Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 175323, 12 pages
doi:10.1155/2011/175323

Research Article

Stability Analysis of Distributed Order Fractional Differential Equations

H. Saberi Najafi,[1] A. Refahi Sheikhani,[1] and A. Ansari[2]

[1] *Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran*
[2] *Department of Mathematics, Faculty of Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran*

Correspondence should be addressed to H. Saberi Najafi, hnajafi@guilan.ac.ir

Received 8 May 2011; Accepted 20 July 2011

Academic Editor: Jinhu Lü

We analyze the stability of three classes of distributed order fractional differential equations (DOFDEs) with respect to the nonnegative density function. In this sense, we discover a robust stability condition for these systems based on characteristic function and new inertia concept of a matrix with respect to the density function. Moreover, we check the stability of a distributed order fractional WINDMI system to illustrate the validity of proposed procedure.

1. Introduction

The fractional differential operator of distributed order

$$ {}_{\mathrm{do}}D^{\alpha} = \int_{l}^{u} b(\alpha) \frac{d^{\alpha}}{dt^{\alpha}} d\alpha, \qquad u > l \geq 0, b(\alpha) \geq 0 \tag{1.1}$$

is a generalization of the single order ${}_{\mathrm{so}}D^{\alpha} = d^{\alpha}/dt^{\alpha}$ which by considering a continuous or discrete distribution of fractional derivative is obtained.

The idea of fractional derivative of distributed order is stated by Caputo [1] and later developed by Caputo himself [2, 3], Bagley and Torvik [4, 5]. Other researchers used this idea, and interesting reviews appeared to describe the related mathematical models of partial fractional differential equation of distributed order.

For example, Diethelm and Ford [6] used a numerical technique along with its error analysis to solve the distributed order differential equation and analyze the physical phenomena and engineering problems, see [6] and references therein.

Furthermore, some investigation on linear distributed order boundary value problems of form

$$\int_0^m b(\alpha) D^{\alpha} u(x,t) d\alpha = B(D) u(x,t), \quad D = \frac{d}{dx}, \ t > 0, x \in R, \tag{1.2}$$

with pseudodifferential operator $B(D)$ and the Cauchy conditions

$$\frac{\partial^k}{\partial t^k} u(x, 0^+) = f_k(x), \quad k = 0, 1, \ldots, m-1, \tag{1.3}$$

have been discussed [7–12].

In particular cases, the characteristics of time-fractional diffusion equation of distributed order were studied for treatises in the sub-, normal, and superdiffusions.

The fractional order applied to dynamical systems is of great importance in applied sciences and engineering [13–19]. The stability results of the fractional order differential equations (FODEs) systems have been a main goal in researches. For example, Matignon considers the stability of FODE system in control processing and Deng has studied the stability of FODE system with multiple time delays [20–23].

Now, in this paper, we consider *the distributed order fractional differential equations systems (DOFDEs)* with respect to the density function $b(\alpha) \geq 0$ as follows:

$${}_{\text{do}}^{\text{C}}D_t^{\alpha} x(t) = A x(t), \quad x(0) = x_0, \ 0 < \alpha \leq 1, \tag{1.4}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and ${}_{\text{do}}^{\text{C}}D_t^{\alpha} = \int_0^1 b(\alpha) \ {}_{\text{so}}^{\text{C}}D_t^{\alpha} x(t) d\alpha$ is the Caputo fractional derivative operator of distributed order with respect to the order-density function $b(\alpha)$.

Since the solution of the above system is rather complicated similar to FODE systems, therefore, the study of stability for DOFDE is a main task.

In this paper, we introduce three classes of DOFDE systems including

(1) distributed order fractional differential systems;

(2) distributed order fractional differential evolution systems with control vector;

(3) distributed order fractional differential evolution systems without control vector.

For studying the stability of these classes of DOFDE systems, first, we introduce a characteristic function of a matrix with respect to the distribute function $B(s)$ where $B(s) = \int_0^1 b(\alpha) s^{\alpha} d\alpha$. Then, we establish a general theory based on new inertia concept for analyzing the stability of distributed order fractional differential equations. The concepts and theorems presented in this paper for DOFDE systems can be considered as generalizations of FODE and ODE systems [21, 24, 25].

In Section 2, we recall some basic definitions of the Caputo fractional derivative operator, the Mittag-Leffler function, and their elementary properties used in this paper. Section 3 contains the main definitions and theorems for checking the stability of DOFDE systems. Also, we study a distributed order fractional WINDMI system [26] generalized from fractional order to distributed order fractional. In Section 4, we introduce *the distributed order fractional evolution systems*

$$ {}_{\mathrm{do}}^{\,C}D_t^{\alpha}x(t) = A\,{}_{\mathrm{do}}^{\,C}D_t^{\beta}x(t) + Bu(t), \quad x(0) = x_0, \quad 0 < \beta < \alpha \leq 1, \tag{1.5} $$

where $u(t)$ is control vector, and generalize the results obtained in Section 3 for this case. Finally, the conclusions are given in the last section.

2. Elementary Definitions and Theorems

In this section, we consider the main definitions and properties of fractional derivative operators of *single and distribute order* and the Mittag-Leffler function. Also, we recall two important theorems in inverse of the Laplace transform.

2.1. Fractional Derivative of Single and Distributed Order

The fractional derivative of single order of $f(t)$ in the Caputo sense is defined as [16, 27]

$$ {}_{\mathrm{so}}^{\,C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau, \tag{2.1} $$

for $m-1 < \alpha \leq m, \ \ m \in \mathbb{N}, \ \ t > 0$. The Caputo's definition has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes. Fortunately, the Laplace transform of the Caputo fractional derivative satisfies

$$ \mathcal{L}\left\{{}_{\mathrm{so}}^{\,C}D_t^{\alpha}f(t)\right\} = s^{\alpha}\mathcal{L}\{f(t)\} - \sum_{k=0}^{m-1} f^{(k)}(0+)s^{\alpha-1-k}, \tag{2.2} $$

where $m-1 < \alpha \leq m$ and s is the Laplace variable. Now, we generalize the above definition in the fractional derivative of distributed order in the Caputo sense with respect to order-density function $b(\alpha) \geq 0$ as follows:

$$ {}_{\mathrm{do}}^{\,C}D_t^{\alpha}f(t) = \int_{m-1}^{m} b(\alpha)\ {}_{\mathrm{do}}^{\,C}D_t^{\alpha}f(t)d\alpha, \tag{2.3} $$

and the Laplace transform of the Caputo fractional derivative of distributed order satisfies

$$ \begin{aligned} \mathcal{L}\left\{{}_{\mathrm{do}}^{\,C}D_t^{\alpha}f(t)\right\} &= \int_{m-1}^{m} b(\alpha)\left[s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k}f^{(k)}(0^+)\right]d\alpha \\ &= B(s)F(s) - \sum_{k=0}^{m-1}\frac{1}{s^{k+1}}B(s)f^{(k)}(0^+), \end{aligned} \tag{2.4} $$

where

$$B(s) = \int_{m-1}^{m} b(\alpha)s^{\alpha}\,d\alpha. \tag{2.5}$$

2.2. Mittag-Leffler Function

The one-parameter Mittag-Leffler function $E_{\alpha}(z)$ and the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$, which are relevant for their connection with fractional calculus, are defined as

$$E_{\alpha}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+1)}, \quad \alpha > 0,\ z \in \mathbb{C}, \tag{2.6}$$

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)} \quad \alpha,\beta > 0,\ z \in \mathbb{C}. \tag{2.7}$$

One of the applicable relations in this paper is the Laplace transforms of the Mittag-leffler function given by

$$\mathcal{L}\Big(t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\Big) = \frac{s^{\alpha-\beta}}{(s^{\alpha}-\lambda)}, \quad \Re(s) > |\lambda|^{1/\alpha}. \tag{2.8}$$

2.3. Main Theorems about Inverse of the Laplace Transform

Theorem 2.1 (Schouten-Vanderpol Theorem [28]). *Suppose that the functions $F(s), \phi(s)$ are analytic in the half plane $\Re(s) > s_0$, then, the Laplace transform inversion of $F(\phi(s))$ can be obtained as*

$$\mathcal{L}^{-1}\{F(\phi(s))\} = \int_{0}^{+\infty} f(\tau)\mathcal{L}^{-1}\Big\{e^{-\phi(s)\tau};t\Big\}d\tau, \tag{2.9}$$

where $f(t)$ is the Laplace transform inversion of the function $F(s)$.

Theorem 2.2 (Titchmarsh Theorem [29]). *Let $F(s)$ be an analytic function which has a branch cut on the real negative semiaxis; furthermore, $F(s)$ has the following properties:*

$$\begin{gathered} F(s) = O(1), \quad |s| \longrightarrow \infty, \\ F(s) = O\left(\frac{1}{|s|}\right), \quad |s| \longrightarrow 0, \end{gathered} \tag{2.10}$$

for any sector $|\arg(s)| < \pi - \eta$ where $0 < \eta < \pi$. Then, the Laplace transform inversion $f(t)$ can be written as the the Laplace transform of the imaginary part of the function $F(re^{-i\pi})$ as follows:

$$f(t) = \mathcal{L}^{-1}\{F(s);t\} = \frac{1}{\pi}\int_{0}^{\infty} e^{-rt}\Im\Big(F\Big(re^{-i\pi}\Big)\Big)dr. \tag{2.11}$$

Theorem 2.3 (Final Value Theorem [28]). *Let $F(s)$ be the Laplace transform of the function $f(t)$. If all poles of $sF(s)$ are in the open left-half plane, then,*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{2.12}$$

3. Stability Analysis of Distributed Order Fractional Systems

In this section, we generalize the main stability properties for the linear system of distributed order fractional differential equations in the following form:

$${}_{\mathrm{do}}^{\mathrm{C}}D_t^{\alpha} x(t) = Ax(t), \quad x(0) = x_0, \quad 0 < \alpha \leq 1, \tag{3.1}$$

where $x \in \mathbb{R}^n$, the matrix $A \in \mathbb{R}^{n\times n}$, and ${}_{\mathrm{do}}^{\mathrm{C}}D_t^{\alpha} = \int_0^1 b(\alpha)\ {}_{\mathrm{so}}^{\mathrm{C}}D_t^{\alpha} x(t) d\alpha$ is the Caputo fractional derivative operator of distributed order with respect to order-density function $b(\alpha) \geq 0$. At first, we obtain the general solution of the system (3.1), and, next, we express the main theorem for checking the stability of this system.

By implementation of the Laplace transform on the above system and using the initial condition and relation (2.4), we have

$$\begin{aligned} B(s)x(s) &= Ax(s) + 1/s\, B(s)x(0), \qquad B(s) = \int_0^1 b(\alpha)s^{\alpha} d\alpha, \\ x(s) &= \frac{B(s)}{s[B(s)I - A]} x(0) = \frac{B(s)I - A + A}{s[B(s)I - A]} x(0) \\ &= \frac{1}{s} x(0) + \frac{A}{s[B(s)I - A]} x(0). \end{aligned} \tag{3.2}$$

Now, by applying the inverse of Laplace transform on the both sides of above relation, we have

$$\begin{aligned} x(t) &= x(0) + \mathcal{L}^{-1}\left\{ \frac{A}{s[B(s)I - A]} x(0) \right\} \\ &= x(0) + \int_0^t \mathcal{L}^{-1}\left\{ \frac{1}{[B(s)I - A]} \right\} Ax(0) dt, \end{aligned} \tag{3.3}$$

which according to the Schouten-Vanderpol and Titchmarsh theorems we get

$$\mathcal{L}^{-1}\left\{ \frac{1}{B(s)\ I - A} \right\} = \int_0^{\infty} e^{A\tau} \mathcal{L}^{-1}\left\{ e^{-B(s)\tau}; t \right\} d\tau, \tag{3.4}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{ e^{-B(s)\tau}; t \right\} &= -\frac{1}{\pi} \int_0^{\infty} e^{-rt} \Im\left(e^{-B(s)\tau} \right) dr \\ &= -\frac{1}{\pi} \int_0^{\infty} e^{-rt} \left[e^{-\rho\cos\pi\gamma} \sin(\rho\sin\pi\gamma) \right] dr, \end{aligned} \tag{3.5}$$

where $B(s) = \rho\cos\pi\gamma + i\rho\sin\pi\gamma$, $\rho = |B(s)|$, $\gamma = (1/\pi)\arg[B(s)]$, and $r = e^{i\pi}$.

Finally, by using (3.4) and (3.5), the general solution of the distributed order fractional systems (3.1) is written by

$$x(t) = x(0) + \frac{1}{\pi}\int_0^t\int_0^\infty\int_0^\infty e^{-rt+A\tau-\rho\cos\pi\gamma}\sin(\rho\sin\pi\gamma)Ax(0)dr\,d\tau\,dt. \tag{3.6}$$

Theorem 3.1. *The distributed order fractional system of* (3.1) *is asymptotically stable if and only if all roots of* $\det(B(s)I - A) = 0$ *have negative real parts.*

Proof. According to the relation (3.2), we have

$$[B(s)I - A]sX(s) = B(s)x(0), \tag{3.7}$$

if all roots of the $\det(B(s)I - A) = 0$ lie in open left half complex plane (i.e., $\Re(s\ ) < 0$), then, we consider (3.7) in $\Re(s) \geq 0$. In this restricted area, the relation (3.7) has a unique solution $sX(s) = (sX_1(s)\ , sX_2(s), \ldots, sX_n(s))$. Since $\lim_{s\to 0}B(s) = 0$, so we have

$$\lim_{s\to 0,\Re(s)\geq 0} sX_i(s) = 0, \quad i = 1, 2, \ldots, n, \tag{3.8}$$

which from the final value Theorem 2.3, we get

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(x_1(t), x_2(t), \ldots, x_n(t)) = \lim_{s\to 0,\Re(s)\geq 0}(sX_1(s), sX_2(s), \ldots, sX_n(s)) = 0. \tag{3.9}$$

The above result shows that the system (3.1) is asymptotically stable. □

Definition 3.2. The value of $\det(B(s)I - A)$ is the characteristic function of the matrix A with respect to the distributed function $B(s)$, where $B(s) = \int_0^1 b(\alpha)s^\alpha d\alpha$ is the distributed function with respect to the density function $b(\alpha) \geq 0$.

Definition 3.3. The eigenvalues of A with respect to the distributed function $B(s)$ are the roots of the characteristic function of A.

The inertia of a matrix is the triplet of the numbers of eigenvalues of A with positive, negative, and zero real parts. In this section, we generalize the inertia concept for analyzing the stability of linear distributed order fractional systems. According to the Theorem (3.1), the transient responses of the system (3.1) are governed by the region where the roots of $\det(B(s)I - A) = 0$ are located in the complex plane.

Definition 3.4. The inertia of a matrix A of order n respect to the order distributed function $B(s)$ is the triplet

$$In_{B(s)}\ (A) = \left(\pi_{B(s)}(A), \nu_{B(s)}(A), \delta_{B(s)}(A)\right), \tag{3.10}$$

where $\pi_{B(s)}(A)$, $\nu_{B(s)}(A)$, and $\delta_{B(s)}(A)$ are, respectively, the number of roots of $\det(B(s)I - A) = 0$ with positive, negative, and zero real parts where $B(s) = \int_0^1 b(\alpha)s^\alpha d\alpha$.

Definition 3.5. The matrix A is called a stable matrix with respect to the order distributed function $B(s)$, if all of the eigenvalue of A with respect to the distributed function $B(s)$ have negative real parts.

Theorem 3.6. *The linear distributed order fractional system* (3.1) *is asymptotically stable if and only if any of the following equivalent conditions holds.*

(1) *The matrix A is stable with respect to the distribute function $B(s)$.*

(2) $\pi_{B(s)}(A) = \delta_{B(s)}(A) = 0$.

(3) *All roots s of the characteristic function of A with respect to the distributed function $B(s)$ satisfy* $|\arg(s)| > \pi/2$.

Proof. According to Theorem 3.1 and the above definitions, proof can be easily obtained. □

Remark 3.7. In special case, if $b(\alpha) = \delta(\alpha - \beta)$, where $0 < \beta \leq 1$ and $\delta(x)$ is the Dirac delta function, then, we have the following linear system of fractional differential equations:

$$\frac{d^{\beta}}{dt^{\beta}}x(t) = Ax(t), \qquad x(0) = x_0, \tag{3.11}$$

and $B(s) = s^{\beta}$. Also, the characteristic matrix and characteristic equation of (3.11) are reduced to $s^{\beta}I - A$ and $\det(s^{\beta}I - A) = 0$, respectively. Let λ be s^{β}, then $s = \lambda^{1/\beta}$, and, by using Theorem 3.6, we have $|\arg(\lambda^{1/\beta})| > \pi/2$. Thus, all the roots λ of equation $\det(\lambda I - A) = 0$ satisfy $|\arg(\lambda)| > \beta\pi/2$. This result is Theorem 2 of [22]. Here, we can very easily prove it by using Theorem 3.6 of the present paper. Particularly, if $\beta = 1$, then, we have a linear system $\dot{x}(t) = Ax(t)$. In this case, $B(s) = s$ and the characteristic function of (3.1) are $\det sI - A$. Also, the inertia of matrix A is a triplet $(\pi(A), \nu(A), \delta(A))$, where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts. This result is a special case of definition (3.4), which agrees with the typical definitions for typical differential equations.

Example 3.8. The solar-wind-driven magnetosphere-ionosphere (WINDMI) system is a complex driven-damped dynamical system which exhibits a variety of dynamical states that include low-level steady plasma convection, episodic releases of geotail stored plasma energy into the ionosphere known broadly as substorms, and states of continuous strong unloading [30, 31]. If we consider the integer-order WINDMI model as follows:

$$\begin{aligned}\frac{dx_1}{dt} &= x_2,\\ \frac{dx_2}{dt} &= x_3,\\ \frac{dx_3}{dt} &= -ax_3 - x_2 + b - e^{x_1},\end{aligned} \tag{3.12}$$

where x_1, x_2, and x_3 are variables and a, b are positive constants, the corresponding distributed order fractional WINDMI system (3.12) can be written in the form:

$$\begin{aligned}
&\int_0^1 b(\alpha)\frac{d^\alpha}{dt^\alpha}x_1(t)d\alpha = x_2,\\
&\int_0^1 b(\alpha)\frac{d^\alpha}{dt^\alpha}x_2(t)d\alpha = x_3,\\
&\int_0^1 b(\alpha)\frac{d^\alpha}{dt^\alpha}x_3(t)d\alpha = -ax_3 - x_2 + b - e^{x_1},
\end{aligned} \tag{3.13}$$

where $b(\alpha) \geq 0$ is the density function. As a generalization of nonlinear autonomous FODE into nonlinear autonomous DOFDE, the linearized form of the system (3.13) at the equilibrium point $\widehat{x} = (\ln b, 0, 0)$, that is, ${}_{\mathrm{do}}^{C}D_t^\alpha \widehat{x}(t) = F(\widehat{x}) = 0$, can be written in the form

$${}_{\mathrm{do}}^{C}D_t^\alpha x(t) = Ax(t), \tag{3.14}$$

where $(t) = (x_1(t), x_2(t), x_3(t))$, ${}_{\mathrm{do}}^{C}D_t^\alpha x(t) = \int_0^1 b(\alpha){}_{\mathrm{so}}^{C}D_t^\alpha x(t)d\alpha$, and $A = (\partial F/\partial x)|_{x=\widehat{x}}$, which is the Jacobian matrix at the equilibrium point [32], is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b & -1 & -a \end{pmatrix}. \tag{3.15}$$

Now, for analyzing the stability of the nonlinear autonomous DFODE, we compute $In_{b(\alpha)}(A)$ in the case that the density function varies. The results are shown in Table 1 for some parameters a and b.

4. Distributed Order Fractional Evolution Systems

In this section, as a generalization of the previous systems, we consider the systems of distributed order fractional differential evolution equations and state two theorems in stability of these systems.

Theorem 4.1. *Consider linear system of distributed order fractional differential evolution equations,*

$${}_{do}^{C}D_t^\alpha x(t) = A{}_{do}^{C}D_t^\beta x(t), \quad x(0) = x_0,\ 0 < \beta < \alpha \leq 1, \tag{4.1}$$

where $A \in \mathbb{R}^{p\times p}$, ${}_{do}^{C}D_t^\alpha x(t) = \int_0^1 b_1(\alpha)\ {}_{so}^{C}D_t^\alpha x(t)d\alpha$, *and* ${}_{do}^{C}D_t^\beta x(t) = \int_0^1 b_2(\beta)\ {}_{so}^{C}D_t^\beta x(t)d\beta$. *Also,* $B_1(s) = \int_0^1 b_1(\alpha)s^\alpha d\alpha$ *and* $B_2(s) = \int_0^1 b_2(\beta)s^\beta d\beta$. *The system* (4.1) *is stable if and only if all roots of characteristic function of matrix* A *with respect to the distributed function* $B_1(s)/B_2(s)$ *have negative real parts.*

Table 1: Stability analysis of distributed order fractional WINDMI system.

Parameters	$b(\alpha)=\delta(\alpha-\beta)$			$b(\alpha)=\delta(\alpha-\beta_1)+\delta(\alpha-\beta_2)$		$b(\alpha)=2\alpha$
	$\beta=1$	$\beta=.95$	$\beta=.65$	$\beta_1=.5,\quad \beta_2=.85$	$\beta_1=.1,\quad \beta_2=.9$	
$a=0,b=0$	$(0,0,3)$	$(0,2,1)$	$(0,2,1)$	$(0,2,1)$	$(0,2,1)$	$(2,0,0)$
$a=0,b=1$	$(2,1,0)$	$(1,0,0)$	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$	$(1,0,0)$
$a=1,b=1$	$(0,1,2)$	$(0,2,0)$	$(0,2,0)$	$(0,2,0)$	$(0,2,0)$	$(3,0,0)$
$a=1,b=0$	$(0,2,1)$	$(0,2,1)$	$(0,0,1)$	$(0,0,1)$	$(0,2,1)$	$(2,0,0)$
$a=1,b=0.001$	$(0,3,0)$	$(0,1,0)$	$(0,0,0)$	$(0,0,0)$	$(0,1,0)$	$(1,0,0)$

Proof. Taking the Laplace transform on both sides of (4.1) gives

$$
\begin{gathered}
B_1(s)X(s)-\frac{B_1(s)}{s}x(0)=A\left[B_2(s)X(s)-\frac{B_2(s)}{s}x(0)\right],\\
[B_1(s)I-AB_2(s)]X(s)=\frac{1}{s}[B_1(s)-AB_2(s)]x(0),\\
\left[\frac{B_1(s)}{B_2(s)}I-A\right](sX(s)-x(0))=0.
\end{gathered}
\tag{4.2}
$$

If all roots of characteristic function of matrix A with respect to the distributed function $B_1(s)/B_2(s)$ have negative real parts,that is, $\Re(s)<0$, then, we consider (4.2) in $\Re(s)\geq 0$. In this restricted area by using final-value theorem of Laplace transform, we have

$$
\lim_{t\to\infty}x(t)=\lim_{s\to 0,\Re(s)\geq 0}sX(s)=x_0.
\tag{4.3}
$$

□

Theorem 4.2. *Consider the linear system of distributed order fractional differential evolution equations*

$$
{}_{do}^{C}D_t^{\alpha}x(t)=A\,{}_{do}^{C}D_t^{\beta}x(t)+Bu(t),\quad x(0)=x_0,\ 0<\beta<\alpha\leq 1,
\tag{4.4}
$$

with the same hypotheses described in Theorem 4.1 where $B\in\mathbb{R}^{n\times n}$ and $u(t)$ is a control vector.The linear distributed order fractional system (4.4) is stabilizable if and only if there exists a linear feedback $u(t)=Y\ {}_{do}^{C}D_t^{\beta}x(t)$, with $Y\in\mathbb{R}^{n\times n}$, such that $A+BY$ is stable with respect to the distributed function $B_1(s)/B_2(s)$.

Proof. The proof can be easily expressed similar to Theorem 4.1. □

Remark 4.3. If $b_1(\alpha)=\delta(\alpha-\alpha_1)$ and $b_2(\beta)=\delta(\beta-\beta_1)$ where $0<\beta_1<\alpha_1\leq 1$ then (4.4) is reduced to the following linear system of fractional differential equations:

$$
\frac{d^{\alpha_1}}{dt^{\alpha_1}}x(t)=A\frac{d^{\beta_1}}{dt^{\beta_1}}x(t)+Bu(t),\quad x(0)=x_0,\quad 0<\beta_1<\alpha_1\leq 1.
\tag{4.5}
$$

By applying the Laplace transform on the above system and using the initial condition, we have

$$s^{\alpha_1} X(s) - s^{\alpha_1 - 1} x(0) = A\left[s^{\beta_1} X(s) - s^{\beta_1 - 1} x(0)\right] + BU(s), \tag{4.6}$$

where $X(s)$ is the Laplace transform of $x(t)$, $U(s)$ is the Laplace transform of $u(t)$, and $B(s) = \int_0^1 b(\alpha) s^{\alpha} d\alpha$. Thus, we can write $X(s)$ as,

$$\begin{aligned} X(s) &= \frac{BU(s) + s^{\alpha_1 - 1} x(0)}{s_1^{\alpha} I - A s_1^{\beta}} + \frac{s^{\beta_1 - 1} A x(0)}{s_1^{\alpha} I - A s_1^{\beta}} \\ &= \frac{s^{-\beta_1}}{s^{\alpha_1 - \beta_1} I - A} BU(s) + \frac{s^{\alpha_1 - \beta_1 - 1}}{s^{\alpha_1 - \beta_1} I - A} x(0) + \frac{s^{-1}}{s^{\alpha_1 - \beta_1} I - A} A x(0). \end{aligned} \tag{4.7}$$

Applying the inverse Laplace transform to (4.7) and using property (2.8), we get

$$\begin{aligned} x(t) = &\int_0^t (t - x)^{\alpha_1 - 1} E_{\alpha_1 - \beta_1, \alpha_1}\left(A(t - x)^{\alpha_1 - \beta_1}\right) Bu(x) dx \\ &+ E_{\alpha_1 - \beta_1, 1}\left(A t^{\alpha_1 - \beta_1}\right) x(0) + t^{\alpha_1 - \beta_1} E_{\alpha_1 - \beta_1, \alpha_1 - \beta_1 + 1}\left(A t^{\alpha_1 - \beta_1}\right) A x(0). \end{aligned} \tag{4.8}$$

Therefore, (4.5) is asymptotically stable if all eigenvalues of A with respect to the distributed function $B_1(s)/B_2(s) = s^{\alpha_1 - \beta_1}$ have negative real parts which is a special case of Theorem 4.2.

5. Conclusions and Future Works

In this work, we introduced three classes of the distributed order fractional differential systems, the distributed order fractional differential evolution systems with control vector, and the distributed order fractional differential evolution systems without control vector. The analysis of the asymptotically stability for such systems based on Theorem 3.1 and several interesting stability criteria are derived according to Theorem 3.6. Moreover, a numerical example was given to verify the effectiveness of the proposed schemes.

In view of the above result, for future works, our attention may be focused on generalizing the numerical methods for computing the eigenvalues of a matrix with respect to the distributed function. The proposed algorithms in [33–35] for computing the eigenvalues of a matrix may be effective in this case.

References

[1] M. Caputo, *Elasticitá e Dissipazione*, Zanichelli, Bologna, Italy, 1969.

[2] M. Caputo, "Mean fractional-order-derivatives differential equations and filters," *Annali dell'Università di Ferrara. Nuova Serie. Sezione VII. Scienze Matematiche*, vol. 41, pp. 73–84, 1995.

[3] M. Caputo, "Distributed order differential equations modelling dielectric induction and diffusion," *Fractional Calculus & Applied Analysis*, vol. 4, no. 4, pp. 421–442, 2001.

[4] R. L. Bagley and P. J. Torvik, "On the existence of the order domain and the solution of distributed order equations," *International Journal of Applied Mathematics*, vol. I, no. 7, pp. 865–882, 2000.

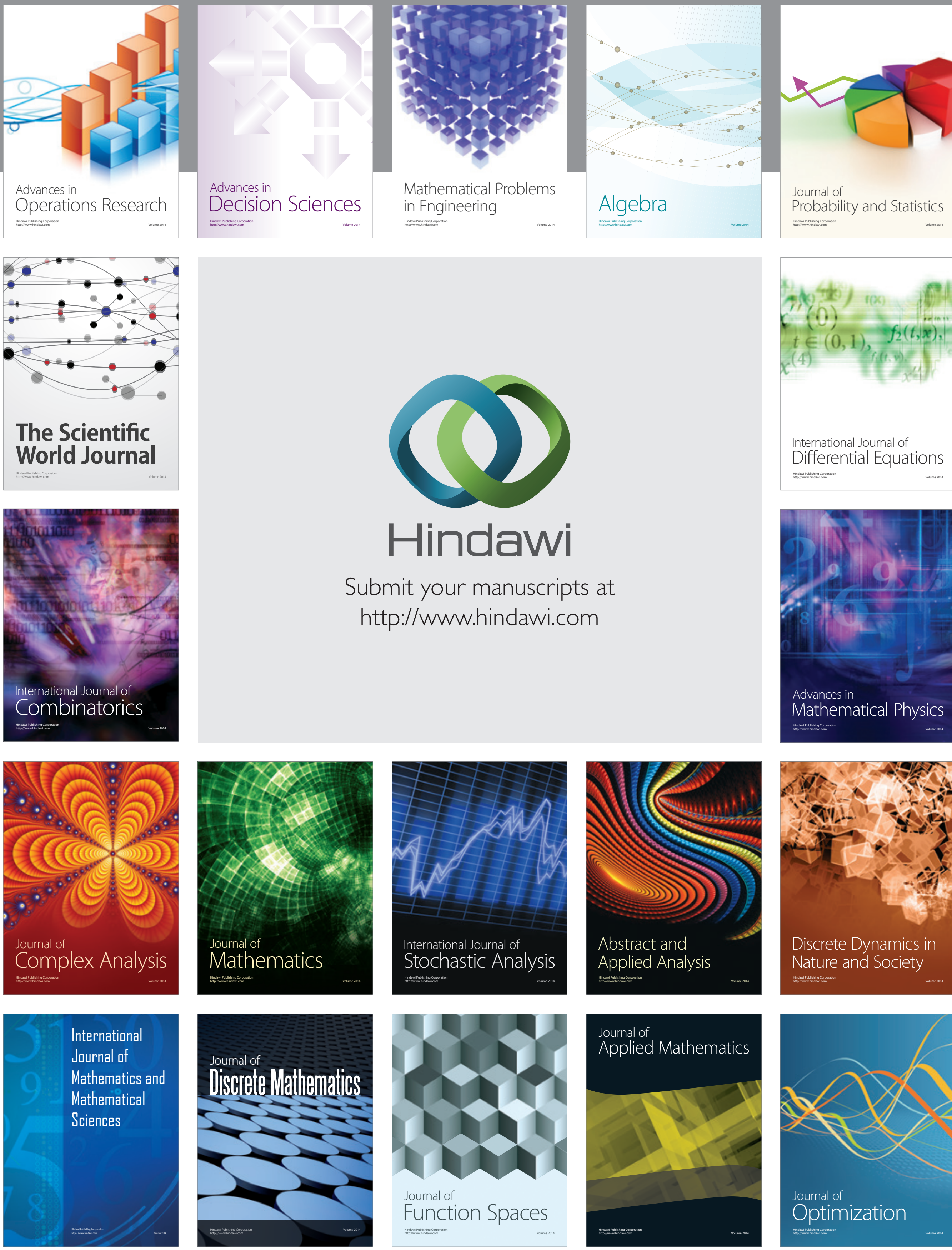
[5] R. L. Bagley and P. J. Torvik, "On the existence of the order domain and the solution of distributed order equations," *International Journal of Applied Mathematics*, vol. II, no. 7, pp. 965–987, 2000.
[6] K. Diethelm and N. J. Ford, "Numerical analysis for distributed-order differential equations," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 96–104, 2009.
[7] A. Aghili and A. Ansari, "Solving partial fractional differential equations using the $\mathcal{L}_A$-transform," *Asian-European Journal of Mathematics*, vol. 3, no. 2, pp. 209–220, 2010, World Scientific Publishing.
[8] A. V. Chechkin, R. Gorenflo, and I. M. Sokolov, "Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equations," *Physical Review E*, vol. 66, no. 4, article 046129, pp. 1–7, 2002.
[9] A. N. Kochubei, "Distributed order calculus and equations of ultraslow diffusion," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 252–281, 2008.
[10] F. Mainardi and G. Pagnini, "The role of the Fox-Wright functions in fractional subdiffusion of distributed order," *Journal of Computational and Applied Mathematics*, vol. 207, no. 2, pp. 245–257, 2007.
[11] F. Mainardi, G. Pagnini, and R. K. Saxena, "Fox H functions in fractional diffusion," *Journal of Computational and Applied Mathematics*, vol. 178, no. 1-2, pp. 321–331, 2005.
[12] S. Umarov and R. Gorenflo, "Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 24, no. 3, pp. 449–466, 2005.
[13] O. P. Agrawal, J. A. Tenreiro Machado, and J. Sabatier, "Introduction," *Nonlinear Dynamics*, vol. 38, no. 1-2, pp. 1–2, 2004.
[14] B. Bonilla, M. Rivero, L. Rodríguez-Germá, and J. J. Trujillo, "Fractional differential equations as alternative models to nonlinear differential equations," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 79–88, 2007.
[15] P. L. Butzer and U. Westphal, *An Introduction to Fractional Calculus*, World Scientific, Singapore, Republic of Singapore, 2000.
[16] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science Publishers, Amsterdam, The Netherlands, 2006.
[17] R. L. Magin, "Fractional calculus in bioengineering," *Critical Reviews in Biomedical Engineering*, vol. 32, no. 1, pp. 1–104, 2004.
[18] T. Matsuzaki and M. Nakagawa, "A chaos neuron model with fractional differential equation," *Journal of the Physical Society of Japan*, vol. 72, no. 10, pp. 2678–2684, 2003.
[19] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, UK, 2005.
[20] W. Deng, "Smoothness and stability of the solutions for nonlinear fractional differential equations," *Nonlinear Analysis: TMA*, vol. 72, no. 3-4, pp. 1768–1777, 2010.
[21] W. Deng, C. Li, and J. Lü, "Stability analysis of linear fractional differential system with multiple time delays," *Nonlinear Dynamics*, vol. 48, no. 4, pp. 409–416, 2007.
[22] D. Matignon, "Stability results of fractional differential equations with applications to control processing," in *Proceedings of the IEEE-SMC International Association for Mathematics and Computers in Simulation (IMACS '96)*, pp. 963–968, Lille,France, 1996.
[23] M. S. Tavazoei and M. Haeri, "A note on the stability of fractional order systems," *Mathematics and Computers in Simulation*, vol. 79, no. 5, pp. 1566–1576, 2009.
[24] B. N. Datta, "Stability and inertia," *Linear Algebra and its Applications*, vol. 302/303, pp. 563–600, 1999.
[25] Z. M. Odibat, "Analytic study on linear systems of fractional differential equations," *Computers and Mathematics with Applications*, vol. 59, no. 3, pp. 1171–1183, 2010.
[26] B. Xin, T. Chen, and Y. Liu, "Synchronization of chaotic fractional-order WINDMI systems via linear state error feedback control," *Mathematical Problems in Engineering*, vol. 2010, Article ID 859685, 10 pages, 2010.
[27] I. Podlubny, *Fractional Differential Equations*, vol. 198, Academic Press, San Diego, Calif, USA, 1999.
[28] D. G. Duffy, *Transform Methods for Solving Partial Differential Equations*, CRC Press, 2nd edition, 2004.
[29] A. V. Bobylev and C. Cercignani, "The inverse laplace transform of some analytic functions with an application to the eternal solutions of the Boltzmann equation," *Applied Mathematics Letters*, vol. 15, no. 7, pp. 807–813, 2002.
[30] W. Horton, R. S. Weigel, and J. C. Sprott, "Chaos and the limits of predictability for the solar-wind-driven magnetosphere-ionosphere system," *Physics of Plasmas*, vol. 8, no. 6, pp. 2946–2952, 2001.
[31] W. Horton and I. Doxas, "A low-dimensional dynamical model for the solar wind driven geotail-ionosphere system," *Journal of Geophysical Research A*, vol. 103, no. A3, pp. 4561–4572, 1998.

[32] Y. Yu, H.-X. Li, S. Wang, and J. Yu, "Dynamic analysis of a fractional-order Lorenz chaotic system," *Chaos, Solitons and Fractals*, vol. 42, no. 2, pp. 1181–1189, 2009.

[33] H. S. Najafi and A. Refahi, "A new restarting method in the Lanczos algorithm for generalized eigenvalue problem," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 421–428, 2007.

[34] H. S. Najafi and A. Refahi, "FOM-inverse vector iteration method for computing a few smallest (largest) eigenvalues of pair (A,B)," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 641–647, 2007.

[35] H. S. Najafi, A. Refahi, and M. Akbari, "Weighted FOM-inverse vector iteration method for computing a few smallest (largest) eigenvalues of pair (A,B)," *Applied Mathematics and Computation*, vol. 192, no. 1, pp. 239–246, 2007.


Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization