Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 208163, 21 pages
doi:10.1155/2011/208163

Research Article

A General System of Euler–Lagrange-Type Quadratic Functional Equations in Menger Probabilistic Non-Archimedean 2-Normed Spaces

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Received 21 April 2011; Revised 20 June 2011; Accepted 21 July 2011

Academic Editor: Wing-Sum Cheung

We prove the generalized Hyers-Ulam-Rassias stability of a general system of Euler-Lagrange-type quadratic functional equations in non-Archimedean 2-normed spaces and Menger probabilistic non-Archimedean-normed spaces.

1. Introduction and Preliminaries

Gähler [1, 2] introduced the concept of linear 2-normed spaces and Gähler and White [3–5] introduced the concept of 2-Banach spaces. In 1999–2003, Lewanwdoska published a series of some papers on 2-normed sets and generalized 2-normed spaces [6, 7]. For more details on linear 2-normed spaces, see the books written by Freese and Cho [8] and Cho et al. [9].

Recently, Park [10] has investigated approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces. We recall and apply notions and notes which are given in [10].

Definition 1.1. Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$ and $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ be a function satisfying the following properties: for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$,

(1) $\|x, y\| = 0$, if and only if x, y are linearly dependent;

(2) $\|x, y\| = \|y, x\|$;

(3) $\|x, \alpha y\| = |\alpha|\|x, y\|$;

(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space.

Lemma 1.2. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $x \in X$ and $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*

Remark 1.3. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space. One can show that the conditions (2) and (4) in Definition 1.1 imply that

$$\left| \|x, z\| - \left\|y, z\right\| \right| \leq \left\|x - y, z\right\| \tag{1.1}$$

for all $x, y, z \in X$. Hence the function $x \to \|x, y\|$ is continuous function of X into $\mathbb{R}$ for any fixed $y \in X$.

Definition 1.4. A sequence $\{x_n\}$ in a linear 2-normed space X is called a Cauchy sequence if there are two linearly independent points $y, z \in X$ such that

$$\lim_{m,n \to +\infty} \left\|x_n - x_m, y\right\| = 0, \qquad \lim_{m,n \to +\infty} \|x_n - x_m, z\| = 0. \tag{1.2}$$

Definition 1.5. A sequence $\{x_n\}$ in a linear 2-normed space X is called a convergent sequence if there exists $x \in X$ such that

$$\lim_{n \to \infty} \left\|x_n - x, y\right\| = 0, \tag{1.3}$$

for all $y \in X$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$, and x is called the limit of $\{x_n\}$. In this case, we also write $\lim_{n \to \infty} x_n = x$.

Lemma 1.6. *For any convergent sequence $\{x_n\}$ in a linear 2-normed space X,*

$$\lim_{n \to \infty} \left\|x_n, y\right\| = \left\| \lim_{n \to \infty} x_n, y \right\| \tag{1.4}$$

for all $y \in X$.

Definition 1.7. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

Hensel [11] has introduced a normed space which does not have the Archimedean property. During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings, and superstrings [12]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are different and require a rather new kind of intuition [13–17]. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles, and there may be no unit vector in a non-Archimedean normed space [15]. These facts show that the non-Archimedean framework is of special interest.

Definition 1.8. Let $\mathbb{K}$ be a field. A valuation mapping on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that, for any $a, b \in \mathbb{K}$,

(a) $|a| \geq 0$ and equality holds if and only if $a = 0$;

(b) $|ab| = |a||b|$;

(c) $|a + b| \leq |a| + |b|$.

A field endowed with a valuation mapping is called a valued field.

If the condition (c) in the definition of a valuation mapping is replaced with the following condition:

$(c)'$ $|a + b| \leq \max\{|a|, |b|\}$,

then the valuation $|\cdot|$ is said to be non-Archimedean.

The condition $(c)'$ is called the strict triangle inequality. By (b), we have $|1| = |-1| = 1$. Thus, by induction, it follows from $(c)'$ that $|n| \leq 1$ for each integer $n \geq 1$. We always assume in addition that $|\cdot|$ is nontrivial, that is, that there exists $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$. The most important examples of non-Archimedean spaces are p–adic numbers.

Example 1.9. Let p be a prime number. For any nonzero rational number $a = p^r(m/n)$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-r}$. Then $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.

Definition 1.10. Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions: for all $x, y \in X$ and $r \in \mathbb{K}$,

(NA1) $\|x\| = 0$ if and only if $x = 0$;

(NA2) $\|rx\| = |r|\|x\|$;

(NA3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ (the strong triangle inequality (ultrametric)).

Then $(X, \|\cdot\|)$ is called a non–Archimedean normed space.

Definition 1.11 (Freese and Cho [8]). Let X be a linear space with dim X > 1 over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot,\cdot\| : X \to \mathbb{R}$ is a non-Archimedean 2-norm (valuation) if it satisfies the following conditions: for any $x, y, z \in X$ and $\alpha \in \mathbb{K}$,

(NA1) $\|x, y\| = 0$ if and only if x, y are linearly dependent;

(NA2) $\|x, y\| = \|y, x\|$;

(NA3) $\|x, \alpha y\| = |\alpha|\|x, y\|$;

(NA4) $\|x + y, z\| \leq \max\{\|x, z\|, \|y, z\|\}$.

Then $(X, \|\cdot,\cdot\|)$ is called a non-Archimedean 2-normed space.

It follows from (NA4) that

$$\|x_m - x_l, y\| \leq \max\{\|x_{j+1} - x_j, y\| : l \leq j \leq m - 1\} \quad (m > l), \tag{1.5}$$

and so a sequence $\{x_m\}$ is a Cauchy sequence in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean 2-normed space.

Theory of probabilistic normed spaces were first defined by Šerstnev in 1962 (see [18]), which was generalized in [19]. We recall and apply the definition of Menger probabilistic normed spaces briefly as given in [20–22].

Definition 1.12. A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F from $[0, +\infty]$ into $[0, 1]$ satisfying $F(0) = 0$, $F(+\infty) = 1$, and F is left-continuous on $(0, +\infty)$. The space of d.d.f.'s is denoted by Δ^+ and the set of all F in Δ^+ for which $\lim_{t \to +\infty^-} F(t) = 1$ by D^+.

The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \le G$ if and only if $F(x) \le G(x)$ for all x in $[0, +\infty]$. For any $a \ge 0$, ε_a is the d.d.f. given by

$$\varepsilon_a(t) = \begin{cases} 0, & \text{if } t \le a, \\ 1, & \text{if } t > a. \end{cases} \tag{1.6}$$

Definition 1.13. A triangular norm (briefly, t-norm) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, and non-decreasing in each variable and has 1 as the unit element.

Basic examples are the Łukasiewicz t-norm T_L, $T_L(a,b) = \max(a+b-1, 0)$, the product t-norm T_P, $T_P(a,b) = ab$, and the strongest triangular norm T_M, $T_M(a,b) = \min(a,b)$.

Definition 1.14. A Menger probabilistic normed space is a triple (X, ν, T), where X is a real vector space, T is continuous t-norm, and ν is a mapping the probabilistic norm from X into Δ^+, such that, for any choice of $p, q \in X$ and $a, s, t \in (0, +\infty)$, the following hold:

(PN1) $\nu(p) = \varepsilon_0$ if and only if $p = \theta$ (θ is the null vector in X);

(PN2) $\nu(ap)(t) = \nu(p)(t/(|a|))$;

(PN3) $\nu(p+q)(s+t) \ge T(\nu(p)(s), \nu(q)(t))$.

Mirmostafaee and Moslehian [23] have introduced a notion of a non-Archimedean fuzzy normed space and, recently, Mihet [24] has restated the definition of them.

Now, we introduce the definition of a Menger probabilistic non-Archimedean 2-normed space by the definitions which is given in [24] and [23].

Definition 1.15. A Menger Probabilistic non-Archimedean 2-normed space is a triple (X, ν, T), where X is a linear space with dim $X > 1$ over a non-Archimedean field $\mathbb{K}$, T is continuous t-norm, and ν is a mapping the probabilistic norm from X^2 into Δ^+, such that, for any choice of $p, q, r \in X$, $\alpha \in \mathbb{K}$, and $s, t \in (0, +\infty)$, the following hold:

(PNA1) $\nu(p, q) = \varepsilon_0$ if and only if p, q are linearly dependent;

(PNA2) $\nu(p, q) = \nu(q, p)$;

(PNA3) $\nu(\alpha p, q)(t) = \nu(p, q)(t/(|\alpha|))$;

(PNA4) $\nu(p+q, r)(\max\{s, t\}) \ge T(\nu(p, r)(s), \nu(q, r)(t))$.

It follows from $\nu(p,q) \in \Delta^+$ that $\nu(p,q)$ is nondecreasing for any fixed $p,q \in X$. So one can show that the condition (PNA4) is equivalent to the following condition:

(PNA4)′ $\nu(p+q,r)(t) \geq T(\nu(p,r)(t), \nu(q,r)(t))$.

Definition 1.16. Let (X,ν,T) be a Menger probabilistic non-Archimedean 2-normed space and $\{x_n\}$ be a sequence in X. Then

(1) the sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} \nu(x_n - x, y)(t) = 1, \tag{1.7}$$

for all $y \in X$ and $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$;

(2) the sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $t > 0$, there exists n_0 and linearly independent elements $y,z \in X$ such that, for all $n \geq n_0$ and $p > 0$,

$$\nu(x_{n+p} - x_n, y)(t) > 1-\varepsilon, \qquad \nu(x_{n+p} - x_n, z)(t) > 1-\varepsilon. \tag{1.8}$$

Let T be a given t-norm. Then, by associativity, a family of mappings $T^n : [0,1] \to [0,1]$ for all $n \in \mathbb{N}$ is defined as follows:

$$T^1(x) = T(x,x), \quad T^n(x) = T\left(T^{n-1}(x), x\right), \tag{1.9}$$

for all $x \in [0,1]$. For three important t-norms T_M, T_P, and T_L, we have

$$T_M^n(x) = x, \quad T_P^n(x) = x^n, \quad T_L^n(x) = \max\{(n+1)x - n, 0\}, \tag{1.10}$$

for all $n \in \mathbb{N}$.

Definition 1.17 (Hadžić [25]). A t-norm T is said to be of H-type if a family of functions $\{T^n(t)\}$ for all $n \in \mathbb{N}$ is equicontinuous at $t = 1$, that is, for all $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that

$$t > 1-\delta \Longrightarrow T^n(t) > 1-\varepsilon \tag{1.11}$$

for all $n \in \mathbb{N}$.

The t-norm T_M is a trivial example of t-norm of H-type, but there are t-norms of H-type with $T \neq T_M$ (see Hadžić [26]).

Lemma 1.18. *Assume that T is a t-norm of H-type. Then the sequence $\{x_n\}$ is a Cauchy sequence if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in N$ and linearly independent elements $y,z \in X$ such that, for all $n \geq n_0$,*

$$\nu(x_{n+1} - x_n, y)(t) > 1-\varepsilon, \qquad \nu(x_{n+1} - x_n, z)(t) > 1-\varepsilon. \tag{1.12}$$

Proof. For any $y \in X$, we have

$$\begin{aligned}
\nu(x_{n+p} - x_n, y)(t) &\geq T(\nu(x_{n+p} - x_{n+p-1}, y)(t), \nu(x_{n+p-1} - x_n, y)(t)) \\
&\geq T(\nu(x_{n+p} - x_{n+p-1}, y)(t), T(\nu(x_{n+p-1} - x_{n+p-2}, y)(t), \nu(x_{n+p-2} - x_n, y)(t))) \\
&\geq \cdots \\
&\geq T(\nu(x_{n+p} - x_{n+p-1}, y)(t), T(\nu(x_{n+p-1} - x_{n+p-2}, y)(t), \ldots, \\
&\qquad T(\nu(x_{n+2} - x_{n+1}, y)(t), \nu(x_{n+1} - x_n, y)(t))) \cdots).
\end{aligned} \tag{1.13}$$

For any $z \in X$, we have the same inequalities to (1.13). Thus, from (1.13), the t-norm T of H-type and the assumption, it follows that the sequence $\{x_n\}$ is a Cauchy sequence in X. This completes the proof. □

It is easy to see that every convergent sequence in a (Menger probabilistic) non-Archimedean 2-normed space X is a Cauchy sequence. If every Cauchy sequence is convergent in X, then the (Menger probabilistic) non-Archimedean 2-normed space X is said to be complete and, further, the space X is called (Menger probabilistic) non-Archimedean 2-Banach space.

The first stability problem concerning group homomorphisms was raised by Ulam [27] in 1940 and solved in the next year by Hyers [28]. Hyers' theorem was generalized by Aoki [29] for additive mappings and by Rassias [30] for linear mappings by considering an unbounded Cauchy difference. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [31] by replacing the unbounded Cauchy difference by a general control function.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.14}$$

is related to a symmetric biadditive function [32, 33]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.14) is said to be a quadratic function. It is well known that a function $f : X \to Y$, where X and Y are real vector spaces, is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$. The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)). \tag{1.15}$$

The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [34]. In [35], Czerwik proved the Hyers-Ulam-Rassias stability of (1.14). Later, Jung [36] has generalized the results obtained by Skof and Czerwik. Rassias [37, 38] has solved the stability problem of Ulam for the generalized Euler-Lagrange type quadratic functional equation of

$$f(ax + by) + f(bx - ay) = \left(a^2 + b^2\right)(f(x) + f(y)). \tag{1.16}$$

Lee [39] proved the stability of (1.16) in the spirit of Hyers, Ulam, Rassias and Găvruta in Banach spaces.

In recent years, many authors proved the stability of various functional equations in various spaces (see, for instance, [10, 40, 41]). Using the method of our paper, one can investigate the stability of many general systems of various functional equations with n functional equations and n variables ($n \in \mathrm{N}$) and our paper notably generalizes previous papers in this area.

We assume that $f : X^n \to Y$ is a mapping and consider a general system of Euler-Lagrange type quadratic functional equations as follows:

$$\begin{aligned} &f(a_1x_1 + b_1y_1, x_2, \ldots, x_n) + f(b_1x_1 - a_1y_1, x_2, \ldots, x_n) \\ &\quad = \left(a_1^2 + b_1^2\right)(f(x_1, \ldots, x_n) + f(y_1, x_2, \ldots, x_n)) \\ &\quad \cdots \\ &f(x_1, \ldots, x_{n-1}, a_nx_n + b_ny_n) + f(x_1, \ldots, x_{n-1}, b_nx_n - a_ny_n) \\ &\quad = \left(a_n^2 + b_n^2\right)(f(x_1, \ldots, x_n) + f(x_1, \ldots, x_{n-1}, y_n)). \end{aligned} \tag{1.17}$$

In Section 2, we establish the generalized Hyers-Ulam–Rassias stability of the system (1.17) in non-Archimedean 2-Banach spaces. In Section 3, we prove the generalized Hyers-Ulam-Rassias stability of the system (1.17) in Menger probabilistic non-Archimedean 2-Banach spaces.

2. Stability of the System (1.17) in Non-Archimedean 2-Banach Spaces

In this section, we prove the generalized Hyers-Ulam-Rassias stability of system (1.17) in non-Archimedean 2-Banach spaces. Throughout this section, we assume that $i, k, n, p \in \mathbb{N} \cup \{0\}$, $\mathbb{K}$ is a non–Archimedean field, Y is a non-Archimedean 2-Banach space over $\mathbb{K}$, and X is a vector space over $\mathbb{K}$. Also assume that $f : X^n \to Y$ is a mapping.

Theorem 2.1. *Let $\varphi_i : X^{n+1} \to [0, \infty)$ for $i \in \{1, \ldots, n\}$ be a function such that*

$$\begin{aligned} &\lim_{k \to \infty} \left| \frac{1}{m_1^{2k} \cdots m_n^{2k}} \right| \\ &\times \max\Bigg\{ \left| \frac{1}{m_1^2 \cdots m_{i-1}^2 m_i} \right| \varphi_i\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, m_i^k x_i, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), \\ &\qquad \left| \frac{1}{m_1^2 \cdots m_i^2} \right| \varphi_i\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, a_i m_i^k x_i, b_i m_i^k x_i, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), \\ &\qquad \left\| \frac{1}{m_1^2 \cdots m_i^2} f\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), w_i \right\|, \\ &\qquad \left\| \frac{1}{m_1^2 \cdots m_{i-1}^2} f\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), w_i \right\| : i = 1, \ldots, n \Bigg\} = 0, \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
\Phi &= \Phi(x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n)\\
&= \lim_{p\to\infty}\max\Bigg\{\left|\frac{1}{m_1^{2k}\cdots m_n^{2k}}\right|\\
&\qquad\times\max\Bigg\{\left|\frac{1}{m_1^2\cdots m_{i-1}^2 m_i}\right|\varphi_i\left(m_1^{k+1}x_1,\ldots,m_{i-1}^{k+1}x_{i-1},m_i^k x_i,0,m_{i+1}^k x_{i+1},\ldots,m_n^k x_n\right),\\
&\qquad\left|\frac{1}{m_1^2\cdots m_i^2}\right|\varphi_i\left(m_1^{k+1}x_1,\ldots,m_{i-1}^{k+1}x_{i-1},a_i m_i^k x_i,b_i m_i^k x_i,m_{i+1}^k x_{i+1},\ldots,m_n^k x_n\right),\\
&\qquad\left\|\frac{1}{m_1^2\cdots m_i^2}f\left(m_1^{k+1}x_1,\ldots,m_{i-1}^{k+1}x_{i-1},0,m_{i+1}^k x_{i+1},\ldots,m_n^k x_n\right),w_i\right\|,\\
&\qquad\left\|\frac{1}{m_1^2\cdots m_{i-1}^2}f\left(m_1^{k+1}x_1,\ldots,m_{i-1}^{k+1}x_{i-1},0,m_{i+1}^k x_{i+1},\ldots,m_n^k x_n\right),w_i\right\|\\
&\qquad: i=1,\ldots,n\Bigg\} : k=0,1,\ldots,p\Bigg\}<\infty,
\end{aligned}
\tag{2.2}
$$

$$
\lim_{k\to\infty}\left|\frac{1}{m_1^{2k}\ldots m_n^{2k}}\right|\varphi_i\left(m_1^k x_1,\ldots,m_i^k x_i,m_i^k y_i,\ldots,m_n^k x_n\right)=0,
\tag{2.3}
$$

for all $x_i, y_i, w_i \in X$, and $a_i, b_i \in \mathbb{K}$ with $m_i = a_i^2 + b_i^2 \neq 0, 1$, $i = 1, \ldots, n$. Let $f : X^n \to Y$ be a mapping satisfying

$$
\begin{aligned}
&\big\|f(a_1x_1+b_1y_1,x_2,\ldots,x_n)+f(b_1x_1-a_1y_1,x_2,\ldots,x_n)\\
&\quad-\left(a_1^2+b_1^2\right)\left(f(x_1,\ldots,x_n)+f(y_1,x_2,\ldots,x_n)\right),w_1\big\|\\
&\quad\le\varphi_1(x_1,y_1,x_2,\ldots,x_n),\\
&\vdots\\
&\big\|f(x_1,\ldots,x_{n-1},a_nx_n+b_ny_n)+f(x_1,\ldots,x_{n-1},b_nx_n-a_ny_n)\\
&\quad-\left(a_n^2+b_n^2\right)\left(f(x_1,\ldots,x_n)+f(x_1,\ldots,x_{n-1},y_n)\right),w_n\big\|\\
&\quad\le\varphi_n(x_1,\ldots,x_{n-1},x_n,y_n),
\end{aligned}
\tag{2.4}
$$

for all $x_i, y_i, w_i \in X$, and $a_i, b_i \in \mathbb{K}$ with $m_i = a_i^2 + b_i^2 \neq 0, 1$, $i = 1, \ldots, n$. Then there exists a unique mapping $T : X^n \to Y$ satisfying (1.17) *and*

$$
\|f(x_1,\ldots,x_n)-T(x_1,\ldots,x_n),w_i\|\le\Phi,
\tag{2.5}
$$

for all $x_i, w_i \in X$, $i = 1, \ldots, n$.

Proof. Fix $i \in \{1, 2, \dots, n\}$ and consider the following inequality.

$$\begin{aligned} &\left\| f\left(x_1, \dots, a_i x_i + b_i y_i, \dots, x_n\right) + f\left(x_1, \dots, b_i x_i - a_i y_i, \dots, x_n\right)\right. \\ &\left. -\left(a_i^2 + b_i^2\right)\left(f\left(x_1, \dots, x_n\right) + f\left(x_1, \dots, y_i, \dots, x_n\right)\right), w_i \right\| \\ &\le \varphi_i\left(x_1, \dots, x_i, y_i, \dots, x_n\right). \end{aligned} \tag{2.6}$$

Putting $y_i = 0$ and dividing by m_i in (2.6), we get

$$\begin{aligned} &\left\| \frac{1}{m_i}\left(f\left(x_1, \dots, a_i x_i, \dots, x_n\right) + f\left(x_1, \dots, b_i x_i, \dots, x_n\right)\right) - f\left(x_1, \dots, x_n\right), w_i \right\| \\ &\le \max\left\{ \left|\frac{1}{m_i}\right| \varphi_i\left(x_1, \dots, x_i, 0, x_{i+1}, \dots, x_n\right),\ \left\| f\left(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n\right), w_i \right\| \right\}. \end{aligned} \tag{2.7}$$

Replacing x_i by $a_i x_i$ and y_i by $b_i x_i$ and dividing by m_i^2 in (2.6), we obtain

$$\begin{aligned} &\left\| \frac{1}{m_i^2} f\left(x_1, \dots, m_i x_i, \dots, x_n\right) - \frac{1}{m_i}\left(f\left(x_1, \dots, a_i x_i, \dots, x_n\right) + f\left(x_1, \dots, b_i x_i, \dots, x_n\right)\right), w_i \right\| \\ &\le \max\left\{ \left|\frac{1}{m_i^2}\right| \varphi_i\left(x_1, \dots, x_{i-1}, a_i x_i, b_i x_i, x_{i+1}, \dots, x_n\right), \left\| \frac{1}{m_i^2} f\left(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n\right), w_i \right\| \right\}. \end{aligned} \tag{2.8}$$

By (2.7) and (2.8), we have

$$\begin{aligned} &\left\| f\left(x_1, \dots, x_n\right) - \frac{1}{m_i^2} f\left(x_1, \dots, m_i x_i, \dots, x_n\right), w_i \right\| \\ &\le \max\left\{ \left|\frac{1}{m_i}\right| \varphi_i\left(x_1, \dots, x_i, 0, x_{i+1}, \dots, x_n\right), \left|\frac{1}{m_i^2}\right| \varphi_i\left(x_1, \dots, x_{i-1}, a_i x_i, b_i x_i, x_{i+1}, \dots, x_n\right), \right. \\ &\qquad \left. \left\| \frac{1}{m_i^2} f\left(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n\right), w_i \right\|,\ \left\| f\left(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n\right), w_i \right\| \right\}. \end{aligned} \tag{2.9}$$

Therefore one can obtain

$$\begin{aligned}
&\left\| \frac{1}{m_1^2\cdots m_{i-1}^2} f(m_1x_1,\ldots,m_{i-1}x_{i-1},x_i,\ldots,x_n) - \frac{1}{m_1^2\cdots m_i^2} f(m_1x_1,\ldots,m_ix_i,x_{i+1},\ldots,x_n), w_i \right\| \\
&\quad \le \max\left\{ \left| \frac{1}{m_1^2\cdots m_{i-1}^2 m_i} \right| \varphi_i(m_1x_1,\ldots,m_{i-1}x_{i-1},x_i,0,x_{i+1},\ldots,x_n), \right. \\
&\qquad \left| \frac{1}{m_1^2\cdots m_i^2} \right| \varphi_i(m_1x_1,\ldots,m_{i-1}x_{i-1},a_ix_i,b_ix_i,x_{i+1},\ldots,x_n), \\
&\qquad \left\| \frac{1}{m_1^2\cdots m_i^2} f(m_1x_1,\ldots,m_{i-1}x_{i-1},0,x_{i+1},\ldots,x_n), w_i \right\|, \\
&\qquad \left. \left\| \frac{1}{m_1^2\cdots m_{i-1}^2} f(m_1x_1,\ldots,m_{i-1}x_{i-1},0,x_{i+1},\ldots,x_n), w_i \right\| \right\}.
\end{aligned} \tag{2.10}$$

So we conclude

$$\begin{aligned}
&\left\| f(x_1,\ldots,x_n) - \frac{1}{m_1^2\cdots m_n^2} f(m_1x_1,\ldots,m_nx_n), w_i \right\| \\
&\quad \le \max\left\{ \left| \frac{1}{m_1^2\cdots m_{i-1}^2 m_i} \right| \varphi_i(m_1x_1,\ldots,m_{i-1}x_{i-1},x_i,0,x_{i+1},\ldots,x_n), \right. \\
&\qquad \left| \frac{1}{m_1^2\cdots m_i^2} \right| \varphi_i(m_1x_1,\ldots,m_{i-1}x_{i-1},a_ix_i,b_ix_i,x_{i+1},\ldots,x_n), \\
&\qquad \left\| \frac{1}{m_1^2\cdots m_i^2} f(m_1x_1,\ldots,m_{i-1}x_{i-1},0,x_{i+1},\ldots,x_n), w_i \right\|, \\
&\qquad \left. \left\| \frac{1}{m_1^2\ldots m_{i-1}^2} f(m_1x_1,\ldots,m_{i-1}x_{i-1},0,x_{i+1},\ldots,x_n), w_i \right\| : i=1,\ldots,n \right\}.
\end{aligned} \tag{2.11}$$

Therefore we get

$$\begin{aligned}
&\left\| \frac{1}{m_1^{2k}\cdots m_n^{2k}} f\left(m_1^kx_1,\ldots,m_n^kx_n\right) - \frac{1}{m_1^{2(k+1)}\cdots m_n^{2(k+1)}} f\left(m_1^{k+1}x_1,\ldots,m_n^{k+1}x_n\right), w_i \right\| \\
&\quad \le \left| \frac{1}{m_1^{2k}\cdots m_n^{2k}} \right| \max\left\{ \left| \frac{1}{m_1^2\ldots m_{i-1}^2 m_i} \right| \varphi_i\left(m_1^{k+1}x_1,\ldots,m_{i-1}^{k+1}x_{i-1},m_i^kx_i,0,m_{i+1}^kx_{i+1},\ldots,m_n^kx_n\right), \right. \\
&\qquad \left| \frac{1}{m_1^2\cdots m_i^2} \right| \varphi_i\left(m_1^{k+1}x_1,\ldots,m_{i-1}^{k+1}x_{i-1},a_im_i^kx_i,b_im_i^kx_i,m_{i+1}^kx_{i+1},\ldots,m_n^kx_n\right),
\end{aligned}$$

$$\left\| \frac{1}{m_1^2 \cdots m_i^2} f\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), w_i \right\|,$$
$$\left\| \frac{1}{m_1^2 \ldots m_{i-1}^2} f\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), w_i \right\|,$$
$$\left. : i = 1, \ldots, n \right\} \tag{2.12}$$

for all $k \in \mathbb{N} \cup \{0\}$. It follows from (2.12) and (2.1) that the sequence

$$\left\{ \frac{1}{m_1^{2k} \cdots m_n^{2k}} f\left(m_1^k x_1, \ldots, m_n^k x_n\right) \right\} \tag{2.13}$$

is Cauchy. Since the space Y is complete, this sequence is convergent. Therefore we can define $T : X^n \rightarrow Y$ by

$$T(x_1, \ldots, x_n) := \lim_{k \to \infty} \frac{1}{m_1^{2k} \cdots m_n^{2k}} f\left(m_1^k x_1, \ldots, m_n^k x_n\right), \tag{2.14}$$

for all $x_i, w_i \in \mathrm{X}$ and $a_i, b_i \in \mathbb{K}$ with $m_i = a_i^2 + b_i^2 \neq 0, 1$, $i = 1, \ldots, n$. Using induction with (2.12) one can show that

$$\left\| f(x_1, \ldots, x_n) - \frac{1}{m_1^{2p} \cdots m_n^{2p}} f\left(m_1^p x_1, \ldots, m_n^p x_n\right), w_i \right\|$$
$$\leq \max\left\{ \left| \frac{1}{m_1^{2k} \cdots m_n^{2k}} \right| \right.$$
$$\times \max\left\{ \left| \frac{1}{m_1^2 \cdots m_{i-1}^2 m_i} \right| \varphi_i\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, m_i^k x_i, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), \right.$$
$$\left| \frac{1}{m_1^2 \cdots m_i^2} \right| \varphi_i\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, a_i m_i^k x_i, b_i m_i^k x_i, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right),$$
$$\left\| \frac{1}{m_1^2 \cdots m_i^2} f\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), w_i \right\|,$$
$$\left\| \frac{1}{m_1^2 \cdots m_{i-1}^2} f\left(m_1^{k+1}x_1, \ldots, m_{i-1}^{k+1}x_{i-1}, 0, m_{i+1}^k x_{i+1}, \ldots, m_n^k x_n\right), w_i \right\|$$
$$\left. \left. : i = 1, \ldots, n \right\} : k = 0, 1, \ldots, p \right\}, \tag{2.15}$$

for all $x_i, w_i \in X$, $i = 1, \dots, n$ and $p \in \mathbb{N} \cup \{0\}$. By taking p to approach infinity in (2.15) and using (2.2) one obtains (2.5).

For fixed $i \in \{1, 2, \dots, n\}$ and by (2.6) and (2.14), we get

$$\begin{aligned}
&\| T(x_1, \dots, a_i x_i + b_i y_i, \dots, x_n) + T(x_1, \dots, b_i x_i - a_i y_i, \dots, x_n) \\
&\quad - \left(a_i^2 + b_i^2\right)\left(T(x_1, \dots, x_n) + T(x_1, \dots, y_i, \dots, x_n)\right), w_i \|, \\
&\quad = \lim_{k \to \infty} \Bigg\| \frac{1}{m_1^{2k} \cdots m_n^{2k}} \Big\{ f\left(m_1^k x_1, \dots, m_i^k (a_i x_i + b_i y_i), \dots, m_n^k x_n\right) \\
&\qquad\qquad + f\left(m_1^k x_1, \dots, m_i^k (b_i x_i - a_i y_i), \dots, m_n^k x_n\right) \\
&\qquad\qquad - \left(a_i^2 + b_i^2\right) \left(f\left(m_1^k x_1, {}_n^k x_n\right) + f\left(m_1^k x_1, \dots, m_i^k y_i, \dots, m_n^k x_n\right)\right) \Big\}, w_i \Bigg\| \\
&\quad \leq \lim_{k \to \infty} \left| \frac{1}{m_1^{2k} \cdots m_n^{2k}} \right| \varphi_i\left(m_1^k x_1, \dots, m_i^k x_i, m_i^k y_i, \dots, m_n^k x_n\right) = 0.
\end{aligned} \tag{2.16}$$

By (2.16) and (2.3), we conclude that T satisfies (1.17).

Suppose that there exists another mapping $T' : X^n \to Y$ which satisfies (1.17) and (2.5). So we have

$$\begin{aligned}
&\| T(x_1, \dots, x_n) - \mathrm{T}'(x_1, \dots, x_n), w_i \| \\
&\quad \leq \left| \frac{1}{m_1^{2k} \cdots m_n^{2k}} \right| \max\Big\{ \left\| T\left(m_1^k x_1, \dots, m_n^k x_n\right) - f\left(m_1^k x_1, \dots, m_n^k x_n\right), w_i \right\|, \\
&\qquad\qquad \left\| f\left(m_1^k x_1, \dots, m_n^k x_n\right) - T'\left(m_1^k x_1, \dots, m_n^k x_n\right), w_i \right\| \Big\} \\
&\quad \leq \left| \frac{1}{m_1^{2k} \cdots m_n^{2k}} \right| \max\left\{ \Phi\left(m_1^k x_1, \dots, m_n^k x_n\right), \Phi\left(m_1^k x_1, \dots, m_n^k x_n\right) \right\}
\end{aligned} \tag{2.17}$$

which tends to zero as $k \to \infty$ by (2.2). Therefore $T = T'$. This completes the proof. □

In the manner of proof of Theorem 2.1, one can prove the following corollary.

Corollary 2.2. *Let $\varphi_i : X^{n+1} \to [0,\infty)$ for $i \in \{1,\dots,n\}$ be a function such that*

$$\lim_{k\to\infty}\left|\frac{1}{m_1^{2k}\cdots m_n^{2k}}\right|\max\Bigg\{\left|\frac{1}{m_1^2\cdots m_{i-1}^2 m_i}\right| \times\varphi_i\Big(m_1^{k+1}x_1,\dots,m_{i-1}^{k+1}x_{i-1},m_i^k x_i,0,m_{i+1}^k x_{i+1},\dots,m_n^k x_n\Big), \left|\frac{1}{m_1^2\dots m_i^2}\right|\varphi_i\Big(m_1^{k+1}x_1,\dots,m_{i-1}^{k+1}x_{i-1},a_i m_i^k x_i,b_i m_i^k x_i,m_{i+1}^k x_{i+1},\dots,m_n^k x_n\Big) : i=1,\dots,n\Bigg\}=0, \tag{2.18}$$

$$\begin{aligned}\Phi &= \Phi(x_1,\dots,x_i,0,x_{i+1},\dots,x_n)\\ &= \lim_{p\to\infty}\max\Bigg\{\left|\frac{1}{m_1^{2k}\cdots m_n^{2k}}\right| \times\max\Bigg\{\left|\frac{1}{m_1^2\dots m_{i-1}^2 m_i}\right|\varphi_i\Big(m_1^{k+1}x_1,\dots,m_{i-1}^{k+1}x_{i-1},m_i^k x_i,0,m_{i+1}^k x_{i+1},\dots,m_n^k x_n\Big),\\ &\qquad \left|\frac{1}{m_1^2\dots m_i^2}\right|\varphi_i\Big(m_1^{k+1}x_1,\dots,m_{i-1}^{k+1}x_{i-1},a_i m_i^k x_i,b_i m_i^k x_i,m_{i+1}^k x_{i+1},\dots,m_n^k x_n\Big)\\ &\qquad : i=1,\dots,n\Bigg\} : k=0,1,\dots,p\Bigg\}<\infty,\end{aligned} \tag{2.19}$$

$$\lim_{k\to\infty}\left|\frac{1}{m_1^{2k}\cdots m_n^{2k}}\right|\varphi_i\Big(m_1^k x_1,\dots,m_i^k x_i,m_i^k y_i,\dots,m_n^k x_n\Big)=0, \tag{2.20}$$

for all $x_i, y_i, w_i \in X$ and $a_i, b_i \in \mathbb{K}$ with $m_i = a_i^2 + b_i^2 \neq 0,1$, $i=1,\dots,n$. Let $f : X^n \to Y$ be a mapping satisfying

$$\begin{aligned}&\Big\| f(a_1x_1+b_1y_1,x_2,\dots,x_n)+f(b_1x_1-a_1y_1,x_2,\dots,x_n)\\ &\quad -\Big(a_1^2+b_1^2\Big)(f(x_1,\dots,x_n)+f(y_1,x_2,\dots,x_n)),w_1\Big\| \le \varphi_1(x_1,y_1,x_2,\dots,x_n),\\ &\vdots\\ &\Big\| f(x_1,\dots,x_{n-1},a_nx_n+b_ny_n)+f(x_1,\dots,x_{n-1},b_nx_n-a_ny_n)\\ &\quad -\Big(a_n^2+b_n^2\Big)(f(x_1,\dots,x_n)+f(x_1,\dots,x_{n-1},y_n)),w_n\Big\| \le \varphi_n(x_1,\dots,x_{n-1},x_n,y_n),\end{aligned} \tag{2.21}$$

for all $x_i, y_i, w_i \in X$, *and* $a_i, b_i \in \mathbb{K}$ *with* $m_i = a_i^2 + b_i^2 \neq 0, 1$, $i = 1, \ldots, n$. *Then there exists a unique mapping* $T : X^n \to Y$ *satisfying* (1.17) *and*

$$\left\| f(x_1, \ldots, x_n) - T(x_1, \ldots, x_n), w_i \right\| \leq \Phi \tag{2.22}$$

for all $x_i, w_i \in X$, $i = 1, \ldots, n$.

3. System (1.17) Stability in Menger Probabilistic Non-Archimedean 2-Banach Spaces

In this section, we prove the generalized Hyers-Ulam-Rassias stability of system (1.17) in Menger probabilistic non-Archimedean 2–Banach spaces. Throughout this section, we assume that $u \in \mathbb{R}$,$i, k, n \in \mathbb{N} \cup \{0\}$, $\mathbb{K}$ is a non–Archimedean field, T is a continuous t–norm of H-type, (Y, ν, T) is a Menger probabilistic non-Archimedean 2-Banach space over $\mathbb{K}$, (Z, ω, T) is a Menger probabilistic non-Archimedean 2-normed space over $\mathbb{K}$, and X is a vector space over $\mathbb{K}$. Also assume that $f : X^n \to Y$ is a mapping.

Theorem 3.1. *Let* $\varphi_i : X^{n+1} \to Z$ *for* $i \in \{1, \ldots, n\}$ *be a mapping such that*

$$\begin{aligned}
\widetilde{\varphi}_i &= \widetilde{\varphi}_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n, u) \\
&= T\Bigg\{T\left(\omega\left(\frac{1}{m_1^2 \cdots m_{i-1}^2 m_i}\varphi_i(m_1x_1, \ldots, m_{i-1}x_{i-1}, x_i, 0, \ldots, x_n), w_i\right)(u)\right., \\
&\qquad \left.\nu\left(\frac{1}{m_1^2 \cdots m_{i-1}^2} f(m_1x_1, \ldots, m_{i-1}x_{i-1}, 0, x_{i+1}, \ldots, x_n), w_i\right)(u)\right), \\
&\qquad T\left(\omega\left(\frac{1}{m_1^2 \cdots m_i^2}\varphi_i(m_1x_1, \ldots, m_{i-1}x_{i-1}, a_ix_i, b_ix_i, x_{i+1}, \ldots, x_n), w_i\right)(u)\right., \\
&\qquad \left.\nu\left(\frac{1}{m_1^2 \cdots m_i^2} f(m_1x_1, \ldots, m_{i-1}x_{i-1}, 0, x_{i+1}, \ldots, x_n), w_i\right)(u)\right)\Bigg\},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\Phi_1 &= \Phi_1(x_1, x_1, x_2, \ldots, x_n, u) \\
&= \widetilde{\varphi}_1(x_1, y_1, x_2, \ldots, x_n, u), \\
\Phi_i &= \Phi_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n, u) \\
&= T(\widetilde{\varphi}_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n, u),\ \Phi_{i-1}(x_1, \ldots, x_{i-1}, x_{i-1}, x_i, \ldots, x_n, u)),
\end{aligned}$$

$$\lim_{k \to \infty} \Phi_n\left(m_1^k x_1, \ldots, m_n^k x_n, m_n^k x_n, \left|m_1^{2k} \cdots m_n^{2k}\right| u\right) = 1,$$

$$\lim_{k\to\infty}\omega\left(\frac{1}{m_1^{2k}\cdots m_n^{2k}}\varphi_i\left(m_1^k x_1,\ldots,m_i^k x_i,m_i^k y_i,\ldots,m_n^k x_n\right),w_i\right)(u)=1, \tag{3.2}$$

$$\begin{aligned}
\Phi_k^* &= \Phi_k^*(x_1,\ldots,x_n,x_n,u)=\Phi_n\left(m_1^k x_1,\ldots,m_n^k x_n,m_n^k x_n,\left|m_1^{2k}\cdots m_n^{2k}\right|u\right),\\
\Psi_0 &= \Phi_0^*(x_1,\ldots,x_n,x_n,u)=\Phi_n(x_1,\ldots,x_n,x_n,u),\\
\Psi_k &= \Psi_k(x_1,\ldots,x_n,x_n,u)=T\left(\Phi_k^*(x_1,\ldots,x_n,x_n,u),\Psi_{k-1}(x_1,\ldots,x_n,x_n,u)\right),\\
\Psi &= \Psi(x_1,\ldots,x_n,x_n,u)=\lim_{k\to\infty}\Psi_k=1,
\end{aligned} \tag{3.3}$$

for all $u>0$, $x_i,w_i\in X$, *and* $a_i,b_i\in\mathbb{K}$ *with* $m_i=a_i^2+b_i^2\neq 0,1$, $i=1,\ldots,n$. *Let* $f:X^n\to Y$ *be a mapping satisfying*

$$\begin{aligned}
&\nu\Big(f(a_1x_1+b_1y_1,x_2,\ldots,x_n)+f(b_1x_1-a_1y_1,x_2,\ldots,x_n)\\
&\quad-\left(a_1^2+b_1^2\right)(f(x_1,\ldots,x_n)+f(y_1,x_2,\ldots,x_n)),w_1\Big)(u)\geq\omega(\varphi_1(x_1,y_1,x_2,\ldots,x_n),w_1)(u),\\
&\vdots\\
&\nu\Big(f(x_1,\ldots,x_{n-1},a_nx_n+b_ny_n)+f(x_1,\ldots,x_{n-1},b_nx_n-a_ny_n)\\
&\quad-\left(a_n^2+b_n^2\right)(f(x_1,\ldots,x_n)+f(x_1,\ldots,x_{n-1},y_n)),w_n\Big)(u)\geq\omega(\varphi_n(x_1,\ldots,x_n,y_n),w_n)(u),
\end{aligned} \tag{3.4}$$

for all $u>0$, $x_i,y_i,w_i\in X$ *and* $a_i,b_i\in\mathbb{K}$ *with* $m_i=a_i^2+b_i^2\neq 0,\ 1$, $i=1,\ldots,n$. *Then there exists a unique mapping* $F:X^n\to Y$ *satisfying* (1.17) *and*

$$\nu(f(x_1,\ldots,x_n)-F(x_1,\ldots,x_n),w_i)(u)\geq\Psi, \tag{3.5}$$

for all $u>0$ *and* $x_i,w_i\in X$, $i=1,\ldots,n$.

Proof. Fix $i\in\{1,2,\ldots,n\}$ and consider the following inequality:

$$\begin{aligned}
&\nu\Big(f\left(x_1,\ldots,a_ix_i+b_iy_i,\ldots,x_n\right)+f(x_1,\ldots,b_ix_i-a_iy_i,\ldots,x_n)\\
&\quad-\left(a_i^2+b_i^2\right)(f(x_1,\ldots,x_n)+f(x_1,\ldots,y_i,\ldots,x_n)),w_i\Big)(u)\\
&\quad\geq\omega(\varphi_i(x_1,\ldots,x_i,y_i,\ldots,x_n),w_i)(u).
\end{aligned} \tag{3.6}$$

Putting $y_i = 0$ and dividing by m_i in (3.6), we get

$$\begin{aligned} &\nu\left(\frac{1}{m_i}(f(x_1,\dots,a_ix_i,\dots,x_n)+f(x_1,\dots,b_ix_i,\dots,x_n))-f(x_1,\dots,x_n),w_i\right)(u) \\ &\quad\geq T\left(\omega\left(\frac{1}{m_i}\varphi_i(x_1,\dots,x_i,0,\dots,x_n),w_i\right)(u),\nu(f(x_1,\dots,0,\dots,x_n),w_i)(u)\right). \end{aligned} \tag{3.7}$$

Replacing x_i by a_ix_i and y_i by b_ix_i and dividing by m_i^2 in (3.6), we obtain

$$\begin{aligned} &\nu\left(\frac{1}{m_i^2}f(x_1,\dots,m_ix_i,\dots,x_n)-\frac{1}{m_i}(f(x_1,\dots,a_ix_i,\dots,x_n)+f(x_1,\dots,b_ix_i,\dots,x_n)),w_i\right)(u) \\ &\quad\geq T\left(\omega\left(\frac{1}{m_i^2}\varphi_i(x_1,\dots,x_{i-1},a_ix_i,b_ix_i,x_{i+1},\dots,x_n),w_i\right)(u)\,,\right. \\ &\qquad\qquad \left.\nu\left(\frac{1}{m_i^2}f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),w_i\right)(u)\right). \end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{aligned} &\nu\left(f(x_1,\dots,x_n)-\frac{1}{m_i^2}f(x_1,\dots,m_ix_i,\dots,x_n),w_i\right)(u) \\ &\quad\geq T\left\{T\left(\omega\left(\frac{1}{m_i}\varphi_i(x_1,\dots,x_i,0,\dots,x_n),w_i\right)(u)\,,\right.\right. \\ &\qquad\quad \nu(f(x_1,\dots,,x_{i-1},0,x_{i+1},\dots,x_n),w_i)(u)), \\ &\qquad\quad T\left(\omega\left(\frac{1}{m_i^2}\varphi_i(x_1,\dots,x_{i-1},a_ix_i,b_ix_i,x_{i+1},\dots,x_n),w_i\right)(u),\right. \\ &\qquad\quad \left.\left.\nu\left(\frac{1}{m_i^2}f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),w_i\right)(u)\right)\right\}. \end{aligned} \tag{3.9}$$

Therefore one can obtain

$$\begin{aligned} &\nu\left(\frac{1}{m_1^2\cdots m_{i-1}^2}f(m_1x_1,\dots,m_{i-1}x_{i-1},x_i,\dots,x_n)-\frac{1}{m_1^2\cdots m_i^2}f(m_1x_1,\dots,m_ix_i,\dots,x_n),w_i\right)(u) \\ &\quad\geq T\left\{T\left(\omega\left(\frac{1}{m_1^2\cdots m_{i-1}^2m_i}\varphi_i(m_1x_1,\dots,m_{i-1}x_{i-1},x_i,0,\dots,x_n),w_i\right)(u),\right.\right. \\ &\qquad\quad \left.\nu\left(\frac{1}{m_1^2\cdots m_{i-1}^2}f(m_1x_1,\dots,m_{i-1}x_{i-1},0,x_{i+1},\dots,x_n),w_i\right)(u)\right), \end{aligned}$$

$$T\left(\omega\left(\frac{1}{m_1^2\cdots m_i^2}\varphi_i(m_1x_1,\ldots,m_{i-1}x_{i-1},a_ix_i,b_ix_i,x_{i+1},\ldots,x_n),w_i\right)(u),\right.$$
$$\left.\nu\left(\frac{1}{m_1^2\cdots m_i^2}f(m_1x_1,\ldots,m_{i-1}x_{i-1},0,x_{i+1},\ldots,x_n),w_i\right)(u)\right)\Bigg\} = \widetilde{\varphi}_i. \tag{3.10}$$

So we conclude

$$\nu\left(f(x_1,\ldots,x_n)-\frac{1}{m_1^2\cdots m_{\mathrm{n}}^2}f(m_1x_1,\ldots,m_nx_n),w_i\right)(u)\geq \Phi_n. \tag{3.11}$$

Therefore we get

$$\begin{aligned}&\nu\left(\frac{1}{m_1^{2k}\cdots m_n^{2k}}f\left(m_1^kx_1,\ldots,m_n^kx_n\right)-\frac{1}{m_1^{2(k+1)}\cdots m_n^{2(k+1)}}f\left(m_1^kx_1,\ldots,m_n^kx_n\right),w_i\right)(u)\\&\geq \Phi_n\left(m_1^kx_1,\ldots,m_n^kx_n,m_n^kx_n,\left|m_1^{2k}\cdots m_{\mathrm{n}}^{2k}\right|u\right),\end{aligned} \tag{3.12}$$

for all $x_i, w_i \in X$ and $k \in \mathbb{N}\cup\{0\}$. It follows from (3.12) and (3.1) that the sequence

$$\left\{\frac{1}{m_1^{2k}\cdots m_n^{2k}}f\left(m_1^kx_1,\ldots,m_n^kx_n\right)\right\} \tag{3.13}$$

is Cauchy. Since the space Y is complete, this sequence is convergent. Therefore we can define $F : X^n \to Y$ by

$$\lim_{k\to\infty}\nu\left(F(x_1,\ldots,x_n)-\frac{1}{m_1^{2k}\cdots m_n^{2k}}f\left(m_1^kx_1,\ldots,m_n^kx_n\right),w_i\right)(u)=1, \tag{3.14}$$

for all $u > 0$, $x_i, w_i \in X$ and $a_i, b_i \in \mathbb{K}$ with $m_i = a_i^2+b_i^2 \neq 0,1$, $i = 1,\ldots,n$. Using induction with (3.12) one can show that

$$\nu\left(f(x_1,\ldots,x_n)-\frac{1}{m_1^{2k}\cdots m_n^{2k}}f\left(m_1^kx_1,\ldots,m_n^kx_n\right),w_i\right)(u)\geq \Psi_k, \tag{3.15}$$

for all $x_i, w_i \in X$, $i = 1,\ldots,n$ and $k \in \mathbb{N}\cup\{0\}$. By taking k to approach infinity in (3.15) and using (3.3) one obtains (3.5).

For fixed $i \in \{1, 2, \ldots, n\}$ and by (3.6) and (3.14), we get

$$\begin{aligned}
&\nu\Big(F(x_1, \ldots, a_i x_i + b_i y_i, \ldots, x_n) + F(x_1, \ldots, b_i x_i - a_i y_i, \ldots, x_n) \\
&\quad -\Big(a_i^2 + b_i^2\Big)(F(x_1, \ldots, x_n) + F(x_1, \ldots, y_i, \ldots, x_n)), w_i\Big)(u) \\
&\quad = \lim_{n\to\infty} \nu\Bigg(\frac{1}{m_1^{2k}\cdots m_n^{2k}}\Big\{f\Big(m_1^k x_1, \ldots, m_i^k(a_i x_i + b_i y_i), \ldots, m_n^k x_n\Big) \\
&\qquad + f\Big(m_1^k x_1, \ldots, m_i^k(b_i x_i - a_i y_i), \ldots, m_n^k x_n\Big) \\
&\qquad -\Big(a_i^2 + b_i^2\Big)\Big(f\Big(m_1^k x_1, {}_n^k x_n\Big) + f\Big(m_1^k x_1, \ldots, m_i^k y_i, \ldots, m_n^k x_n\Big)\Big)\Big\}, w_i\Bigg)(u) \\
&\quad \geq \lim_{k\to\infty} \omega\Bigg(\frac{1}{m_1^{2k}\cdots m_n^{2k}}\varphi_i\Big(m_1^k x_1, \ldots, m_i^k x_i, m_i^k y_i, \ldots, m_n^k x_n\Big), w_i\Bigg)(u) = 1.
\end{aligned} \tag{3.16}$$

By (3.2) and (3.16), we conclude that F satisfies (1.17).

Suppose that there exists another mapping $F' : X^n \to Y$ which satisfies (1.17) and (3.5). So we have

$$\begin{aligned}
&\nu(F(x_1, \ldots, x_n) - F'(x_1, \ldots, x_n), w)(u) \\
&\quad = \nu\Bigg(\frac{1}{m_1^{2k}\ldots m_n^{2k}}\Big\{F\Big(m_1^k x_1, \ldots, m_n^k x_n\Big) - f\Big(m_1^k x_1, \ldots, m_n^k x_n\Big) \\
&\qquad + f\Big(m_1^k x_1, \ldots, m_n^k x_n\Big) - F'\Big(m_1^k x_1, \ldots, m_n^k x_n\Big)\Big\}, w\Bigg)(u) \\
&\quad \geq T\Big\{\Psi\Big(m_1^k x_1, \ldots, m_n^k x_n, m_n^k x_n, \Big|m_1^{2k}\ldots m_n^{2k}\Big|u\Big),\ \Psi\Big(m_1^k x_1, \ldots, m_n^k x_n, m_n^k x_n, \Big|m_1^{2k}\ldots m_n^{2k}\Big|u\Big)\Big\},
\end{aligned} \tag{3.17}$$

which tends to 1 as $k \to \infty$ by (3.3). Therefore $F = F'$. This completes the proof. □

In the manner of proof of Theorem 3.1, one can investigate the following corollary.

Corollary 3.2. *Let $\varphi_i : X^{n+1} \to Z$ for $i \in \{1, \ldots, n\}$ be a mapping such that*

$$\begin{aligned}
\tilde{\varphi}_i &= \tilde{\varphi}_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n, u) \\
&= T\Bigg\{\omega\Bigg(\frac{1}{m_1^2\cdots m_{i-1}^2 m_i}\varphi_i(m_1 x_1, \ldots, m_{i-1}x_{i-1}, x_i, 0, \ldots, x_n), w_i\Bigg)(u), \\
&\qquad \omega\Bigg(\frac{1}{m_1^2\cdots m_i^2}\varphi_i(m_1 x_1, \ldots, m_{i-1}x_{i-1}, a_i x_i, b_i x_i, x_{i+1}, \ldots, x_n), w_i\Bigg)(u)\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\Phi_1 &= \Phi_1(x_1, x_1, x_2, \ldots, x_n, u) = \widetilde{\varphi}_1(x_1, y_1, x_2, \ldots, x_n, u), \\
\Phi_i &= \Phi_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n, u) \\
&= T(\widetilde{\varphi}_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n, u), \Phi_{i-1}(x_1, \ldots, x_{i-1}, x_{i-1}, x_i, \ldots, x_n, u)), \\
&\lim_{k\to\infty} \Phi_n\left(m_1^k x_1, \ldots, m_n^k x_n, m_n^k x_n, \left|m_1^{2k} \ldots m_n^{2k}\right| u\right) = 1, \\
&\lim_{k\to\infty} \omega\left(\frac{1}{m_1^{2k} \cdots m_n^{2k}} \varphi_i\left(m_1^k x_1, \ldots, m_i^k x_i, m_i^k y_i, \ldots, m_n^k x_n\right), w_i\right)(u) = 1, \\
\Phi_k^* &= \Phi_k^*(x_1, \ldots, x_n, x_n, u) = \Phi_n\left(m_1^k x_1, \ldots, m_n^k x_n, m_n^k x_n, \left|m_1^{2k} \cdots m_n^{2k}\right| u\right), \\
\Psi_0 &= \Phi_0^*(x_1, \ldots, x_n, x_n, u) = \Phi_n(x_1, \ldots, x_n, x_n, u), \\
\Psi_k &= \Psi_k(x_1, \ldots, x_n, x_n, u) = T(\Phi_k^*(x_1, \ldots, x_n, x_n, u), \Psi_{k-1}(x_1, \ldots, x_n, x_n, u)), \\
\Psi &= \Psi(x_1, \ldots, x_n, x_n, u) = \lim_{k\to\infty} \Psi_k = 1,
\end{aligned} \tag{3.18}$$

for all $u > 0$, $x_i, w_i \in X$ *and* $a_i, b_i \in \mathbb{K}$ *with* $m_i = a_i^2 + b_i^2 \neq 0, 1$, $i = 1, \ldots, n$. *Let* $f : X^n \to Y$ *be a mapping satisfying*

$$\begin{aligned}
&\nu\Big(f(a_1x_1 + b_1y_1, x_2, \ldots, x_n) + f(b_1x_1 - a_1y_1, x_2, \ldots, x_n) \\
&\quad -\left(a_1^2 + b_1^2\right)(f(x_1, \ldots, x_n) + f(y_1, x_2, \ldots, x_n)), w_1\Big)(u), \\
&\geq \omega(\varphi_1(x_1, y_1, x_2, \ldots, x_n), w_1)(u); \\
&\vdots \\
&\nu\Big(f(x_1, \ldots, x_{n-1}, a_nx_n + b_ny_n) + f(x_1, \ldots, x_{n-1}, b_nx_n - a_ny_n) \\
&\quad -\left(a_n^2 + b_n^2\right)(f(x_1, \ldots, x_n) + f(x_1, \ldots, x_{n-1}, y_n)), w_n\Big)(u), \\
&\geq \omega(\varphi_n(x_1, \ldots, x_n, y_n), w_1)(u);
\end{aligned} \tag{3.19}$$

for all $u > 0$, $x_i, y_i, w_i \in X$ *and* $a_i, b_i \in \mathbb{K}$ *with* $m_i = a_i^2 + b_i^2 \neq 0, 1$, $i = 1, \ldots, n$. *Assume that* $f(x_1, x_2, \ldots, x_n) = 0$ *if* $x_i = 0$ *for some* $i = 1, \ldots, n$. *Then there exists a unique mapping* $F : X^n \to Y$ *satisfying* (1.17) *and*

$$\nu(f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n), w_i)(u) \geq \Psi, \tag{3.20}$$

for all $u > 0$ *and* $x_i, w_i \in X$, $i = 1, \ldots, n$.

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