(X)$ is a natural uniform algebra on X.

Example 2.13. Let X be a compact plane set such that $\mathbb{C} \setminus X$ is strongly accessible from the interior. If $R(X) \subseteq B \subseteq C(X)$, then X has a peak boundary with respect to B.

Proof. Let $z_0 \in \mathbb{C} \setminus X$. Since $\mathbb{C} \setminus X$ is strongly accessible from the interior, for each $c \in \partial(\mathbb{C} \setminus X)$, there exists a $\delta > 0$ such that $|c - z_0| = \delta$ and $\overline{\mathbb{D}(z_0, \delta)} \subseteq \operatorname{int}(\mathbb{C} \setminus X)$. Now, we define the function $h : X \to \mathbb{C}$ by

$$h(z) = \frac{\delta^2}{(\overline{c} - \overline{z_0})(z - z_0)}. \tag{2.13}$$

Then $h \in B$, $\|h\|_X = h(c) = 1$. □

Theorem 2.14. *Let X_1 be a compact plane set such that $G_1 = \operatorname{int}(X_1)$ is connected, and $\overline{G_1} = X_1$. Suppose that X_1 has peak boundary with respect to $Lip_A(X_1, 1)$. Let $\Omega_1 \subseteq G_1$ be a bounded connected open set in the complex plane, and let $K_1 = \overline{\Omega_1}$. Let Ω_2 be a bounded connected open set in the complex plane, and let $K_2 = \overline{\Omega_2}$ such that K_2 is strongly accessible from the interior. Suppose that X_2 is a compact plane set such that $K_2 \subseteq X_2$. If $T : Lip_A(X_1, K_1, 1) \to Lip_A(X_2, K_2, 1)$ is a unital compact homomorphism, then T is induced by a continuous mapping $\varphi : X_2 \to X_1$ such that φ is constant on K_2 or $\varphi(K_2) \subseteq G_1 = \operatorname{int}(X_1)$. Moreover, $\varphi = TZ$.*

Proof. Since $\text{Lip}_A(X_1, K_1, 1)$ and $\text{Lip}_A(X_2, K_2, 1)$ are, respectively, natural Banach function algebras on X_1 and X_2, $T : \text{Lip}_A(X_1, K_1, 1) \to \text{Lip}_A(X_2, K_2, 1)$ is a unital homomorphism, X_1 is a compact plane set, and $Z \in \text{Lip}_A(X_1, K_1, 1)$, we conclude that T is induced by $\varphi = TZ$ and so $\varphi \in \text{Lip}_A(X_2, K_2, 1)$ by Proposition 1.2. Suppose that φ is nonconstant on Ω_2. Since φ is analytic on Ω_2, we deduce that $\varphi(\Omega_2)$ is an open subset of X_1 and so $\varphi(\Omega_2) \subseteq G_1$. We now show that $\varphi(K_2) \subseteq G_1$. Suppose that $\varphi(K_2) \nsubseteq G_1$. Then there exists $c \in \partial K_2$ such that $\varphi(c) \in \partial X_1$. Since X_1 has peak boundary with respect to $\text{Lip}_A(X_1, 1)$, there exists a nonconstant function $h \in \text{Lip}_A(X_1, 1)$ such that $\|h\|_{X_1} = h(\varphi(c)) = 1$. We now define the sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions on X_1 by $f_n(z) = (1/n)h^n(z)$. Let $n \in \mathbb{N}$. Then

$$\|f_n\|_{X_1} = \frac{1}{n}\left(\|h\|_{X_1}\right)^n = \frac{1}{n}, \tag{2.14}$$

$$\begin{aligned} p_{1,K_1}(f_n) &= \sup\left\{\frac{|h^n(z) - h^n(w)|}{n|z - w|} : z, w \in K_1, z \neq w\right\} \\ &\leq \sup\left\{\frac{|h(z) - h(w)|}{|z - w|} : z, w \in K_1, z \neq w\right\} \\ &\leq \sup\left\{\frac{|h(z) - h(w)|}{|z - w|} : z, w \in X_1, z \neq w\right\} \\ &\leq p_{1,X_1}(h). \end{aligned} \tag{2.15}$$

Thus

$$\|f_n\|_{\text{Lip}(X_1, K_1, 1)} \leq \frac{1}{n} + p_{1,X_1}(h) \leq 1 + p_{1,X_1}(h), \tag{2.16}$$

by (2.14) and (2.15). This implies that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $\text{Lip}_A(X_1, K_1, 1)$. The compactness of homomorphism T implies that there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and a function g in $\text{Lip}_A(X_2, K_2, 1)$ such that

$$\lim_{j \to \infty}\left\|Tf_{n_j} - g\right\|_{\text{Lip}(X_2, K_2, 1)} = 0. \tag{2.17}$$

This implies that

$$\lim_{j \to \infty}\left\|Tf_{n_j} - g\right\|_{X_2} = 0. \tag{2.18}$$

On the other hand, we have $\|Tf_{n_j}\|_{X_2} \leq 1/n_j$ for all $j \in \mathbb{N}$ by (2.14). Hence,

$$\lim_{j \to \infty}\left\|Tf_{n_j}\right\|_{X_2} = 0. \tag{2.19}$$

By (2.18) and (2.19), $g = 0$. Therefore, by (2.17) we have

$$\lim_{j \to \infty}\left\|Tf_{n_j}\right\|_{\text{Lip}(X_2, K_2, 1)} = 0. \tag{2.20}$$

This implies that

$$\lim_{j\to\infty} p_{1,K_2}\left(f_{n_j}\circ\varphi\right) = 0. \tag{2.21}$$

Assume that $\varepsilon > 0$. By (2.21), there exists a natural number N such that for each $j \in \mathbb{N}$ with $j \geq N$

$$\sup\left\{\frac{\left|\left(f_{n_j}\circ\varphi\right)(z) - \left(f_{n_j}\circ\varphi\right)(w)\right|}{|z-w|} : z, w \in K_2, z \neq w\right\} < \varepsilon. \tag{2.22}$$

In particular,

$$\sup\left\{\frac{\left|((h\circ\varphi)(z))^{n_N} - ((h\circ\varphi)(w))^{n_N}\right|}{n_N|z-w|} : z, w \in K_2, z \neq w\right\} < \varepsilon. \tag{2.23}$$

This implies that

$$\frac{1}{n_N}\sup\left\{\frac{\left|((h\circ\varphi)(z))^{n_N} - ((h\circ\varphi)(c))^{n_N}\right|}{|z-c|} : z \in K_2, z \neq c\right\} < \varepsilon. \tag{2.24}$$

Since $c \in \partial K_2$ and K_2 is strongly accessible from the interior, there exists an open disc $D = \mathbb{D}(z_0, r)$ such that $c \in \partial D$ and $\overline{D}\setminus\{c\} \subseteq \operatorname{int}(K_2)$. Since φ is analytic on $\operatorname{int}(\overline{D}) \subseteq \operatorname{int}(K_2)$ and h is analytic on $\varphi(\overline{D}) \subseteq \operatorname{int}(X_1)$, we deduce that $h\circ\varphi$ is analytic on $\operatorname{int}(\overline{D})$. On the other hand, we can easily show that

$$p_{1,\overline{D}}(h\circ\varphi) \leq p_{1,X_1}(h)\,p_{1,K_2}(\varphi) < \infty. \tag{2.25}$$

Therefore, $h\circ\varphi \in \operatorname{Lip}_A(X_2, \overline{D}, 1)$. Since $\|h\|_{X_1} = h(\varphi(c)) = 1$, we conclude that

$$(h\circ\varphi)(c) = \|h\circ\varphi\|_{\overline{D}} = 1. \tag{2.26}$$

We claim that $h\circ\varphi$ is constant on D. If $h\circ\varphi$ is nonconstant on D, then, by Proposition 2.10, $\angle(h\circ\varphi)'(c)$ exists and is nonzero and since $(h\circ\varphi)^{n_N}(c) = 1$, $(h\circ\varphi)^{n_N}(c) \in \partial D$. If Γ is a sector in D at $c \in \partial D$, then

$$\frac{1}{n_N}\angle\lim_{\substack{z\to c\\ z\in\Gamma}}\left|\frac{(h\circ\varphi)^{n_N}(z) - (h\circ\varphi)^{n_N}(c)}{z-c}\right| \leq \varepsilon \tag{2.27}$$

by (2.24). Thus

$$\frac{1}{n_N}\angle\left((h\circ\varphi)^{n_N}\right)'(c) \leq \varepsilon. \tag{2.28}$$

But

$$\angle((h\circ\varphi)^{n_N})'(c) = n_N (h\circ\varphi)^{n_N-1}(c)\cdot\angle(h\circ\varphi)'(c). \tag{2.29}$$

Hence, by (2.28) we have

$$\angle(h\circ\varphi)'(c) = \frac{1}{n_N}\angle((h\circ\varphi)^{n_N})'(c) \le \varepsilon. \tag{2.30}$$

Since ε is assumed to be a positive number, we conclude that $\angle(h\circ\varphi)'(c) = 0$, contradicting to $\angle(h\circ\varphi)'(c) \neq 0$. Hence, our claim is justified. Since φ is nonconstant on K_2, φ is a nonconstant analytic function on connected open set D. This implies that $\varphi(D)$ is a connected open set in the complex plane. This implies that h is constant on connected open set G_1. The continuity of h on $X_1 = \overline{G_1}$ follows that h is constant on $\overline{G_1} = X_1$. This contradicting to h is nonconstant on X_1. Therefore, $\varphi(K_2) \subseteq G_1$. □

Corollary 2.15. *Let X be a compact plane set such that $G = \text{int}(X)$ is connected and $\overline{G} = X$. Let $\Omega \subseteq G$ be a bounded connected open set in the complex plane, and let $K = \overline{\Omega}$. Suppose that K is strongly accessible from the interior and X has peak boundary with respect to $\text{Lip}_A(X,1)$. If T is a unital compact endomorphism of $\text{Lip}_A(X,K,1)$, then T is induced by a continuous self-map φ of X such that φ is constant on K or $\varphi(K) \subseteq G = \text{int}(X)$. Moreover, $\varphi = TZ$.*

Lemma 2.16. *Let G and Ω be bounded connected open sets in the complex plane with $\Omega \subseteq G$, and let $X = \overline{G}$ and $K = \overline{\Omega}$. Then for each $c \in G\setminus K$ there exists a function $f_c \in \text{Lip}_A(X,K,1)$ such that f_c is not analytic at c.*

Proof. Let $c \in G\setminus K$. Then there is a positive number r such that

$$\{z\in\mathbb{C} : |z-c| \le r\} \subseteq G\setminus K. \tag{2.31}$$

We now define the function $f_c : X \to \mathbb{C}$ by

$$f_c(z) = \begin{cases} z-c & z\in X,\ |z-c|\ge r, \\ \dfrac{(1+r)(z-c)}{1+|z-c|} & z\in X,\ |z-c|<r. \end{cases} \tag{2.32}$$

It is easily seen that $f_c \in \text{Lip}_A(X,K,1)$ and f_c is not analytic at c. □

Definition 2.17. Let X and K be compact plane sets such that $K \subseteq X$. We say that K has *peak K-boundary* with respect to $B \subseteq A(X,K)$ if for each $c \in \partial K$ there is a function $h \in B$ such that h is nonconstant on K and $\|h\|_X = h(c) = 1$.

Example 2.18. Let $r \in (0,1]$ and $K = \overline{\mathbb{D}_r}$. Suppose that $\text{Lip}_A(\overline{\mathbb{D}},K,1) \subseteq B \subseteq A(\overline{\mathbb{D}},K)$. Then K has peak K-boundary with respect to B.

Proof. We first assume that $r = 1$. If for each $c \in \partial\overline{\mathbb{D}}$ the function $h : \overline{\mathbb{D}} \to \mathbb{C}$ defined by $h(z) = (1/2)(1+\bar{c}z)$, then $h \in B$, h is nonconstant on $K = \overline{\mathbb{D}}$ and $h(c) = 1 = \|h\|_{\overline{\mathbb{D}}}$.

We now assume that $0 < r < 1$. For each $c \in \partial K$, set $z_0 = (1+r)c/r$. Then $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Define the function $h : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$h(z) = \begin{cases} -\dfrac{r}{c(z-z_0)} & z \in \overline{\mathbb{D}},\ |z-z_0| \geq 1, \\ -\dfrac{r|z-z_0|}{\overline{c}(z-z_0)} & z \in \overline{\mathbb{D}},\ |z-z_0| < 1. \end{cases} \tag{2.33}$$

It is easily seen that $h \in \operatorname{Lip}_A(\overline{\mathbb{D}}, K, 1)$ and $\|h\|_{\overline{\mathbb{D}}} = 1 = h(c)$. □

Lemma 2.19. *Let Ω be a connected open set in the complex plane, and let φ be a one-to-one analytic function on Ω. If f is a continuous complex-valued function on $\varphi(\Omega)$ and $f \circ \varphi$ is analytic on Ω, then f is an analytic function on $\varphi(\Omega)$.*

Proof. By [8, Theorem 7.5 and Corollary 7.6 in Chapter IV], we deduce that $\varphi(\Omega)$ is a connected open set in the complex plane, $\varphi'(z) \neq 0$ for all $z \in \Omega$, and $\varphi^{-1} : \varphi(\Omega) \to \Omega$ is an analytic function on $\varphi(\Omega)$. Since $f = f \circ \varphi \circ \varphi^{-1}$ on $\varphi(\Omega)$, we conclude that f is analytic on $\varphi(\Omega)$. □

Theorem 2.20. *Let X_1 be a compact plane set such that $G_1 = \operatorname{int}(X_1)$ is connected and $\overline{G_1} = X_1$. Suppose that K_1 is a compact subset of X_1 such that $\Omega_1 = \operatorname{int}(K_1)$ is connected, $K_1 = \overline{\Omega_1}$, and K_1 has peak K_1-boundary with respect to $Lip_A(X_1, K_1, 1)$. Let Ω_2 be a bounded connected open set in the complex plane, and let $K_2 = \overline{\Omega_2}$ such that K_2 is strongly accessible from the interior. Suppose that X_2 is a compact plane set such that $K_2 \subseteq X_2$. If $T : Lip_A(X_1, K_1, 1) \to Lip_A(X_2, K_2, 1)$ is a unital compact homomorphism and TZ is one-to-one on Ω_2, then T is induced by a continuous mapping $\varphi : X_2 \to X_1$ such that $\varphi = TZ$ and $\varphi(K_2) \subseteq \Omega_1 = \operatorname{int}(K_1)$.*

Proof. Since $\operatorname{Lip}_A(X_1, K_1, 1)$ and $\operatorname{Lip}_A(X_2, K_2, 1)$ are, respectively, natural Banach function algebras on X_1 and X_2, $T : \operatorname{Lip}_A(X_1, K_1, 1) \to \operatorname{Lip}_A(X_2, K_2, 1)$ is a unital homomorphism, X_1 is a compact plane set, and $Z \in \operatorname{Lip}_A(X_1, K_1, 1)$, we conclude that T is induced by $\varphi = TZ$ and so $\varphi \in \operatorname{Lip}_A(X_2, K_2, 1)$ by Proposition 1.2.

To prove $\varphi(K_2) \subseteq \Omega_1$, we first show that $\varphi(\Omega_2) \subseteq K_1$. Since φ is a one-to-one analytic mapping on Ω_2, we conclude that $\varphi(\Omega_2)$ is an open set in the complex plane. This follows that $\varphi(\Omega_2) \subseteq \operatorname{int}(X_1) = G_1$ since $\varphi(\Omega_2) \subseteq X_1$. Suppose that $\varphi(\Omega_2) \nsubseteq K_1$. Then there exists $\lambda \in \Omega_2$ such that $\varphi(\lambda) \in G_1 \setminus K_1$. By Lemma 2.16, there exists a function $f \in \operatorname{Lip}_A(X_1, K_1, 1)$ such that f is not analytic at $\varphi(\lambda)$. But $f \circ \varphi = Tf \in \operatorname{Lip}_A(X_2, K_2, 1)$, so that $f \circ \varphi$ is analytic on Ω_2. Since f is continuous on $\varphi(\Omega_2)$ and φ is a one-to-one analytic function on Ω_2, we conclude that f is analytic on $\varphi(\Omega_2)$ by Lemma 2.19. This contradicts to the fact f is not analytic at $\varphi(\lambda) \in \varphi(\Omega_2)$. Therefore, $\varphi(\Omega_2) \subseteq K_1$ so $\varphi(\Omega_2) \subseteq \operatorname{int}(K_1) = \Omega_1$ since $\varphi(\Omega_2)$ is an open set in the complex plane. Since φ is continuous on K_2, $\varphi(\Omega_2) \subseteq \Omega_1$, $K_2 = \overline{\Omega_2}$, and $K_1 = \overline{\Omega_1}$, we can easily show that $\varphi(K_2) \subseteq K_1$. We now show that $\varphi(K_2) \subseteq \Omega_1$. Suppose that $\varphi(K_2) \nsubseteq \Omega_1$. Then there exists $c \in \partial K_2$ such that $\varphi(c) \in \partial K_1$. Since K_1 has peak K_1-boundary with respect to $\operatorname{Lip}_A(X_1, K_1, 1)$, there exists a function $h \in \operatorname{Lip}_A(X_1, K_1, 1)$ such that h is not constant on K_1 and

$$\|h\|_{X_1} = h(\varphi(c)) = 1. \tag{2.34}$$

Applying the similar argument used in the proof of Theorem 2.14, we can prove that h is constant on K_1. This contradiction shows that $\varphi(K_2) \subseteq \Omega_1$. □

Corollary 2.21. *Let X be a compact plane set such that $G = \operatorname{int}(X)$ is connected and $\overline{G} = X$. Let K be a compact subset of X such that $\Omega = \operatorname{int}(K)$ is connected and $K = \overline{\Omega}$. Suppose that K has peak K-boundary with respect to $\mathrm{Lip}_A(X, K, 1)$ and K is strongly accessible from the interior. If T is a unital compact endomorphism of $\mathrm{Lip}_A(X, K, 1)$ and TZ is a one-to-one mapping on Ω, then T is induced by a continuous self-map φ of X such that $\varphi = TZ$ and $\varphi(K) \subseteq \Omega = \operatorname{int}(K)$.*

3. Spectrum of Unital Compact Endomorphisms

In this section we determine the spectrum of a unital compact endomorphism of a subalgebra of $\mathrm{Lip}_A(X, K, \alpha)$ which is a natural Banach function algebra with the norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$.

The following result is a modification of [9, Theorem 1.7] for unital compact endomorphisms of natural Banach function algebras.

Theorem 3.1. *Let X be a compact Hausdorff space and B a natural Banach function algebra on X. If T is a unital compact endomorphism of B induced by a self-map $\varphi : X \to X$, then $\bigcap_{n=0}^{\infty} \varphi_n(X)$ is finite, and if X is connected, $\bigcap_{n=0}^{\infty} \varphi_n(X)$ is singleton where φ_n is the nth iterate of φ, that is, $\varphi_0(x) = x$ and $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$. If $\bigcap_{n=0}^{\infty} \varphi_n(X) = \{x_0\}$, then x_0 is a fixed point for φ. In fact, if $F = \bigcap_{n=0}^{\infty} \varphi_n(X)$, then $\varphi(F) = F$.*

Theorem 3.2. *Suppose that X is a compact plane set with $\operatorname{int}(X) \neq \emptyset$, Ω is a connected open set in the complex plane with $\Omega \subseteq \operatorname{int}(X)$, and $K = \overline{\Omega}$. Let B be a subalgebra of $A(X, K)$ containing the coordinate function Z which is a natural Banach function algebra on X with an algebra norm $\|\cdot\|_B$. Let T be a unital compact endomorphism of B induced by a self-map φ of X. If $\varphi(X) \subseteq \operatorname{int}(K)$ and z_0 is a fixed point of φ, then*

$$\sigma(T) = \{0, 1\} \cup \left\{ \left(\varphi'(z_0)\right)^n : n \in \mathbb{N} \right\}. \tag{3.1}$$

Proof. Clearly 0 and also $1 \in \sigma(T)$ since $T(1_X) = 1_X$. If φ is constant then the proof is complete. Let $\lambda \in \sigma(T) \setminus \{0, 1\}$. The compactness of T implies that there exists $f \in B \setminus \{0\}$ such that $Tf = f \circ \varphi = \lambda f$. Since $\varphi(z_0) = z_0 \in \operatorname{int}(K)$, $f(z_0) = 0$. We claim that $f^{(j)}(z_0) \neq 0$ for some $j \in \mathbb{N}$. If $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ on an open disc with center z_0 and so on Ω. By maximum modules principle, it follows that $f = 0$ on X since $\varphi(X) \subseteq \Omega$, $\lambda \in \mathbb{C} \setminus \{0\}$, and $\lambda f(z) = f(\varphi(z))$ for all $z \in X$. This contradicts to $f \neq 0$. Hence, our claim is justified. Let $m = \min\{n \in \mathbb{N} : f^{(n)}(z_0) \neq 0\}$. Then $f^{(k)}(z_0) = 0$ for all $k \in \{0, \ldots, n-1\}$ and $f^{(m)}(z_0) \neq 0$. By m times differentiation of $f \circ \varphi = \lambda f$, we have $(\varphi'(z_0))^m f^{(m)}(\varphi(z_0)) = \lambda f^{(m)}(z_0)$, and therefore $\lambda = (f'(z_0))^m$. Then $\sigma(T) \setminus \{0, 1\} \subseteq \{(\varphi'(z_0))^n : n \in \mathbb{N}\}$.

Conversely, first we show that, if $\lambda \in \sigma(T)$ with $|\lambda| = 1$, then $\lambda = 1$. Let $\lambda \in \sigma(T)$ and $|\lambda| = 1$. The compactness of T implies that there exists $g \in B \setminus \{0\}$ such that $g \circ \varphi = \lambda g$. It follows that $|g \circ \varphi| = |g|$. Since $\varphi(K) \subseteq \operatorname{int}(K) = \Omega$ and g is analytic on the connected open set Ω, we conclude that g is constant on Ω by maximum modules principle. Since $\varphi(X) \subseteq \Omega$, $g \circ \varphi = \lambda g$, and $\lambda \in \mathbb{C} \setminus \{0\}$, we deduce that g is constant on X. Applying again $g \circ \varphi = \lambda g$ implies that $\lambda = 1$ since $g \in B \setminus \{0\}$.

We now claim that $\varphi'(z_0) \in \sigma(T)$. If $\varphi'(z_0) \notin \sigma(T)$, then there exists a nonzero linear operator $S : B \to B$ such that

$$(T - \varphi'(z_0)I)S = I. \tag{3.2}$$

Since $Z - z_0 1_X \in B$, $h = S(Z - z_0 1_X) \in B$ and so

$$h \circ \varphi - \varphi'(z_0)h = Z - z_0 1_X, \tag{3.3}$$

by (3.2). By differentiation at z_0, we have

$$0 = h'(\varphi(z_0))\varphi'(z_0) - \varphi'(z_0)h'(z_0) = 1, \tag{3.4}$$

this is a contradiction. Hence, our claim is justified.

We now show that $(\varphi'(z_0))^n \in \sigma(T)$ for all $n \in \mathbb{N}$. If $\varphi'(z_0) = 0$ or $|\varphi'(z_0)| = 1$, the proof is complete. Suppose that $\varphi'(z_0) \neq 0$ and $|\varphi'(z_0)| \neq 1$. If $(\varphi'(z_0))^j \notin \sigma(T)$ for some $j \in \mathbb{N}$ with $j > 1$, then there exists a nonzero linear operator $S_j : B \to B$ such that

$$\left(T - (\varphi'(z_0))^j I\right)S_j = I. \tag{3.5}$$

Since $(Z - z_0 1_X)^j \in B$, $h_j = S_j(Z - z_0 1_X)^j \in B$ and so

$$h_j \circ \varphi - (\varphi'(z_0))^j h_j = (Z - z_0 1_X)^j, \tag{3.6}$$

by (3.5). By $j - 1$ times differentiation at z_0, we have

$$h_j(z_0) = h_j'(z_0) = \cdots = h_j^{(j-1)}(z_0) = 0, \tag{3.7}$$

and by j times differentiation at z_0, we have

$$0 = (\varphi'(z_0))^j h_j^{(j)}(\varphi(z_0)) - (\varphi'(z_0))^j h_j^{(j)}(z_0) = j!, \tag{3.8}$$

this is a contradiction. Thus, $(\varphi'(z_0))^n \in \sigma(T)$ for all $n \in \mathbb{N}$. This completes the proof. □

Corollary 3.3. *Let B and T satisfy the conditions of Theorem 3.2, and let B be a natural Banach function algebra with the norm $\|\cdot\|_{\alpha,K}$. If F is a finite set such that $\varphi(F) = F$, then there exist $z_0 \in F$ and $m \in \mathbb{N}$ such that*

$$\{\lambda^m : \lambda \in \sigma(T)\} = \{0, 1\} \cup \left\{(\varphi_m'(z_0))^n : n \in \mathbb{N}\right\}. \tag{3.9}$$

Proof. Since F is a finite set and $\varphi(F) = F$, there exist $z_0 \in F$ and $m \in \mathbb{N}$ such that $\varphi_m(z_0) = z_0$. Since $\varphi(X) \subseteq \text{int}(K)$, so $z_0 \in \text{int}(K)$. If φ is constant, then the proof is complete. When φ is nonconstant, we define $\widetilde{T} : B \to B$ by $\widetilde{T}f = f \circ \varphi_m$. Then $\widetilde{T}$ is a compact endomorphism of B induced by φ_m by Corollary 2.6 and $\varphi_m(z_0) = z_0$. Therefore,

$$\sigma\left(\widetilde{T}\right) = \{0,1\} \cup \left\{ \left(\varphi'_m(z_0)\right)^n : n \in \mathbb{N} \right\} \tag{3.10}$$

by Theorem 3.2. Since $Tf = f \circ \varphi$ and $\widetilde{T}f = f \circ \varphi_m$ for all $f \in B$, we have $\widetilde{T} = T^m$. By Spectral Mapping Theorem, $\sigma(T^m) = \{\lambda^m : \lambda \in \sigma(T)\}$. Therefore,

$$\{\lambda^m : \lambda \in \sigma(T)\} = \{0,1\} \cup \left\{ \left(\varphi'_m(z_0)\right)^n : n \in \mathbb{N} \right\}. \tag{3.11}$$

□

Corollary 3.4. *Suppose that X is a compact plane set with* $\mathrm{int}(X) \neq \emptyset$, *$\Omega$ is a connected open set in the complex plane with $\Omega \subseteq \mathrm{int}(X)$, and $K = \overline{\Omega}$. Take $\alpha \in (0,1]$. Let φ be a self-map of X with $\varphi(X) \subseteq \Omega$ such that $\varphi \in Lip_A(X, K, \alpha)$, and let $\varphi(z_0) = z_0$ for some $z_0 \in \Omega$. If T is a unital endomorphism of $Lip_A(X, K, \alpha)$ induced by φ, then T is compact and*

$$\sigma(T) = \{0,1\} \cup \left\{ \left(\varphi'(z_0)\right)^n : n \in \mathbb{N} \right\}. \tag{3.12}$$

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