Research Article

Mean-Variance Hedging and Forward-Backward Stochastic Differential Filtering Equations

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This paper is concerned with a mean-variance hedging problem with partial information, where the initial endowment of an agent may be a decision and the contingent claim is a random variable. This problem is explicitly solved by studying a linear-quadratic optimal control problem with non-Markov control systems and partial information. Then, we use the result as well as filtering to solve some examples in stochastic control and finance. Also, we establish *backward* and *forward-backward* stochastic differential filtering equations which are *different* from the classical filtering theory introduced by Liptser and Shiryayev (1977), Xiong (2008), and so forth.

1. Introduction and Problem Formulation

We begin with a finite time horizon $[0,T]$ for $T > 0$, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which an $\mathbf{R}^m$-valued standard Brownian motion $(W(\cdot))$ is defined. Moreover, we let the natural filtration $\mathcal{F}_t = \sigma\{W(s); 0 \le s \le t\}$, $0 \le t \le T$, and $\mathcal{F} = \mathcal{F}_T$.

Suppose there is a financial market in which $m + 1$ securities can be continuously traded. One of them is a bond whose price $B(\cdot)$ satisfies

$$dB(t) = r(t)B(t)dt, \tag{1.1}$$

where $r(t)$ is the interest rate of the bond at time t. The other m assets are stocks whose dynamicses are subject to the following stochastic differential equations (SDEs):

$$dS_i(t) = \mu_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dW_i(t), \tag{1.2}$$

where $\mu_i(t)$ and $\sigma_i(t)$ are called the appreciation rate of return and volatility coefficient of the ith stock.

Suppose there is an agent who invests in the bond and stocks, whose decision cannot influence the prices in the financial market. We assume that the trading of the agent is self-financed, that is, there is no infusion or withdrawal of funds over $[0, T]$. We denote by $\pi_i(t)$ the amount that the agent invests in the ith stock and by $x^\pi(t)$ the wealth of the agent with an initial endowment $x_0 > 0$. Then the agent has $x^\pi(t) - \sum_{i=1}^{m} \pi_i(t)$ savings in a bank. Under the forgoing notations and interpretations, the wealth $x^\pi(\cdot)$ is modeled by

$$
dx^\pi(t) = \left[r(t)x^\pi(t) + \sum_{i=1}^{m}(\mu_i(t) - r(t))\pi_i(t) \right] dt + \sum_{i=1}^{m} \sigma_i(t)\pi_i(t)dW_i(t),
$$

$$
x^\pi(0) = x_0.
$$

(1.3)

Generally speaking, it is impossible for the agent to know all the events occurred in the financial market. For instance, if the agent has not enough time or great vigor to observe all the prices of the $m + 1$ assets, then the agent will only observe some data of all the prices. Without loss of generality, we denote by $\mathscr{Z}_t$ the information available to the agent at time t, which is a subfiltration of $\mathscr{F}_t$. Suppose a process only adapted to $\mathscr{Z}_t$ is called observable. Therefore, the agent has to choose a portfolio strategy according to the observable filtration $\mathscr{Z}_t$. A portfolio strategy $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_m(\cdot))$ is called admissible if $\pi_i(t)$ is a $\mathscr{Z}_t$-adapted, square-integrable process with values in $\mathbf{R}$. The set of the admissible portfolio strategies is denoted by $\mathscr{U}_{ad}$.

We give the following hypothesis.

(H1) *The coefficients $r(\cdot)$, $\mu_i(\cdot)$, $\sigma_i(\cdot)$, and $\sigma_i(\cdot)^{-1}$ are uniformly bounded and deterministic functions with values in $\mathbf{R}$.*

For any $\pi(\cdot) \in \mathscr{U}_{ad}$, (1.3) admits a unique solution under Hypothesis (H1). If we define $v_i(\cdot) = \sigma_i(\cdot)\pi_i(\cdot)$, then (1.3) is rewritten as

$$
dx^v(t) = \left[r(t)x^v(t) + \sum_{i=1}^{m} \frac{\mu_i(t) - r(t)}{\sigma_i(t)} v_i(t) \right] dt + \sum_{i=1}^{m} v_i(t)dW_i(t),
$$

$$
x^v(0) = x_0.
$$

(1.4)

Let $\xi > 0$ be a given contingent claim, which is a $\mathscr{Z}_T$-measurable, square-integrable random variable. Furthermore, we suppose ξ is larger than or equal to $x_0 e^{\int_0^T r(t)dt}$, where the value $x_0 e^{\int_0^T r(t)dt}$ coincides with the amount that the agent would earn when the initial wealth x_0 was invested in the bond at the interest rate $r(\cdot)$ for the entire investment period.

Define a cost functional

$$
J(v(\cdot); x_0) = \frac{1}{2}\mathbb{E}|x^v(T) - \xi|^2.
$$

(1.5)

Note that the above ξ can contain $\mathbb{E}[x^v(T) \mid \mathscr{Z}_T]$ as a special case. For a priori given initial wealth x_0, (1.5) measures the risk that the contingent claim ξ cannot be reached. The agent's

objective is

$$\min \quad J(v(\cdot); x_0)$$

$$\text{subject to} \quad v(\cdot) \in \mathcal{U}_{\text{ad}}, \tag{PIMV}$$

$$(x^v(\cdot); v(\cdot)) \text{ satisfies (1.3) or (1.4)}.$$

The above problem formulates a mean-variance hedging problem with partial information. For simplicity, hereinafter we denote it by the notation "Problem (PIMV)", short for the "partial information mean-variance hedging problem". In particular, if we let $\mathcal{F}_t = \mathcal{Z}_t$, $0 \le t \le T$, then Problem (PIMV) reduces to the case with complete information. See, for example, Kohlmann and Zhou [1] for more details.

Because the contingent claim ξ in (1.5) is random and the initial endowment x_0 in (1.3) may be a decision, our Problem (PIMV) is distinguished from the existing literature. See, for example, Pham [2], Xiong and Zhou [3], Hu and Øksendal [4], and so forth. Motivated by Problem (PIMV), we study a general linear-quadratic (LQ) optimal control problem with partial information in Section 2. By a combination of the martingale representation theorem, the technique of "completing the square", and conditional expectation, we derive a corresponding optimal control which is denoted by a related optimal state equation, a Riccati differential equation and a backward stochastic differential equation (BSDE). To demonstrate the applications of our results, we work out some partial information LQ examples and obtain some explicitly observable optimal controls by filtering for BSDEs. Also, we establish some backward and forward-backward stochastic differential filtering equations which are different from the classical ones.

In Section 3, we use the result established in Section 2 to derive an optimal portfolio strategy of Problem (PIMV), which is denoted by the sum of a replicating portfolio strategy for the contingent claim ξ and a Merton's portfolio strategy. To explicitly illustrate Problem (PIMV), we provide a special but nontrivial example in this section. In terms of filtering theory, we derive the corresponding risk measure. Furthermore, we use some numerical simulations and three figures to illustrate the risk measure and the optimal portfolio strategy.

In Section 4, we compare our results with the existing ones.

Finally, for the convenience of the reader, we state a classical filtering equation for SDEs which is used in Section 3 of this paper.

2. An LQ Optimal Control Problem with Partial Information

In this section, we study a partial information LQ optimal control problem, which is a generalization of Problem (PIMV).

Let us now begin to formulate the LQ problem. Consider a stochastic control system

$$dx^v(t) = \left[A(t)x^v(t) + \sum_{i=1}^{m} B_i(t)v_i(t) + g(t) \right] dt + \sum_{i=1}^{m} v_i(t) dW_i(t), \tag{2.1}$$

$$x^v(0) = x_0.$$

Here $x^v(t)$, x_0, $v_i(t)$, $g(t) \in \mathbf{R}^n$, $A(t)$ and $B_i(t) \in \mathbf{R}^{n \times n}$; $v(\cdot) = (v_1(\cdot), \ldots, v_m(\cdot))$ is a control (process) with values in $\mathbf{R}^{n \times m}$. We suppose $v(t)$ is $\mathcal{Z}_t$-adapted, where $\mathcal{Z}_t$ is a given

subfiltration of $\mathcal{F}_t$ representing the information available to a policymaker at time t. We say that the control $v(\cdot)$ is admissible and write $v(\cdot) \in \mathcal{U}_{\mathrm{ad}}$ if $v(\cdot) \in L^2_{\mathcal{Z}}(0,T;\mathbf{R}^{n\times m})$, that is, $v(t)$ is a $\mathcal{Z}_t$-adapted process with values in $\mathbf{R}^{n\times m}$ and satisfies

$$\mathbb{E}\int_0^T |v(t)|^2 \mathrm{d}t < +\infty. \tag{2.2}$$

The following basic hypothesis will be in force throughout this section.

(H2) $A(\cdot)$, $B_i(\cdot)$ *are uniformly bounded and deterministic functions*, x_0 *is* $\mathcal{F}_0$-*adapted, and* $g(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^n)$.

For any $v(\cdot) \in \mathcal{U}_{\mathrm{ad}}$, control system (2.1) admits a unique solution under Hypothesis (H2). The associated cost functional is

$$J(v(\cdot);x_0) = \frac{1}{2}\mathbb{E}|x^v(T) - \xi|^2, \tag{2.3}$$

where ξ is a given $\mathcal{F}_T$-measurable, square-integrable random variable. The LQ optimal control problem with partial information is

$$\begin{aligned}
\min \quad & J(v(\cdot);x_0) \\
\text{subject to} \quad & v(\cdot) \in \mathcal{U}_{\mathrm{ad}}, \\
& (x^v(\cdot);v(\cdot)) \text{ satisfies (2.1).}
\end{aligned} \tag{PILQ}$$

An admissible control $u(\cdot)$ is called optimal if it satisfies

$$J(u(\cdot);x_0) = \min_{v(\cdot)\in\mathcal{U}_{\mathrm{ad}}} J(v(\cdot);x_0). \tag{2.4}$$

The solution $x(\cdot)$ and cost functional (2.3) along with $u(\cdot)$ are called the optimal state and the value function, respectively.

Problem (PILQ) is related to the recent work by Hu and Øksendal [4], where an LQ control for jump diffusions with partial information is investigated. Due to some characteristic setup, our Problem (PILQ) is not covered by [4]. See, for example, Section 4 in this paper for some detailed comments. Since the nonhomogeneous term in the drift of (2.1) is random and the observable filtration $\mathcal{Z}_t$ is very general, it is not easy to solve Problem (PILQ). To overcome the resulting difficulty, we shall adopt a combination method of the martingale representation theorem, the technique of "completing the square", and conditional expectation. This method is inspired by Kohlmann and Zhou [1], where an LQ control problem with complete information is studied.

To simplify the cost functional (2.3), we define

$$y^v(t) = x^v(t) - \mathbb{E}[\xi \mid \mathcal{F}_t]. \tag{2.5}$$

Since $\mathbb{E}[\xi \mid \mathcal{F}_t]$ is an $\mathcal{F}_t$-martingale, by the martingale representation theorem (see e.g., Liptser and Shiryasyev [5]), there is a unique $z_i(\cdot) \in L^2_{\mathcal{F}}(0, T; R^n)$ such that

$$\mathbb{E}[\xi \mid \mathcal{F}_t] = \mathbb{E}\xi + \sum_{i=1}^{m} \int_0^t z_i(s) \mathrm{d}W_i(s). \tag{2.6}$$

Applying Itô's formula to (2.1) and (2.5)-(2.6), we have

$$\mathrm{d}y^v(t) = \left[A(t)y^v(t) + \sum_{i=1}^{m} B_i(t)v_i(t) + h(t) \right] \mathrm{d}t + \sum_{i=1}^{m} (v_i(t) - z_i(t)) \mathrm{d}W_i(t),$$
$$y^v(0) = y_0 = x_0 - \mathbb{E}\xi \tag{2.7}$$

with

$$h(t) = g(t) + A(t)\mathbb{E}[\xi \mid \mathcal{F}_t], \tag{2.8}$$

and cost functional (2.3) reduces to

$$J(v(\cdot); y_0) = \frac{1}{2} \mathbb{E} |y^v(T)|^2. \tag{2.9}$$

Then Problem (PILQ) is equivalent to minimize (2.9) subject to (2.6)-(2.7) and $\mathcal{U}_{\mathrm{ad}}$. To solve the resulting problem, we first introduce a Riccati differential equation on $\mathbf{R}^{n \times n}$

$$\dot{P}(t) + P(t)A(t) + A(t)^{\tau}P(t) - \sum_{i=1}^{m} P(t)B_i(t)P(t)^{-1}B_i(t)^{\tau}P(t) = 0,$$
$$P(T) = I, \tag{2.10}$$
$$P(t) > 0, \quad 0 \le t \le T.$$

Note that (2.7) contains a nonhomogeneous term $h(\cdot)$. For this, we also introduce a BSDE on $\mathbf{R}^n$,

$$-\mathrm{d}\alpha(t) = \left\{ \left[A(t)^{\tau} - \sum_{i=1}^{m} P(t)B_i(t)P(t)^{-1}B_i(t)^{\tau} \right] \alpha(t) - \sum_{i=1}^{m} P(t)B_i(t)P(t)^{-1}\beta_i(t) \right.$$
$$\left. + P(t) \left(h(t) + \sum_{i=1}^{m} B_i(t)z_i(t) \right) \right\} \mathrm{d}t - \sum_{i=1}^{m} \beta_i(t)\mathrm{d}W_i(t), \tag{2.11}$$
$$\alpha(T) = 0.$$

Assume that the following hypothesis holds.

(H3) *For any* $0 \leq t \leq T$,

$$A(t) + A(t)^\tau \geq \sum_{i=1}^{m} B_i(t) B_i(t)^\tau. \tag{2.12}$$

Under Hypotheses (H2) and (H3), according to [1, Theorem 4.2], it is easy to see that (2.10) admits a unique solution, and then (2.11) admits a unique $\mathcal{F}_t$-adapted solution $(\alpha(\cdot), \beta_1(\cdot), \ldots, \beta_m(\cdot))$.

For any admissible pair $(v(\cdot), y^v(\cdot))$, using Itô's formula to $(1/2) y^v(\cdot)^\tau P(\cdot) y^v(\cdot) + \alpha(\cdot)^\tau y^v(\cdot)$, integrating from 0 to T, taking the expectations and trying to complete a square, then we have

$$
\begin{aligned}
J(v(\cdot); y_0) &= J_{\mathcal{F}}(y_0) + \frac{1}{2}\mathbb{E} \int_0^T \sum_{i=1}^m (v_i(t) + L_i^v(t))^\tau P(t) (v_i(t) + L_i^v(t)) \mathrm{d}t \\
&= J_{\mathcal{F}}(y_0) + \frac{1}{2}\mathbb{E} \int_0^T \sum_{i=1}^m \mathbb{E}[(v_i(t) + L_i^v(t))^\tau P(t) (v_i(t) + L_i^v(t)) \mid \mathcal{Z}_t] \mathrm{d}t,
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
J_{\mathcal{F}}(y_0) &= \frac{1}{2} y_0^\tau P(0) y_0 + y_0^\tau \alpha(0) \\
&\quad + \frac{1}{2}\mathbb{E} \int_0^T \Bigg[2\alpha(t)^\tau h(t) - 2\sum_{i=1}^m \beta_i(t) z_i(t) + \sum_{i=1}^m z_i(t)^\tau P(t) z_i(t) \\
&\qquad\qquad - \sum_{i=1}^m \left(P(t)^{-1} B_i(t)^\tau \alpha(t) + P(t)^{-1} \beta_i(t) - z_i(t) \right)^\tau \\
&\qquad\qquad \times P(t) \left(P(t)^{-1} B_i(t)^\tau \alpha(t) + P(t)^{-1} \beta_i(t) - z_i(t) \right) \Bigg] \mathrm{d}t,
\end{aligned}
\tag{2.14}
$$

$$L_i^v(t) = P(t)^{-1} B_i(t)^\tau P(t) y^v(t) + P(t)^{-1} B_i(t)^\tau \alpha(t) + P(t)^{-1} \beta_i(t) - z_i(t). \tag{2.15}$$

Since $J_{\mathcal{F}}(y_0)$ is independent of $v_i(\cdot)$, the integrand in (2.13) is quadratic with respect to $v_i(\cdot)$ and $P(\cdot) > 0$, then it follows from the property of conditional expectation that the minimum of

$$\frac{1}{2}\mathbb{E} \int_0^T \sum_{i=1}^m \mathbb{E}[(v_i(t) + L_i^v(t))^\tau P(t) (v_i(t) + L_i^v(t)) \mid \mathcal{Z}_t] \mathrm{d}t \tag{2.16}$$

over all $\mathcal{Z}_t$-adapted $v_i(t)$ is attained at

$$
\begin{aligned}
u_i(t) &= -\mathbb{E}[L_i(t) \mid \mathcal{Z}_t] \\
&= -P(t)^{-1} B_i(t)^\tau \{ P(t)\mathbb{E}[y(t) \mid \mathcal{Z}_t] + \mathbb{E}[\alpha(t) \mid \mathcal{Z}_t] \} \\
&\quad - P(t)^{-1}\mathbb{E}[\beta_i(t) \mid \mathcal{Z}_t] + \mathbb{E}[z_i(t) \mid \mathcal{Z}_t],
\end{aligned}
\tag{2.17}
$$

where

$$L_i(t) = P(t)^{-1} B_i(t)^\tau P(t) y(t) + P(t)^{-1} B_i(t)^\tau \alpha(t) + P(t)^{-1} \beta_i(t) - z_i(t) \tag{2.18}$$

and $y(\cdot)$ is the solution of the SDE with $u_i(\cdot)$ replaced by (2.17)

$$dy(t) = \left[A(t)y(t) + \sum_{i=1}^{m} B_i(t)u_i(t) + h(t) \right] dt + \sum_{i=1}^{m} (u_i(t) - z_i(t)) dW_i(t),$$
$$y(0) = y_0 = x_0 - \mathbb{E}\xi. \tag{2.19}$$

Now, we are in the position to derive an optimal feedback control in terms of the original optimal state variable $x(\cdot)$. Substituting (2.5) into (2.17), we get

$$u_i(t) = -\mathbb{E}[L_i(t) \mid \mathscr{Z}_t]$$
$$= -P(t)^{-1} B_i(t)^\tau \{ P(t)\mathbb{E}[x(t) \mid \mathscr{Z}_t] - P(t)\mathbb{E}[\xi \mid \mathscr{Z}_t] + \mathbb{E}[\alpha(t) \mid \mathscr{Z}_t] \} \tag{2.20}$$
$$- P(t)^{-1}\mathbb{E}[\beta_i(t) \mid \mathscr{Z}_t] + \mathbb{E}[z_i(t) \mid \mathscr{Z}_t],$$

where $x(\cdot)$ satisfies the SDE with $u_i(\cdot)$ replaced by (2.20)

$$dx(t) = \left[A(t)x(t) + \sum_{i=1}^{m} B_i(t)u_i(t) + g(t) \right] dt + \sum_{i=1}^{m} u_i(t) dW_i(t),$$
$$x(0) = x_0. \tag{2.21}$$

Furthermore, we define for any $0 \le t \le T$

$$p(t) = \mathbb{E}[\xi \mid \mathscr{F}_t] - P(t)^{-1}\alpha(t),$$
$$q_i(t) = z_i(t) - P(t)^{-1}\beta_i(t). \tag{2.22}$$

Applying Itô's formula to (2.6) and (2.10)-(2.11), we can check that $(p(\cdot), q_1(\cdot), \ldots, q_m(\cdot))$ is the unique solution of the BSDE

$$dp(t) = \left[A(t)p(t) + \sum_{i=1}^{m} B_i(t)q_i(t) + g(t) \right] dt + \sum_{i=1}^{m} q_i(t) dW_i(t),$$
$$p(T) = \xi. \tag{2.23}$$

Finally, substituting (2.17) into (2.13), we get the value function

$$J_{\mathcal{Z}}(y_0) = J_{\mathcal{F}}(y_0) + \frac{1}{2}\mathbb{E}\int_0^T \sum_{i=1}^m \left\{ \mathbb{E}\big[L_i(t)^\tau P(t)L_i(t) \mid \mathcal{Z}_t\big] - \mathbb{E}\big[L_i(t)^\tau \mid \mathcal{Z}_t\big] P(t)\mathbb{E}\big[L_i(t) \mid \mathcal{Z}_t\big] \right\} dt,$$

(2.24)

where $J_{\mathcal{F}}(y_0)$ and $L_i(\cdot)$ are defined by (2.14) and (2.18), respectively.

Theorem 2.1. *Let Hypotheses (H2) and (H3) hold. Then the optimal control of Problem* (PILQ) *is*

$$u_i(t) = -P(t)^{-1}B_i(t)^\tau P(t)\big\{\mathbb{E}[x(t) \mid \mathcal{Z}_t] - \mathbb{E}[p(t) \mid \mathcal{Z}_t]\big\} + \mathbb{E}[q_i(t) \mid \mathcal{Z}_t],$$

(2.25)

where $x(\cdot)$ and $(p(\cdot), q_1(\cdot), \ldots, q_m(\cdot))$ are the solutions of (2.21) *and* (2.23)*, respectively; the corresponding value function is given by* (2.24)*.*

Remark 2.2. Note that the dynamics of BSDE (2.23) is similar to control system (2.1) except for the state constraint, which shows a perfect relationship between stochastic control and BSDE. This interesting phenomenon is first found by [1], to our best knowledge. Also, [1] finds that the solution $(p(\cdot), q_1(\cdot), \ldots, q_m(\cdot))$ of (2.23) can be regarded as the optimal state-control pair $(x(\cdot), u(\cdot))$ of an LQ control problem with complete information, in which the initial state x_0 is an additional decision. That is, $p(\cdot) = x(\cdot)$ and $q_i(\cdot) = u_i(\cdot)$ with $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$. However, this conclusion is not true in our partial information case. For clarity, we shall illustrate it by the following example.

Example 2.3. Without loss of generality, we let Hypothesis (H2) hold and $n = 1$ in Problem (PILQ).

Since $P(\cdot)$ defined by (2.10) is a scalar, it is natural that (2.10) admits a unique solution. Consequently, (2.11) also admits a unique solution. Note that Hypothesis (H3) is not used in this setup. Define

$$\Delta(\cdot) = x(\cdot) - p(\cdot).$$

(2.26)

From Itô's formula, (2.21)-(2.23) and (2.25), we get

$$d\Delta(t) = \left\{ A(t)\Delta(t) - \sum_{i=1}^m B_i(t)^2 \mathbb{E}[\Delta(t) \mid \mathcal{Z}_t] + \sum_{i=1}^m B_i(t)\big(\mathbb{E}[q_i(t) \mid \mathcal{Z}_t] - q_i(t)\big) \right\} dt$$

$$+ \sum_{i=1}^m \left\{ B_i(t)\big(\mathbb{E}[q_i(t) \mid \mathcal{Z}_t] - q_i(t)\big) - B_i(t)\mathbb{E}[\Delta(t) \mid \mathcal{Z}_t] \right\} dW_i(t),$$

(2.27)

$$\Delta(0) = x_0 - p(0).$$

Hereinafter, we set

$$\widehat{Y}(t) = \mathbb{E}[Y(t) \mid \mathcal{Z}_t], \qquad Y(\cdot) = x(\cdot), p(\cdot), q_1(\cdot), q_2(\cdot), g(\cdot), X(\cdot) \text{ or } \Delta(\cdot), \quad 0 \le t \le T,$$

(2.28)

where the signal $Y(t)$ is an $\mathcal{F}_t$-adapted and square-integrable stochastic progress, while the observation is the component of the m-dimensional Brownian motion $(W(\cdot))$. Without loss of generality, we let the observable filtration $\mathcal{Z}_t$ be

$$\mathcal{Z}_t = \sigma\{(W_1(s),\ldots,W_l(s));\ 0 \le s \le t\}, \quad 0 \le t \le T,\ 1 \le l \le m - 1. \tag{2.29}$$

In this setting, we call (2.28) the optimal filtering of the signal $Y(t)$ with respect to the observable filtration $\mathcal{Z}_t$ in the sense of square error. See, for example, [5, 6] for more details.

Note that $(W_1(\cdot),\ldots,W_l(\cdot))$ is independent of $(W_{l+1}(\cdot),\ldots,W_m(\cdot))$, x_0 and $p(0)$ are deterministic. Taking the conditional expectations on both sides of (2.27), we get the optimal filtering equation of $\Delta(t)$ with respect to $\mathcal{Z}_t$

$$\mathrm{d}\widehat{\Delta}(t) = \left[A(t) - \sum_{i=1}^{m} B_i(t)^2\right]\widehat{\Delta}(t)\mathrm{d}t - \sum_{i=1}^{l} B_i(t)\widehat{\Delta}(t)\mathrm{d}W_i(t),$$

$$\widehat{\Delta}(0) = x_0 - p(0). \tag{2.30}$$

Note that $\widehat{\Delta}(\cdot)$ satisfies a homogeneous linear SDE and hence must be identically zero if $x_0 = p(0)$.

Thereby, if the decision x_0 takes the value $p(0)$ in Example 2.3, then the next corollary follows from Theorem 2.1.

Corollary 2.4. *The optimal control of Example 2.3 is*

$$u_i(t) = \mathbb{E}\big[q_i(t) \mid \mathcal{Z}_t\big], \quad 0 \le t \le T. \tag{2.31}$$

In particular, if $\mathcal{Z}_t = \mathcal{F}_t$, $0 \le t \le T$, then it reduces to the case of [1], that is, $u_i(\cdot) = q_i(\cdot)$.

From Theorem 2.1 and Corollary 2.4, we notice that the optimal control strongly depends on the conditional expectation of $(p(t), q_1(t), \ldots, q_m(t))$ with respect to $\mathcal{Z}_t$, $0 \le t \le T$, where $(p(\cdot), q_1(\cdot), \ldots, q_m(\cdot))$ is the solution of BSDE (2.23). Since $\mathcal{Z}_t$ is very general, the conditional expectation is, in general, infinite dimensional. Then it is very hard to find an *explicitly* observable optimal control by some usual methods. However, it is well known that such an optimal control plays an important role in theory and reality. For this, we desire to seek some new technique to further research the problem in the rest of this section. Recently, Wang and Wu [7] investigate the filtering of BSDEs and use a backward separation technique to explicitly solve an LQ optimal control problem with partial information. Please refer to Wang and Wu [8] and Huang et al. [9] for more details about BSDEs with partial information. Inspired by [7, 9], we shall apply the filtering of BSDEs to study the conditional expectation mentioned above. Note that there is no general filtering result for BSDEs in the published literature. In the rest of this section, we shall present two examples of such filtering problems. Combining Theorem 2.1 with a property of conditional expectation, we get some explicitly observable optimal controls. As a byproduct, we establish two *new* kinds of filtering equations, which are called as *backward* and *forward-backward* stochastic differential filtering equations. The result enriches and develops the classical filtering-control theory (see e.g., Liptser and Shiryayev [5], Bensoussan [10], Xiong [6], and so on).

Example 2.5. Let Hypothesis (H2) hold, $n = 1$, and $m = 2$ in Problem (PILQ). Suppose the observable filtration is denoted by

$$\mathcal{Z}_t = \sigma\{W_1(s); \ 0 \leq s \leq t\}, \quad 0 \leq t \leq T. \tag{2.32}$$

From Theorem 2.1, the optimal control is

$$u_i(t) = -B_i(t)\{\mathbb{E}[x(t) \mid \mathcal{Z}_t] - \mathbb{E}[p(t) \mid \mathcal{Z}_t]\} + \mathbb{E}[q_i(t) \mid \mathcal{Z}_t]$$
$$= -B_i(t)[\widehat{x}(t) - \widehat{p}(t)] + \widehat{q}_i(t), \tag{2.33}$$

where $(p(\cdot), q_1(\cdot), q_2(\cdot))$ is the unique solution of

$$dp(t) = \left[A(t)p(t) + \sum_{i=1}^{2} B_i(t)q_i(t) + g(t)\right]dt + \sum_{i=1}^{2} q_i(t)dW_i(t),$$
$$p(T) = \xi \tag{2.34}$$

and $x(\cdot)$ satisfies (2.21) with $m = 2$.

Similar to Example 2.3, the optimal filtering equation of $x(\cdot)$ is

$$d\widehat{x}(t) = \left[A(t)\widehat{x}(t) + \sum_{i=1}^{2} B_i(t)u_i(t) + \widehat{g}(t)\right]dt + u_1(t)dW_1(t),$$
$$\widehat{x}(0) = x_0. \tag{2.35}$$

We proceed to calculate the optimal filtering of $(p(\cdot), q_1(\cdot), q_2(\cdot))$. Recalling BSDE (2.34) and noting that the observable filtration is $\mathcal{Z}_t$, it follows that

$$d\widehat{p}(t) = \left[A(t)\widehat{p}(t) + \sum_{i=1}^{2} B_i(t)\widehat{q}_i(t) + \widehat{g}(t)\right]dt + \widehat{q}_1(t)dW_1(t),$$
$$\widehat{p}(T) = \mathbb{E}[\xi \mid \mathcal{Z}_T]. \tag{2.36}$$

As (2.36) is a (non-Markov) BSDE, we call it a *backward* stochastic differential filtering equation which is different from the classical filtering equation for SDEs. Since $\widehat{q}_2(\cdot)$ is absent from the diffusion term in (2.36), then we are uncertain that if (2.36) admits a unique solution $(\widehat{p}(\cdot), \widehat{q}_1(\cdot), \widehat{q}_2(\cdot))$. But, we are sure that it is true in some special cases. See the following example, in which we establish a *forward-backward* stochastic differential filtering equation and obtain a unique solution of this equation.

Example 2.6. Let all the assumptions hold and $g(\cdot) \equiv 0$ in Example 2.5. For simplicity, we set the random variable $\xi = X(T)$, where $X(\cdot)$ is the solution of

$$dX(t) = K(t)X(t)dt + M_1(t)dW_1(t) + M_2(t)dW_2(t),$$
$$X(0) = X_0. \tag{2.37}$$

Assume that $K(\cdot)$, $M_1(\cdot)$, and $M_2(\cdot)$ are bounded and deterministic functions with values in $\mathbf{R}$; X_0 is a constant.

Similar to Example 2.5, the optimal control is

$$u_i(t) = -B_i(t)\left[\widehat{x}(t) - \widehat{p}(t)\right] + \widehat{q}_i(t), \tag{2.38}$$

where $(\widehat{x}(\cdot), \widehat{p}(\cdot), \widehat{q}_1(\cdot), \widehat{q}_2(\cdot), \widehat{X}(\cdot))$ is the solution of

$$d\widehat{x}(t) = \left[A(t)\widehat{x}(t) + \sum_{i=1}^{2} B_i(t)u_i(t)\right]dt + u_1(t)dW_1(t),$$
$$\widehat{x}(0) = x_0, \tag{2.39}$$

$$d\widehat{p}(t) = \left[A(t)\widehat{p}(t) + \sum_{i=1}^{2} B_i(t)\widehat{q}_i(t)\right]dt + \widehat{q}_1(t)dW_1(t),$$
$$\widehat{p}(T) = \widehat{X}(T), \tag{2.40}$$

$$d\widehat{X}(t) = K(t)\widehat{X}(t)dt + M_1(t)dW_1(t),$$
$$\widehat{X}(0) = X_0. \tag{2.41}$$

It is remarkable that (2.35) together with (2.40)-(2.41) is a *forward-backward* stochastic differential filtering equation. To our best knowledge, this is also a *new* kind of filtering equation.

We now desire to give a more explicitly observable representation of $u_i(\cdot)$. Due to the terminal condition of (2.40), we get by Itô's formula and the method of undetermined coefficients,

$$\widehat{p}(\cdot) = \Phi(\cdot)\widehat{X}(\cdot) + \Psi(\cdot),$$
$$\widehat{q}_i(\cdot) = \Phi(\cdot)M_i(\cdot). \tag{2.42}$$

Here $\widehat{X}(\cdot)$ is the solution of (2.41), and

$$\Psi(t) = -\int_t^T \sum_{i=1}^{2} e^{-\int_t^s A(r)dr}\Phi(s)B_i(s)M_i(s)ds,$$
$$\Phi(t) = e^{\int_t^T (K(s)-A(s))ds}. \tag{2.43}$$

Thus, the optimal control is

$$u_i(t) = -B_i(t)\left[\widehat{x}(t) - \Phi(t)\widehat{X}(t) - \Psi(t)\right] + \Phi(t)M_i(t), \tag{2.44}$$

where $\widehat{X}(\cdot)$ satisfies (2.41) and $\widehat{x}(\cdot)$ is the solution of

$$
\begin{aligned}
\mathrm{d}\widehat{x}(t) &= \left\{ \left[A(t) - \sum_{i=1}^{2} B_i(t)^2 \right] \widehat{x}(t) + \sum_{i=1}^{2} B_i(t) [\Phi(t) M_i(t) + B_i(t)(\Phi(t)\widehat{x}(t) + \Psi(t))] \right\} \mathrm{d}t \\
&\quad + \left\{ B_1(t) \left[\Phi(t)\widehat{X}(t) - \widehat{x}(t) + \Psi(t) \right] + \Phi(t) M_1(t) \right\} \mathrm{d}W_1(t),
\end{aligned}
\tag{2.45}
$$

$\widehat{x}(0) = x_0.$

Since $\widehat{X}(\cdot)$ is the solution of (2.41), it is easy to see that the above equation admits a unique solution $\widehat{x}(\cdot)$. Now $u_i(t)$, $0 \le t \le T$, defined by (2.44) is an explicitly observable optimal control.

Remark 2.7. BSDE theory plays an important role in many different fields. Then we usually treat some backward stochastic systems with partial information. For instance, to get an explicitly observable optimal control in Theorem 2.1, it is necessary to estimate $(p(t), q_1(t), \ldots, q_m(t))$ depending on the observable filtration $\mathcal{Z}_t$. However, there are short of some effective methods to deal with these estimates. In this situation, although the filtering of BSDEs is very restricted, it can be regarded as an alternative technique (just as we see in Examples 2.2-2.3). By the way, the study of Problem (PILQ) motivates us to establish some general filtering theory of BSDEs in future work. To our best knowledge, this is a new and unexplored research field.

3. Solution to the Problem (PIMV)

We now regard Problem (PIMV) as a special case of Problem (PILQ). Consequently, we can apply the result there to solve the Problem (PIMV). From Theorem 2.1, we get the optimal portfolio strategy

$$
\pi_i^*(t) = \pi_{i1}^*(t) + \pi_{i2}^*(t)
\tag{3.1}
$$

with

$$
\pi_{i1}^*(t) = -\frac{\mu_i(t) - r(t)}{\sigma_i(t)^2} \left\{ \mathbb{E}[x^*(t) \mid \mathcal{Z}_t] - \mathbb{E}[p(t) \mid \mathcal{Z}_t] \right\},
$$

$$
\pi_{i2}^*(t) = \frac{1}{\sigma_i(t)} \mathbb{E}[q_i(t) \mid \mathcal{Z}_t].
\tag{3.2}
$$

Here $(p(\cdot), q_1(\cdot), \ldots, q_m(\cdot))$ and $x^*(\cdot)$ are the solutions of

$$
\mathrm{d}p(t) = \left[r(t)p(t) + \sum_{i=1}^{m} \frac{\mu_i(t) - r(t)}{\sigma_i(t)} q_i(t) \right] \mathrm{d}t + \sum_{i=1}^{m} q_i(t)\mathrm{d}W_i(t),
$$

$$
p(T) = \xi,
\tag{3.3}
$$

$$
\mathrm{d}x^*(t) = \left[r(t)x^*(t) + \sum_{i=1}^{m} (\mu_i(t) - r(t))\pi_i^*(t) \right] \mathrm{d}t + \sum_{i=1}^{m} \sigma_i(t)\pi_i^*(t)\mathrm{d}W_i(t),
$$

$$
x^*(0) = x_0.
\tag{3.4}
$$

So we have the following theorem.

Theorem 3.1. *If Hypothesis (H1) holds, then the optimal portfolio strategy of Problem* (PIMV) *is given by* (3.1).

We now give a straightforward economic interpretation of (3.1). Introduce an adjoint equation

$$-\mathrm{d}\theta(s) = r(s)\theta(s)ds + \sum_{i=1}^{m} \frac{\mu_i(s) - r(s)}{\sigma_i(s)}\theta_i(s)\mathrm{d}W_i(s), \qquad \theta(t) = 1, \quad 0 \le t \le s \le T. \tag{3.5}$$

Applying Itô's formula to $p(\cdot)\theta(\cdot)$,

$$p(t) = \mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\mathbb{E}\left[\xi\mathrm{e}^{-\int_t^T \sum_{i=1}^{m}(\mu_i(s)-r(s))/(\sigma_i(s))\mathrm{d}W_i(s)-(1/2)\int_t^T \sum_{i=1}^{m}((\mu_i(s)-r(s))/\sigma_i(s))^2 ds} \mid \mathcal{F}_t\right]. \tag{3.6}$$

Note that $\mathcal{Z}_t$ is a subfiltration of $\mathcal{F}_t$, $0 \le t \le T$. Then we have

$$\mathbb{E}[p(t) \mid \mathcal{Z}_t] = \mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\mathbb{E}\left[\xi\mathrm{e}^{-\int_t^T \sum_{i=1}^{m}(\mu_i(s)-r(s))/(\sigma_i(s))\mathrm{d}W_i(s)-(1/2)\int_t^T \sum_{i=1}^{m}((\mu_i(s)-r(s))/(\sigma_i(s)))^2 \mathrm{d}s} \mid \mathcal{Z}_t\right], \tag{3.7}$$

which is the partial information option price for the contingent claim ξ. According to Corollary 2.4, $\pi_{i2}^*(\cdot)$ is the partial information replicating portfolio strategy for the contingent claim ξ when the initial endowment x_0 is the initial option price $p(0)$. Then $\pi_{i1}^*(\cdot)$ defined by (3.2) is exactly the partial information Merton's portfolio strategy for the terminal utility function $U(x) = x^2$ (see e.g., Merton [11]). That is, the optimal portfolio strategy (3.1) is the sum of the partial information replicating portfolio strategy for the contingent claim ξ and the partial information Merton's portfolio strategy. Consequently, if the initial endowment x_0 is different from the initial option price $p(0)$ necessary to hedge the contingent claim ξ, then $x_0 - p(0)$ should be invested according to Merton's portfolio strategy.

In particular, suppose the contingent claim ξ is a constant. In this case, it is easy to see that the solution $(p(\cdot), q_1(\cdot), \ldots, q_m(\cdot))$ of (3.3) is

$$p(t) = \mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\xi, \quad q_i(t) = 0, \quad 0 \le t \le T. \tag{3.8}$$

So we have the following corollary.

Corollary 3.2. *Let Hypothesis (H1) hold and ξ be a constant. Then the optimal portfolio strategy of Problem* (PIMV) *is*

$$\pi_i^*(t) = -\frac{\mu_i(t) - r(t)}{\sigma_i(t)^2}\left\{\mathbb{E}[x^*(t) \mid \mathcal{Z}_t] - e^{-\int_t^T r(s)ds}\xi\right\}, \quad 0 \le t \le T. \tag{3.9}$$

Remark 3.3. The solution $(p(\cdot), q_1(\cdot), \ldots, q_m(\cdot))$ defined by (3.8) has a straightforward interpretation in financial terms. That is, to achieve a deterministic wealth level ξ at the terminal time T, the agent should only invest a risk-free asset (a bond) and cannot invest any risky assets (stocks). Therefore, the optimal portfolio strategy obtained in Corollary 3.2 is only the partial information Merton's portfolio strategy.

The left part of this section will focus on a special mean-variance hedging problem with partial information. By virtue of filtering theory, we get an explicitly observable optimal portfolio strategy as well a risk measure. We also plot three figures and give numerical simulations to illustrate the theoretical result.

Example 3.4. Let $m = 2$ and all the conditions in Corollary 3.2 hold. Suppose the observable filtration of an agent is

$$\mathcal{Z}_t = \sigma\{S_1(s) : 0 \leq s \leq t\}, \quad 0 \leq t \leq T. \tag{3.10}$$

It implies that the agents can observe all the past prices of $S_1(\cdot)$, but due to some limit factors (e.g., bad behavior of the stock $S_2(\cdot)$ or time and energy of the investor) they cannot (do not want to) observe $S_2(\cdot)$.

Set $\mathcal{S}_1(\cdot) = \log S_1(\cdot)$, where $\log x$ $(x > 0)$ denotes a logarithm function. It follows from Itô's formula that

$$d\mathcal{S}_1(t) = \left(\mu_1(t) - \frac{1}{2}\sigma_1(t)^2\right)dt + \sigma_1(t)dW_1(t). \tag{3.11}$$

Since $\mu_1(\cdot)$ and $\sigma_1(\cdot)$ are deterministic functions (see Hypothesis (H1)), the above filtration $\mathcal{Z}_t$ is equivalently rewritten as

$$\mathcal{Z}_t = \sigma\{\mathcal{S}_1(s) : 0 \leq s \leq t\} = \sigma\{W_1(s) : 0 \leq s \leq t\}. \tag{3.12}$$

Similar to Example 2.5, we get from Corollary 3.2

$$\pi_i^*(t) = -\frac{\mu_i(t) - r(t)}{\sigma_i(t)^2}\left(\widehat{x}^*(t) - e^{-\int_t^T r(s)ds}\xi\right), \tag{3.13}$$

where $\widehat{x}^*(\cdot)$ is the solution of

$$d\widehat{x}^*(t) = \left\{\left[r(t) - \sum_{i=1}^{2}\left(\frac{\mu_i(t) - r(t)}{\sigma_i(t)}\right)^2\right]\widehat{x}^*(t) + \sum_{i=1}^{2}\left(\frac{\mu_i(t) - r(t)}{\sigma_i(t)}\right)^2 e^{-\int_t^T r(s)ds}\xi\right\}dt$$

$$- \frac{\mu_1(t) - r(t)}{\sigma_1(t)}\left(\widehat{x}^*(t) - e^{-\int_t^T r(s)ds}\xi\right)dW_1(t), \tag{3.14}$$

$$\widehat{x}^*(0) = x_0.$$

Now $\widehat{\pi}_i^*(t), 0 \leq t \leq T$, defined by (3.13), is an observable optimal portfolio strategy.

We now calculate the risk measure (or the value function) of the agent's goal

$$\text{RM}^2 := 2J(u(\cdot); x_0) = \min_{v(\cdot) \in \mathcal{U}_{ad}} 2J(v(\cdot); x_0). \tag{3.15}$$

From (3.14), we derive

$$\mathbb{E}\widehat{x}^*(t) = x_0 e^{\int_0^t [r(s) - \sum_{i=1}^2 ((\mu_i(s)-r(s))/(\sigma_i(s)))^2] ds}$$

$$+ \xi \int_0^t \sum_{i=1}^2 \left(\frac{\mu_i(s)-r(s)}{\sigma_i(s)}\right)^2 e^{\int_s^t [r(v) - \sum_{i=1}^2 ((\mu_i(v)-r(v))/(\sigma_i(v)))^2] dv - \int_s^T r(v) dv}\, ds,$$

$$\mathbb{E}\widehat{x}^*(t)^2 = x_0^2 e^{\int_0^t [2r(s) - \sum_{i=1}^2 ((\mu_i(s)-r(s))/(\sigma_i(s)))^2 - ((\mu_2(s)-r(s))/(\sigma_2(s)))^2] ds}$$

$$+ \int_0^t e^{\int_s^t [2r(v) - \sum_{i=1}^2 ((\mu_i(v)-r(v))/(\sigma_i(v)))^2 - ((\mu_2(v)-r(v))/(\sigma_2(v)))^2] dv}$$

$$\times e^{-\int_s^T r(v) dv} \xi \left[2\left(\frac{\mu_2(s)-r(s)}{\sigma_2(s)}\right)^2 \mathbb{E}\widehat{x}^*(s) \right.$$

$$\left. + \left(\frac{\mu_1(t)-r(t)}{\sigma_1(t)}\right)^2 e^{-\int_s^T r(v) dv} \xi \right] ds. \tag{3.16}$$

Combining (3.4), (3.14), Itô's formula with Lemma A.1,

$$d\widehat{x^*(t)^2} = \left[2r(t)\widehat{x^*(t)^2} + \left(e^{-2\int_t^T r(s)ds}\xi^2 - \widehat{x}^*(t)^2\right) \sum_{i=1}^2 \left(\frac{\mu_i(t)-r(t)}{\sigma_i(t)}\right)^2 \right] dt$$

$$+ 2\frac{\mu_1(t)-r(t)}{\sigma_1(t)} \left(e^{-2\int_t^T r(s)ds}\xi - \widehat{x}^*(t)\right) \widehat{x}^*(t) dW_1(t), \tag{3.17}$$

$$\widehat{x^*(0)^2} = x_0^2.$$

Solving the above equation,

$$\mathbb{E}\widehat{x^*(T)^2} = x_0^2 e^{2\int_0^T r(s)ds} + \int_0^T \sum_{i=1}^2 \left(\frac{\mu_i(t)-r(t)}{\sigma_i(t)}\right)^2 \left[\xi^2 - e^{2\int_t^T r(s)ds}\mathbb{E}\widehat{x}^*(t)^2\right] dt. \tag{3.18}$$

Applying a property of conditional expectation and throughout integration by parts, we get

$$RM^2 = \mathbb{E}\widehat{x^*(T)^2} - 2\xi\mathbb{E}\widehat{x}^*(T) + \xi^2$$

$$= \left(\xi - x_0 e^{\int_0^T r(t)dt}\right)^2 \left(1 - \int_0^T \rho(t)dt\right) \tag{3.19}$$

with

$$\rho(t) = \sum_{i=1}^2 \left(\frac{\mu_i(t)-r(t)}{\sigma_i(t)}\right)^2 e^{-\int_0^t [\sum_{i=1}^2 ((\mu_i(s)-r(s))/(\sigma_i(s)))^2 + ((\mu_2(s)-r(s))/(\sigma_2(s)))^2] ds}. \tag{3.20}$$
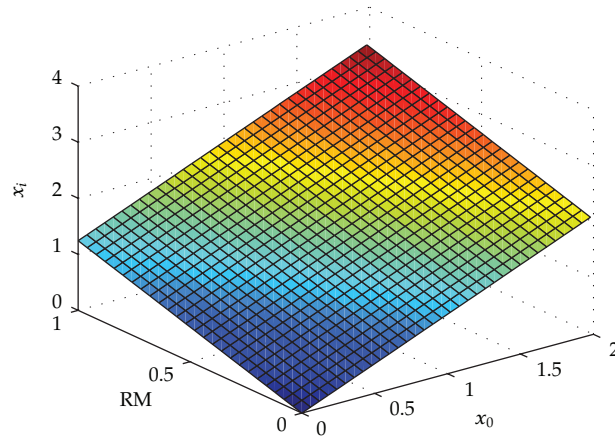
So we have the following proposition.

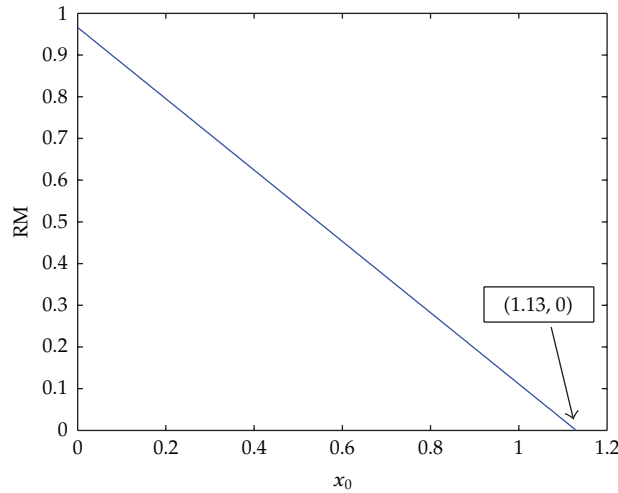**Figure 1:** The relationship among ξ, x_0 and RM.

Figure 2: The relationship between RM and x_0.

Proposition 3.5. *The optimal portfolio strategy and the risk measure are given by* (3.13) *and* (3.19), *respectively.*

To further illustrate the optimal portfolio strategy (3.13) and the risk measure (3.19), we plot three figures and give some numerical results here. Suppose $r = 0.06$, $\mu_1 = 0.12$, $\mu_2 = 0.18$, $\sigma_1 = 0.12$, and $\sigma_2 = 0.24$. Taking $T = 1$ year, we get from (3.19)

$$\xi = x_0 e^{0.06} + \frac{1}{\sqrt{\dfrac{1}{3}\left(1 + 2e^{-(3/4)}\right)}} \text{RM}. \tag{3.21}$$

In Figure 1, we let \$ $0 \leq x_0 \leq$ \$ 2 million and \$ $0 \leq$ RM $\leq$ \$ 1 million. The plane explicitly describes the relationship among ξ, x_0, and RM. In detail, the bigger x_0 and RM, the bigger ξ.

In Figure 2, we let the investment goal of the agent $\xi = $ \$ 1.2 million. The beeline shows that RM is a deceasing function of the initial endowment x_0. In particular, when $x_0 = $ \$ 1.13
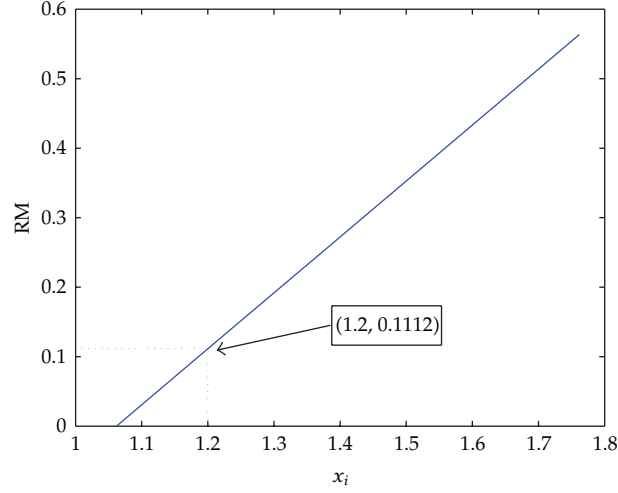
Figure 3: The relationship between RM and ξ.

million, we have RM = \$ 0. This means that to achieve the investment goal \$ 1.2 million at the end of one year, the agent only needs to invest \$ 1.13 million in the bond at the interest rate 6%; moreover, there is no risk for the investment strategy.

In Figure 3, we let $x_0 = \$ 1$ million. The beeline implies that RM is an increasing function of the investment goal ξ. Consider now the agent who has an initial endowment $x_0 = \$ 1$ million and wishes to obtain an expected return rate 20% in one year. Taking $x_0 = \$ 1$ million and $\xi = \$ 1.2$ million, we get RM = \$ 0.1112 million, meaning that the risk of the investment goal is as high as 11.12%.

Furthermore, we calculate the Merton's portfolio strategy. Let $r = 0.06$, $T = 1$, $x_0 = \$ 1$ million, and $\xi = \$ 1.2$ million.

(1) Set $\mu_1 = 0.12$, $\mu_2 = 0.18$, $\sigma_1 = 0.12$, and $\sigma_2 = 0.24$. By (3.13), the amount of money the agent should invest in the ith stock is

$$\pi_i^*(t) = \frac{\mu_i - 0.06}{\sigma_i^2}\left(1.2e^{-0.06(1-t)} - \widehat{x}^*(t)\right). \tag{3.22}$$

In particular, at the initial time $t = 0$, $\pi_1^*(0) = \$ 0.5422$ million and $\pi_2^*(0) = \$ 0.2711$ million, which implies that the agent needs to invest \$ 0.5422 million and \$ 0.2711 million in the stocks $S_1(\cdot)$ and $S_2(\cdot)$, respectively, and invest in the bond for an amount of

$$1 - (0.5422 + 0.2711) = \$ 0.1867 \text{ million.} \tag{3.23}$$

(2) Set $\mu_1 = 0.12$, $\mu_2 = 0.18$, $\sigma_1 = 0.12$, and $\sigma_2 = 0.17$. Similarly, we have $\pi_1^*(0) = \$ 0.5422$ million, $\pi_2^*(0) = \$ 0.5403$ million, and

$$(0.5422 + 0.5403) - 1 = \$ 0.0825 \text{ million,} \tag{3.24}$$

which implies that the agent needs to borrow \$ 0.0825 million from a bank and invest the amount \$ 1.0825 million in the two stocks $S_1(\cdot)$ and $S_2(\cdot)$. This is indeed an aggressive policy.

4. Comparison with Existing Results

The subject of stochastic control with partial information has been discussed by many researchers, such as Bensoussan [10] and Xiong and Zhou [3]. Usually, they made one of the following two assumptions: (i) the filtration $\mathcal{E}$ is a sigma algebra generated by some observable process; (ii) the control systems are Markovian. From this viewpoint, our work cannot be covered by their results.

Note that our work is related to the recent paper by Hu and Øksendal [4]. In what follows, we shall give some detailed comparisons between them.

(1) Comparing with [4], the distinctive characteristics of our Problem (PIMV) are the following four points. First, since ξ in our cost functional (1.5) is a random variable, then it partly generalizes that of [4]; meanwhile, our cost functional (1.5) can measure the risk that the contingent claim cannot be reached. Second, we give a possible formulation of partial information in the setting of finance and interpret the economic meaning of the optimal portfolio with partial information. Third, in terms of filtering theory, we explicitly compute the observable optimal portfolio strategy and the risk measure of an agent in Example 3.4. Last but not least, we present some numerical results and figures to illustrate the optimal portfolio strategy and the risk measure. Although the example is special, it is nontrivial and contains some filtering techniques. These results show the practical sense of our paper.

(2) Since the initial state x_0 in control system (2.1) may be a decision and ξ in cost functional (2.3) is a random variable, our Problem (PILQ) is different from that of [4]. In particular, if the foregoing x_0 is also a decision, then the partial information optimal control can be denoted by a conditional expectation of the solution of the corresponding adjoint equation (recall Corollary 2.4). Obviously, this is different from [1, 4].

(3) Filtering theory plays an important role in optimal control with partial information. To get an explicitly observable optimal control, it is necessary to compute the conditional expectation of the solution of BSDEs. However, it is not doable in most cases. In this paper, we try to solve some stochastic control problems by using filtering theory. As a byproduct, we establish backward and forward-backward stochastic differential filtering equations which are different from the existing literature about filtering theory. Although the technique of solving stochastic control is restricted and the filtering equations are linear, we can regard them as a contribution to filtering-control theory. By the way, the study about Problem (PILQ) motivates us to establish some general filtering theory of BSDEs in future work.

In [4], Hu and Øksendal obtain some optimal controls and value functions with complete and partial information. Note that they only denote the optimal controls by conditional expectation. We also notice there is not any filtering result in [4]. This is different from ours.

Appendix

A Classical Filtering Equation for SDEs

We present here a classical filtering result for the readers' convenience which was employed in this paper. For a detailed discussion of filtering, we refer to the books [5, 6].

Consider the following 1-dimensional state and observation equations:

$$\theta(t) = \theta_0 + \int_0^t h(s)ds + x(t),$$

$$\xi(t) = \xi_0 + \int_0^t a(s,\omega)ds + \int_0^t b(s,\xi)dW(s).$$

(A.1)

Here $(W(\cdot))$ is a 1-dimensional standard Brownian motion defined on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ equipped with a natural filtration $\mathcal{F}_t = \sigma\{W(s); 0 \le s \le t\}$, $\mathcal{F} = \mathcal{F}_T$, $0 \le t \le T$; $x(t)$ is an $\mathcal{F}_t$-martingale; $h(t)$ is an $\mathcal{F}_t$-adapted process with $\int_0^T |h(s)|ds < +\infty$; the functional $b(t, y)$, $y \in C_T$, $0 \le t \le T$, is $\mathcal{B}_t$-measurable.

We need the following hypothesis.

(H) *For any $y, \overline{y} \in C_T$, $0 \le t \le T$, the functional $b(t, \cdot)$ satisfies*

$$\left| b(t,y) - b(t,\overline{y}) \right|^2 \le L_1 \int_0^t \left[y(s) - \overline{y}(s) \right]^2 dK(s) + L_2 \left[y(t) - \overline{y}(t) \right]^2,$$

$$b(t,y)^2 \le L_1 \int_0^t \left(1 + y(s)^2 \right) dK(s) + L_2 \left(1 + y(t)^2 \right),$$

(A.2)

where L_1 and L_2 are two nonnegative constants and $0 \le K(\cdot) \le 1$ is a nondecreasing right continuous function. Moreover, $\sup_{0 \le t \le T} \mathbb{E}\theta(t)^2 < +\infty$, $\int_0^T \mathbb{E}h(t)^2 dt < +\infty$, $\int_0^T \mathbb{E}a(t,\omega)^2 dt < +\infty$, $b(t,y)^2 \ge C > 0$.

The following result is due to of [5, Theorem 8.1].

Lemma A.1. *Define $\widehat{\theta}(t) = \mathbb{E}[\theta(t) \mid \mathcal{F}_t^{\xi}]$. Here $\theta(t)$ can take $\theta(t)$, $h(t)$, $D(t)$, $a(t,\omega)$, and $\theta(t)a(t,\omega)$; $\mathcal{F}_t^{\xi} = \sigma\{\xi(s) : 0 \le s \le t\}$, $0 \le t \le T$. If Hypothesis (H) holds, then the optimal nonlinear filtering equation is*

$$\widehat{\theta}(t) = \widehat{\theta}_0 + \int_0^t \widehat{h}(s)ds + \int_0^t \left\{ \widehat{D}(s) + \left[\widehat{\theta(s)a(s,\omega)} - \widehat{\theta}(s)\widehat{a}(s,\omega) \right] b(s,\xi)^{-1} \right\} d\overline{W}(s),$$

(A.3)

where

$$\overline{W}(t) = \int_0^t \frac{d\xi(s) - \widehat{a}(s,\omega)ds}{b(s,\xi)}$$

(A.4)

is a standard Brownian motion (with respect to $\mathcal{F}_t^{\xi}$), and $D(t)$ is an $\mathcal{F}_t$-adapted process with

$$D(t) = \frac{d\langle x(t), W(t) \rangle}{dt}.$$

(A.5)

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