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Research Article

A Fold Bifurcation Theorem of Degenerate Solutions in a Perturbed Nonlinear Equation

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We consider a nonlinear equation $F(\varepsilon, \lambda, u) = 0$, where the parameter ε is a perturbation parameter, F is a differentiable mapping from $\mathbf{R}\times\mathbf{R}\times X$ to Y, and X, Y are Banach spaces. We obtain an abstract bifurcation theorem by using the generalized saddle-node bifurcation theorem.

1. Introduction

In [1, 2], Crandall and Rabinowitz proved two celebrated theorems which are now regarded as foundation of the analytical bifurcation theory in infinite-dimensional spaces and both results are based on the implicit function theorem. In [3], we obtained the generalized saddle-node bifurcation theorem by the generalized inverse. In [4], we proved a perturbed problem using Morse Lemma. For a more general introduction to bifurcation theory and other related methods in nonlinear analysis, see, for example, [5–7]. On the other hand, [8–11] provide a more detailed introduction to mathematical models in some recent new results in the application of bifurcation theory including chemical reactions, population ecology, and nonautonomous differential equations.

In this paper, we continue the work of [3] and obtain an abstract bifurcation theorem under the opposite condition in [4]. We consider the solution set of

$$F(\varepsilon, \lambda, u) = 0, \tag{1.1}$$

where ε indicates the perturbation. Fix $\varepsilon = \varepsilon_0$; let (λ_0, u_0) be a solution of $F(\varepsilon_0, \cdot, \cdot) = 0$. From the implicit function theorem, a necessary condition for bifurcation is that $F_u(\varepsilon_0, \lambda_0, u_0)$ is not invertible; we call $(\varepsilon_0, \lambda_0, u_0)$ a degenerate solution. In [12], Shi shows the persistence and the bifurcation of degenerate solutions when ε varies near ε_0 by the implicit function theorem and the saddle-node bifurcation theorem. In this paper, we prove a new perturbed bifurcation theorem by the generalized saddle-node bifurcation theorem.

In the paper, we use $\|\cdot\|$ as the norm of Banach space X and $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u. For a linear operator L, we use $N(L)$ as the null space of L and $R(L)$ as the range of L.

2. Preliminaries

Definition 2.1 (see [13]). Let X, Y be Banach spaces, and let $A \in \mathcal{L}(X, Y)$ be a linear operator. Then, $A^+ \in \mathcal{L}(Y, X)$ is called the generalized inverse of A if it satisfies

(i) $AA^+A = A$,

(ii) $A^+AA^+ = A^+$.

Definition 2.2 (see [13]). Let X, Y, and A be the same as in Definition 2.1. If $A \in \mathcal{L}(X, Y)$ has the bounded linear generalized inverse A^+, then A is called a generalized regular operator.

Lemma 2.3 (see [13]). *Let $A \in \mathcal{L}(X, Y)$, then A is a generalized regular operator if and only if $N(A), R(A)$ are topologically complemented in X, Y, respectively. In this case, $I - A^+A$, AA^+ are bounded linear projectors from X, Y into $N(A)$, $R(A)$, respectively.*

We recall the generalized saddle-node bifurcation in [3] and give an alternate proof here using the generalized Lyapunov-Schmidt reduction.

Theorem 2.4 (generalized saddle-node bifurcation). *Let $V \subset \mathbf{R} \times X$ be a neighborhood of (λ_0, u_0), $F \in C^1(V, Y)$. Suppose that*

(i) $F(\lambda_0, u_0) = 0$;

(ii) $F_u(\lambda_0, u_0) : X \to Y$ *is a generalized regular operator, and*

$$\dim N(F_u(\lambda_0, u_0)) \geq \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1, \tag{2.1}$$

(iii) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

Let $Z = R((F_u(\lambda_0, u_0))^+)$, then the subset $\{(\lambda, u) | F(\lambda, u) = 0\}$ contains the curve $(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s))$ near (λ_0, u_0), where $w_0 \in N(F_u(\lambda_0, u_0)) \setminus \{\theta\}$, the mapping $z(s)$ is continuously differentiable near $s = 0$, and $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, $z'(0) = z(0) = \theta$.

Proof. Since $A = F_u(\lambda_0, u_0)$ is a generalized regular operator, there exist closed subspaces Z in X, Y_1 in Y satisfing $X = Z \oplus N(A)$, $Y = R(A) \oplus Y_1$.

Taking an arbitrary $w_0 \in N(A) \setminus \{\theta\}$, from Lemma 2.3, $F(\lambda, u) = 0$ is equivalent to

$$\begin{gathered} (I - AA^+)F(\lambda, u_0 + sw_0 + z) = 0, \\ AA^+F(\lambda, u_0 + sw_0 + z) = 0, \end{gathered} \tag{2.2}$$

where $s \in \mathbf{R}$, $z \in Z$.

Define $G : \mathbf{R} \times \mathbf{R} \times Z \to R(A)$ as

$$\begin{aligned} G(s,\lambda,z) &= AA^{+}F(\lambda, u_0 + sw_0 + z), \\ G_{(\lambda,z)}(0,\lambda_0,0)\left[(\tau,\psi)\right] &= AA^{+}\left(\tau F_{\lambda}(\lambda_0,u_0) + F_u(\lambda_0,u_0)\left[\psi\right]\right), \\ &= AA^{+}A\left[\psi\right] = A\left[\psi\right], \end{aligned} \tag{2.3}$$

because of (iii), then $G_{(\lambda,z)}(0,\lambda_0,0) : R \times Z \to R(A)$ is an isomorphism.

For the equation $G(s,\lambda,z) = 0$, by the implicit function theorem, there exist $\varepsilon > 0$ and $(\lambda(s), z(s)) \in C^1(-\varepsilon,\varepsilon)$, with $\lambda(0) = \lambda_0$, $z(0) = 0$ satisfying

$$G(s,\lambda(s),z(s)) = 0. \tag{2.4}$$

From (2.2), we have

$$F(\lambda(s), u_0 + sw_0 + z(s)) = 0, \quad s \in (-\varepsilon,\varepsilon). \tag{2.5}$$

Differentiating (2.5) with respect to s, we have

$$F_{\lambda}(\lambda(s), u_0 + sw_0 + z(s))\lambda'(s) + F_u(\lambda(s), u_0 + sw_0 + z(s))\left[w_0 + z'(s)\right] = 0. \tag{2.6}$$

Setting $s = 0$,

$$F_{\lambda}(\lambda_0,u_0)\lambda'(0) + F_u(\lambda_0,u_0)\left[w_0 + z'(0)\right] = 0. \tag{2.7}$$

Thus, $\lambda'(0) = 0$ since (iii) and we have

$$F_u(\lambda_0,u_0)\left[z'(0)\right] = 0, \tag{2.8}$$

that is, $z'(0) \in N(A) \cap Z$, we have $z'(0) = 0$. □

Corollary 2.5. *Assume the conditions in Theorem 2.4 are satisfied and*

$$\dim N(F_u(\lambda_0,u_0)) = n, \qquad N(F_u(\lambda_0,u_0)) = \operatorname{span}\{w_1, w_2, \ldots, w_n\}, \tag{2.9}$$

then the direction of the solution curves is determined by

$$\lambda_i''(0) = -\frac{\langle l, F_{uu}(\lambda_0,u_0)[w_i,w_i]\rangle}{\langle l, F_{\lambda}(\lambda_0,u_0)\rangle}, \tag{2.10}$$

where $l \in R(F_u(\lambda_0,u_0))^{\perp}$, $i = 1,2,\ldots,n$. Furthermore, when

$$F_{uu}(\lambda_0,u_0)[w_i,w_i] \notin R(F_u(\lambda_0,u_0)) \tag{2.11}$$

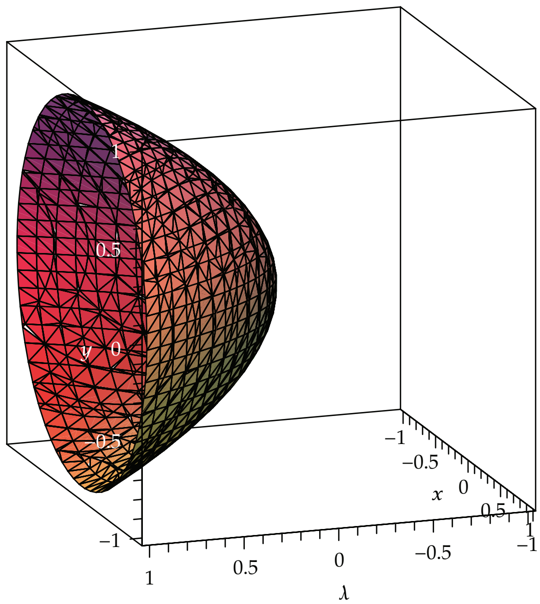

Figure 1: Bifurcation diagram of the equation $\lambda - x^2 - y^2 = 0$ in Example 2.6.

is satisfied, $\lambda_i''(0) \neq 0$*, and the solution curve* $\{(\lambda_i(s), u_i(s)) : |s| < \delta\}$ *is a parabola-like curve which reaches an extreme point at* (λ_0, u_0)*.*

We illustrate our result by a simple example.

Example 2.6. Define

$$F\left(\lambda, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \lambda - x^2 - y^2 = 0, \tag{2.12}$$

where $U = \binom{x}{y} \in \mathbf{R}^2$, $\lambda \in \mathbf{R}$. From simple calculations, we obtain

$$F_U = (-2x, -2y), \qquad F_\lambda = 1, \qquad F_{UU} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{2.13}$$

We analyze the bifurcation at $(0, \binom{0}{0})$. It is easy to see that $N(F_U) = \operatorname{span}\{w_1, w_2\}$, where $w_1 = \binom{1}{0}$, $w_2 = \binom{0}{1}$, $R(F_U) = \{0\}$. So, obviously, $\dim N(F_U) = 2$, $\operatorname{codim} R(F_U) = 1$, and $F_\lambda \notin R(F_U)$. From the above calculation,

$$F_{UU}[w_1, w_1] = -2, \qquad F_{UU}[w_2, w_2] = -2. \tag{2.14}$$

Obviously, $F_{UU}(0, \binom{0}{0})[w_i, w_i] \notin R(F_U(0, \binom{0}{0}))$ and $\lambda_i''(0) = -2$, $i = 1, 2$. Thus, we can apply Corollary 2.5 to (2.12). In fact, all solution curves for all $w_i \in N(F_U)$ form a surface (see Figure 1).

3. Main Theorems

Applying Theorem 2.4, we discuss the bifurcation of solutions of the perturbed problem. We consider the solution set of

$$F(\varepsilon, \lambda, u) = 0, \tag{3.1}$$

where the parameter ε indicates the perturbation, $F \in C^1(M, Y)$, $M \equiv \mathbf{R} \times \mathbf{R} \times X$, and X, Y are Banach spaces. Let

$$H(\varepsilon, \lambda, u, w) = \begin{pmatrix} F(\varepsilon, \lambda, u) \\ F_u(\varepsilon, \lambda, u)[w] \end{pmatrix}. \tag{3.2}$$

Suppose that $(\varepsilon_0, \lambda_0, u_0, w_0)$ is a solution of $H(\varepsilon, \lambda, u, w) = 0$. For $(\varepsilon_0, \lambda_0, u_0) \in M$ and

$$w_0 \in X_1 \equiv \{x \in X : \|x\| = 1\}, \tag{3.3}$$

by Hahn-Banach theorem, there exists a closed subspace X_3 of X with codimension 1 such that $X = L(w_0) \oplus X_3$, where $L(w_0) = \operatorname{span}\{w_0\}$ and $d(w_0, X_3) = \inf\{\|w_0 - x\| : x \in X_3\} > 0$. Let $X_2 = w_0 + X_3 = \{w_0 + x : x \in X_3\}$. Then, X_2 is a closed hyperplane of X with codimension 1. Since X_3 is a closed subspace of X and X_3 is also a Banach space in the subspace topology, Hence we can regard $M_1 = M \times X_2$ as a Banach space with product topology. Moreover, the tangent space of M_1 is homeomorphic to $M \times X_3$ (see [12] for more on the setting).

In the following, we will still use the conditions (Fi) on F defined in [12].

(F1) $\dim N(F_u(\varepsilon_0, \lambda_0, u_0)) = \operatorname{codim} R(F_u(\varepsilon_0, \lambda_0, u_0)) = 1$, and $N(F_u(\varepsilon_0, \lambda_0, u_0)) = \operatorname{span}\{w_0\}$;

(F2) $F_\lambda(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$;

(F3) $F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$;

(F4) $F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$;

(F5) $F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$.

We use the convention that (Fi') means that the condition defined in (Fi) does not hold.

Theorem 3.1. *Let $F \in C^2(M, Y)$, $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$ such that $H(T_0) = (0, 0)$. Suppose that the operator F satisfies $(F1)$, $(F2')$, $(F3)$, $(F4')$, and $(F5)$ at T_0. One also assumes that*

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_1, w_0] + F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \in R(F_u(\varepsilon_0, \lambda_0, u_0)), \tag{3.4}$$

where $v_1 \in X_3 \setminus \{0\}$ is the unique solution of

$$F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0. \tag{3.5}$$

Then, there exists $\delta > 0$ *such that all the solutions of* $H(\varepsilon, \lambda, u, w) = (0,0)$ *near* T_0 *form two* C^2 *curves:*

$$\{T_1(s) = (\varepsilon_1(s), \lambda_1(s), u_1(s), w_1(s)), s \in I = (-\delta, \delta)\},$$
$$\{T_2(s) = (\varepsilon_2(s), \lambda_2(s), u_2(s), w_2(s)), s \in I = (-\delta, \delta)\}, \tag{3.6}$$

where $\varepsilon_i(s) = \varepsilon_0 + \tau_i(s)$, $s \in I$; $\tau_i(\cdot) \in C^2(I, \mathbf{R})$; $\tau_i(0) = \tau_i'(0) = 0$, *and*

$$\begin{aligned} \lambda_1(s) &= \lambda_0 + z_{11}(s), \quad \lambda_2(s) = \lambda_0 + s + z_{21}(s), \quad s \in I, \\ u_1(s) &= u_0 + s w_0 + z_{12}(s), \quad u_2(s) = u_0 + s v_1 + z_{22}(s), \quad s \in I, \\ w_1(s) &= w_0 + s\psi_0 + z_{13}(s), \quad w_2(s) = w_0 + s\psi_1 + z_{23}(s), \quad s \in I, \end{aligned} \tag{3.7}$$

where $z_{ij}(\cdot) \in C^2(I, Z)$, $z_{ij}(0) = z_{ij}'(0) = 0$ $(i = 1, 2,\ j = 1, 2, 3)$, $\psi_0 \in X_3$, $\psi_1 \in X_3$ *are the unique solution of*

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = 0, \tag{3.8}$$

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_1, w_0] + F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = 0, \tag{3.9}$$

respectively.

Remark 3.2. Theorem 2.4 complements Theorem 3.2 in [4], where the opposite condition (3.4) is imposed.

Proof. We apply Theorem 2.4 to the operator H, so we need to verify all the conditions. We define a differential operator $K : \mathbf{R} \times X \times X_3 \to Y \times Y$,

$$\begin{aligned} K[\tau, v, \psi] &= H_{(\lambda,u,w)}(\varepsilon_0, \lambda_0, u_0, w_0)[\tau, v, \psi] \\ &= \begin{pmatrix} \tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] \\ \tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] \end{pmatrix}. \end{aligned} \tag{3.10}$$

(1) $\dim N(K) = 2$. Suppose that $(\tau, v, \psi) \in N(K)$ and $(\tau, v, \psi) \neq 0$. If $\tau = 0$, from $F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0$ and $(F1)$, then we have $v = k w_0$ and

$$k F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = 0. \tag{3.11}$$

From $(F4')$, we can define $\psi_0 \in X_3$ is the unique solution of (3.8). Thus, $(0, w_0, \psi_0) \in N(K)$ and $(\tau, v, \psi) = k(0, w_0, \psi_0)$.

Next, we consider $\tau \neq 0$. Without loss of generality, we assume that $\tau = 1$. Notice that $F_\lambda(\varepsilon_0, \lambda_0, u_0) \in R(F_u(\varepsilon_0, \lambda_0, u_0))$ from $(F2')$, we can define that $v_1 \in X_3 \setminus \{0\}$ is unique solution of (3.5). Substituting $\tau = 1$, $v = v_1$ into (3.10), we have

$$F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_1, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = 0. \tag{3.12}$$

From (3.4), there exists a unique $\psi_1 \in X_3$ satisfies (3.9). Then,

$$N(K) = \operatorname{span}\{(0, w_0, \psi_0), (1, v_1, \psi_1)\}, \tag{3.13}$$

that is, $\dim N(K) = 2$.

(2) $\operatorname{codim} R(K) = 1$. We only claim that

$$R(K) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y. \tag{3.14}$$

Let $(h, g) \in R(K)$ and $(\tau, v, \psi) \in \mathbf{R} \times X \times X_3$ satisfy

$$\tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] = h, \tag{3.15}$$

$$\tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = g. \tag{3.16}$$

Using (3.15) and $(F2')$, then $(h, g) \in R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$ and $R(K) \subset R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$.

Conversely, for any $(h, g) \in R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$, from $(F3)$, set

$$\tau_1 = \frac{\langle l, g\rangle}{\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle}, \tag{3.17}$$

where $l \in R(F_u(\varepsilon_0, \lambda_0, u_0))^\perp \subset Y^*$. From $(F2')$, we have

$$h - \tau_1 F_\lambda(\varepsilon_0, \lambda_0, u_0) \in R(F_u(\varepsilon_0, \lambda_0, u_0)). \tag{3.18}$$

Set $v_2 = [F_u|_{X_3}]^{-1}[h - \tau_1 F_\lambda(\varepsilon_0, \lambda_0, u_0)] \in X_3$, we obtain that

$$\tau_1 F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v_2] = h. \tag{3.19}$$

Substituting $\tau = \tau_1$, $v = v_2$ into (3.16), we have

$$\tau_1 F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_2, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = g. \tag{3.20}$$

Using $(F1)$, $(F3)$, then there exists $v_3 \in X_3$, $\tau_2 \in \mathbf{R}$ satisfies

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_2, w_0] = \tau_2 F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[v_3]. \tag{3.21}$$

Substituting (3.21) into (3.20), we have

$$(\tau_1 + \tau_2) F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi + v_3] = g. \tag{3.22}$$

Applying l to (3.22), we have $\tau_2 = 0$ because of the definition of τ_1 and

$$g - \tau_1 F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \in R(F_u(\varepsilon_0, \lambda_0, u_0)). \tag{3.23}$$

Thus we can define

$$\psi_2 = \left[F_u|_{X_3}\right]^{-1}\{g - \tau_1 F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\} - v_3 \in X_3. \tag{3.24}$$

Therefore, $K(\tau_1, v_2, \psi_2) = (h, g)$, that is, $R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y \subset R(K)$. Hence, $R(K) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$. That is, codim $R(K) = 1$.

(3) $H_\varepsilon(\varepsilon_0, \lambda_0, u_0, w_0) \notin R(K)$. Since $R(K) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$, we need only to show that $F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$ but that is exactly assumed in $(F5)$. So, the statement of the theorem follows from Theorem 2.4. □

4. Calculations of Bifurcation Directions

In Theorem 3.1, we have $\varepsilon_1(0) = \varepsilon_2(0) = \varepsilon_0$, $\varepsilon_1'(0) = \varepsilon_2'(0) = 0$, $\lambda_1(0) = \lambda_2(0) = \lambda_0$, $u_1(0) = u_2(0) = u_0$, $w_1(0) = w_2(0) = w_0$, $\lambda_1'(0) = 0$, $u_1'(0) = w_0$, $w_1'(0) = \psi_0$, $\lambda_2'(0) = 1$, $u_2'(0) = v_1$, $w_2'(0) = \psi_1$.

To completely determine the turning direction of curve of degenerate solutions, we need some calculations.

Let $\{T_i(s) = (\varepsilon_i(s), \lambda_i(s), u_i(s), w_i(s)) : s \in (-\delta, \delta)\}$ be a curve of degenerate solutions which we obtain in Theorem 3.1. Differentiating $H(\varepsilon_i(s), \lambda_i(s), u_i(s), w_i(s)) = 0$ with respect to s, we obtain

$$\begin{gathered} F_\varepsilon \varepsilon_i'(s) + F_\lambda \lambda_i'(s) + F_u\left[u_i'(s)\right] = 0, \\ F_{\varepsilon u}[w_i(s)]\varepsilon_i'(s) + F_{\lambda u}[w_i(s)]\lambda_i'(s) + F_{uu}\left[w_i(s), u_i'(s)\right] + F_u\left[w_i'(s)\right] = 0. \end{gathered} \tag{4.1}$$

Setting $s = 0$ in (4.1), we get exactly $F_u[w_0] = 0$, (3.5), (3.8), and (3.9). We differentiate (4.1) again, and we have (omit the subscript i in the equation)

$$\begin{aligned} &F_{\varepsilon\varepsilon}\left[\varepsilon'(s)\right]^2 + F_\varepsilon \varepsilon''(s) + F_{\lambda\lambda}\left[\lambda'(s)\right]^2 + F_\lambda \lambda''(s) + F_{uu}\left[u'(s), u'(s)\right] \\ &\quad + F_u\left[u''(s)\right] + 2F_{\varepsilon\lambda}\varepsilon'(s)\lambda'(s) + 2F_{\varepsilon u}\left[u'(s)\right]\varepsilon'(s) + 2F_{\lambda u}\left[u'(s)\right]\lambda'(s) = 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &F_{\varepsilon\varepsilon u}[w(s)]\left[\varepsilon'(s)\right]^2 + F_{\varepsilon\lambda u}[w(s)]\varepsilon'(s)\lambda'(s) + F_{\varepsilon uu}\left[u'(s), w(s)\right]\varepsilon'(s) \\ &\quad + F_{\varepsilon u}\left[w'(s)\right]\varepsilon'(s) + F_{\varepsilon u}[w(s)]\varepsilon''(s) + F_{\varepsilon\lambda u}[w(s)]\varepsilon'(s)\lambda'(s) \\ &\quad + F_{\lambda\lambda u}[w(s)]\left[\lambda'(s)\right]^2 + F_{\lambda uu}\left[u'(s), w(s)\right]\lambda'(s) + F_{\lambda u}\left[w'(s)\right]\lambda'(s) \\ &\quad + F_{\lambda u}[w(s)]\lambda''(s) + F_{\varepsilon uu}\left[u'(s), w(s)\right]\varepsilon'(s) + F_{\lambda uu}\left[u'(s), w(s)\right]\lambda'(s) \\ &\quad + F_{uuu}\left[u'(s), u'(s), w(s)\right] + F_{uu}\left[w'(s), u'(s)\right] \\ &\quad + F_{uu}\left[w(s), u''(s)\right] + F_{\varepsilon u}\left[w'(s)\right]\varepsilon'(s) + F_{\lambda u}\left[w'(s)\right]\lambda'(s) \\ &\quad + F_{uu}\left[w'(s), u'(s)\right] + F_u\left[w''(s)\right] = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned}
&F_{\varepsilon\varepsilon u}[w(s)][\varepsilon'(s)]^2 + F_{\varepsilon u}[w(s)]\varepsilon''(s) + F_{\lambda u}[w(s)]\lambda''(s) \\
&\quad + F_{\lambda\lambda u}[w(s)][\lambda'(s)]^2 + F_{uuu}[u'(s), u'(s), w(s)] + F_{uu}[w(s), u''(s)] \\
&\quad + F_u[w''(s)] + 2F_{\varepsilon\lambda u}[w(s)]\varepsilon'(s)\lambda'(s) + 2F_{\varepsilon uu}[u'(s), w(s)]\varepsilon'(s) \\
&\quad + 2F_{\lambda uu}[u'(s), w(s)]\lambda'(s) + 2F_{\varepsilon u}[w'(s)]\varepsilon'(s) + 2F_{\lambda u}[w'(s)]\lambda'(s) \\
&\quad + 2F_{uu}[w'(s), u'(s)] = 0.
\end{aligned} \tag{4.4}$$

Setting $s = 0$ in (4.2), we obtain

$$F_\varepsilon \varepsilon_1''(0) + F_\lambda \lambda_1''(0) + F_{uu}[w_0, w_0] + F_u[u_1''(0)] = 0, \tag{4.5}$$

$$F_\varepsilon \varepsilon_2''(0) + F_{\lambda\lambda} + F_\lambda \lambda_2''(0) + F_{uu}[v_1, v_1] + F_u[u_2''(0)] + 2F_{\lambda u}[v_1] = 0. \tag{4.6}$$

And applying l to it, we have

$$\varepsilon_1''(0) = 0, \tag{4.7}$$

$$\varepsilon_2''(0) = -\frac{\langle l, F_{\lambda\lambda} + F_{uu}[v_1, v_1] + 2F_{\lambda u}[v_1]\rangle}{\langle l, F_\varepsilon\rangle}, \tag{4.8}$$

Using $(F2')$, $(F4')$, $(F5)$. From (4.7), (4.5) implies $u_1''(0) = \lambda_1''(0)v_1 + \psi_0 + kw_0$. Setting $s = 0$ in (4.4),

$$F_{\lambda u}[w_0]\lambda_1''(0) + F_{uuu}[w_0, w_0, w_0] + F_{uu}[w_0, u_1''(0)] + F_u[w_1''(0)] + 2F_{uu}[\psi_0, w_0] = 0, \tag{4.9}$$

$$\begin{aligned}
&F_{\varepsilon u}[w_0]\varepsilon_2''(0) + F_{\lambda u}[w_0]\lambda_2''(0) + F_{\lambda\lambda u}[w_0] + F_{uuu}[v_1, v_1, w_0] + F_{uu}[w_0, u_2''(0)] \\
&\quad + F_u[w_2''(0)] + 2F_{\lambda uu}[v_1, w_0] + 2F_{\lambda u}[\psi_1] + 2F_{uu}[\psi_1, v_1] = 0.
\end{aligned} \tag{4.10}$$

Substituting the expression of $u_1''(0)$ into (4.9), we have

$$\begin{aligned}
&\lambda_1''(0)(F_{\lambda u}[w_0] + F_{uu}[v_1, w_0]) + 3F_{uu}[\psi_0, w_0] + F_{uuu}[w_0, w_0, w_0] \\
&\quad + kF_{uu}[w_0, w_0] + F_u[w_1''(0)] = 0.
\end{aligned} \tag{4.11}$$

And applying l to it, we obtain $\langle l, F_{uuu}[w_0, w_0, w_0] + 3F_{uu}[\psi_0, w_0]\rangle = 0$, that is,

$$F_{uuu}[w_0, w_0, w_0] + 3F_{uu}[\psi_0, w_0] \in R(F_u(\varepsilon_0, \lambda_0, u_0)). \tag{4.12}$$

Assume $F_{\varepsilon u}[w_0] \in R(F_u(\varepsilon_0, \lambda_0, u_0))$ and applying l to (4.10),

$$\lambda_2''(0) = -\frac{\langle l, F_{\lambda\lambda u}[w_0] + F_{uuu}[v_1, v_1, w_0] + F_{uu}[w_0, u_2''(0)] + 2F_{\lambda uu}[v_1, w_0] + 2F_{\lambda u}[\psi_1] + 2F_{uu}[\psi_1, v_1]\rangle}{\langle l, F_{\lambda u}[w_0]\rangle}. \tag{4.13}$$

We differentiate (4.2) again:

$$\begin{aligned} &F_{\varepsilon\varepsilon\varepsilon}[\varepsilon'(s)]^3 + F_{\varepsilon}\varepsilon'''(s) + F_{\lambda\lambda\lambda}[\lambda'(s)]^3 + F_{\lambda}\lambda'''(s) + F_{uuu}[u'(s), u'(s), u'(s)] \\ &\quad + F_u[u'''(s)] + 3F_{\varepsilon\varepsilon}\varepsilon'(s)\varepsilon''(s) + 3F_{\lambda\lambda}\lambda'(s)\lambda''(s) + 3F_{uu}[u''(s), u'(s)] \\ &\quad + 3F_{\varepsilon\lambda}\varepsilon''(s)\lambda'(s) + 3F_{\varepsilon\lambda}\varepsilon'(s)\lambda''(s) + 3F_{\varepsilon u}[u'(s)]\varepsilon''(s) + 3F_{\varepsilon u}[u''(s)]\varepsilon'(s) \\ &\quad + 3F_{\lambda u}[u''(s)]\lambda'(s) + 3F_{\lambda u}[u'(s)]\lambda''(s) + 3F_{\varepsilon\lambda\lambda}\varepsilon'(s)(\lambda'(s))^2 \\ &\quad + 3F_{\varepsilon\varepsilon\lambda}(\varepsilon'(s))^2\lambda'(s) + 3F_{\varepsilon\varepsilon u}[u'(s)](\varepsilon'(s))^2 + 3F_{\varepsilon uu}[u'(s), u'(s)]\varepsilon'(s) \\ &\quad + 3F_{\lambda\lambda u}[u'(s)](\lambda'(s))^2 + 3F_{\lambda uu}[u'(s), u'(s)]\lambda'(s) + 6F_{\varepsilon\lambda u}[u'(s)]\varepsilon'(s)\lambda'(s) = 0. \end{aligned} \tag{4.14}$$

Setting $s = 0$ in (4.14), we obtain

$$\begin{aligned} &F_{\varepsilon}\varepsilon_1'''(0) + F_{\lambda}\lambda_1'''(0) + F_{uuu}[w_0, w_0, w_0] + Fu[u_1'''(0)] \\ &\quad + 3F_{uu}[u_1''(0), w_0] + 3F_{\lambda u}[w_0]\lambda_1''(0) = 0, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &F_{\lambda}\lambda_2'''(0) + F_{\varepsilon}\varepsilon_2'''(0) + 3\lambda_2''(0)(F_{\lambda\lambda} + F_{\lambda u}[v_1]) + 3\varepsilon_2''(0)(F_{\varepsilon\lambda} + F_{\varepsilon u}[v_1]) \\ &\quad + F_{\lambda\lambda\lambda} + F_{uuu}[v_1, v_1, v_1] + F_u[u_2'''(0)] + 3F_{uu}[u_2''(0), v_1] + 3F_{\lambda u}[u_2''(0)] \\ &\quad + 3F_{\lambda\lambda u}[v_1] + 3F_{\lambda uu}[v_1, v_1] = 0. \end{aligned} \tag{4.16}$$

Substituting the expression of $u_1''(0)$ into (4.15) and applying l to it, we have $\varepsilon_1'''(0) = 0$ using (3.4), (4.12), and ($F5$).

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References

[1] M. G. Crandall and P. H. Rabinowitz, “Bifurcation from simple eigenvalues,” *Journal of Functional Analysis*, vol. 8, pp. 321–340, 1971.

[2] M. G. Crandall and P. H. Rabinowitz, “Bifurcation, perturbation of simple eigenvalues and linearized stability,” *Archive for Rational Mechanics and Analysis*, vol. 52, pp. 161–180, 1973.

[3] P. Liu and Y.-W. Wang, "The generalized saddle-node bifurcation of degenerate solution," *Polskiego Towarzystwa Matematycznego*, vol. 45, no. 2, pp. 145–150, 2005.

[4] P. Liu, J. Shi, and Y. Wang, "Imperfect transcritical and pitchfork bifurcations," *Journal of Functional Analysis*, vol. 251, no. 2, pp. 573–600, 2007.

[5] K.-C. Chang, *Methods in Nonlinear Analysis*, Springer Monographs in Mathematics, Springer, Berlin, Germany, 2005.

[6] H. Kielhöfer, *Bifurcation Theory: An Introduction with Applications to PDEs*, vol. 156 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2004.

[7] J. López-Gómez, *Spectral Theory and Nonlinear Functional Analysis*, vol. 426 of *Chapman & Hall/CRC Research Notes in Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2001.

[8] C. Pötzsche, "Nonautonomous bifurcation of bounded solutions I: a Lyapunov-Schmidt approach," *Discrete and Continuous Dynamical Systems B*, vol. 14, no. 2, pp. 739–776, 2010.

[9] C. Pötzsche, "Persistence and imperfection of nonautonomous bifurcation patterns," *Journal of Differential Equations*, vol. 250, no. 10, pp. 3874–3906, 2011.

[10] J. Jang, W.-M. Ni, and M. Tang, "Global bifurcation and structure of Turing patterns in the 1-D Lengyel-Epstein model," *Journal of Dynamics and Differential Equations*, vol. 16, no. 2, pp. 297–320, 2004.

[11] P. Gray and S. K. Scott, *Chemical Oscillations and Instabilities: Nonlinear Chemical Kinetics*, Proceedings in Nonlinear Science, Clarendon Press, Oxford, UK, 1990.

[12] J. Shi, "Persistence and bifurcation of degenerate solutions," *Journal of Functional Analysis*, vol. 169, no. 2, pp. 494–531, 1999.

[13] M. Z. Nashed, Ed., *Generalized Inverse and Applications*, Academic Press, New York, NY, USA, 1976.