

## Research Article

# Some Properties of Subclasses of Multivalent Functions

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The authors introduce two new subclasses denoted by  $\mathcal{T}^*(\Omega, \lambda, p, \alpha)$  and  $\mathcal{T}_\varepsilon^*(\Omega, \lambda, p, \alpha)$  of the class  $A(p, n)$  of  $p$ -valent analytic functions. They obtain coefficient inequality for the class  $\mathcal{T}^*(\Omega, \lambda, p, \alpha)$ . They investigate various properties of classes  $\mathcal{T}^*(\Omega, \lambda, p, \alpha)$  and  $\mathcal{T}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ . Furthermore, they derive partial sums associated with the class  $\mathcal{T}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ .

## 1. Introduction and Definition

Let  $A(p, n)$  denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (n, p \in \mathbb{N} = \{1, 2, \dots\}), \quad (1.1)$$

which are analytic and  $p$ -valent in the open unit disc  $U = \{z : z \in \mathbb{C}, |z| < 1\}$ . We write  $A(1, 1) = A$ .

A function  $f \in A(p, n)$  is said to be in the class  $S(p, n, \alpha)$  of  $p$ -valently star-like functions of order  $\alpha$  if it satisfies the condition

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{C}; 0 \leq \alpha < p). \quad (1.2)$$

Furthermore, a function  $f \in A(p, n)$  is said to be in the class  $K(p, n, \alpha)$  of  $p$ -valently convex functions of order  $\alpha$  if it satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U; 0 \leq \alpha < p). \quad (1.3)$$

The classes  $S(p, n, \alpha)$  and  $K(p, n, \alpha)$  were studied by Owa [1]. The class  $S^*(p, \alpha) := S(p, 1, \alpha)$  was considered by Patil and Thakare [2].

We denote by  $T(p, n)$  the subclass of the class  $A(p, n)$  consisting of functions of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0; n, p \in \mathbb{N}) \quad (1.4)$$

and define two further classes  $T^*(p, n, \alpha)$  and  $C(p, n, \alpha)$  by

$$T^*(p, n, \alpha) := S(p, n, \alpha) \cap T(p, n), \quad C(p, n, \alpha) := K(p, n, \alpha) \cap T(p, n). \quad (1.5)$$

For the classes  $T^*(p, \alpha) := S^*(p, \alpha) \cap T(p)$ ,  $C(p, \alpha) := K(p, \alpha) \cap T(p)$ .

The following lemmas were given by Owa [3].

**Lemma 1.1.** *Let the function  $f$  be defined by*

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0; p \in \mathbb{N}). \quad (1.6)$$

*Then,  $f$  is in the class  $T^*(p, \alpha)$  if and only if*

$$\sum_{k=1}^{\infty} (p+k-\alpha) a_{p+k} \leq p-\alpha. \quad (1.7)$$

*The result is sharp.*

**Lemma 1.2.** *Let the function  $f$  be defined by (1.6). Then,  $f$  is in the class  $C(p, \alpha)$  if and only if*

$$\sum_{k=1}^{\infty} (p+k)(p+k-\alpha) a_{p+k} \leq p(p-\alpha). \quad (1.8)$$

*The result is sharp.*

For a function  $f$  defined by (1.6) and in the class  $T^*(p, \alpha)$ , Lemma 1.1 yields

$$a_{p+1} \leq \frac{p-\alpha}{p+1-\alpha}. \quad (1.9)$$

On the other hand, for a function  $f$  defined by (1.6) and in the class  $C(p, \alpha)$ , Lemma 1.2 yields

$$a_{p+1} \leq \frac{p(p - \alpha)}{(p + 1)(p + 1 - \alpha)}. \tag{1.10}$$

In view of the coefficient inequalities (1.9) and (1.10), it would seem to be natural to introduce and study here two further classes  $T_\varepsilon^*(p, \alpha)$  and  $C_\varepsilon(p, \alpha)$  of analytic and  $p$ -valent functions, where  $T_\varepsilon^*(p, \alpha)$  denotes the subclass of  $T^*(p, \alpha)$  consisting of functions of the form

$$f(z) = z^p - \frac{(p - \alpha)\varepsilon}{p + 1 - \alpha} z^{p+1} - \sum_{k=2}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0, p \in \mathbb{N}, k \in \mathbb{N} - \{1\}; 0 \leq \alpha < p; 0 \leq \varepsilon < 1) \tag{1.11}$$

and  $C_\varepsilon(p, \alpha)$  denotes the subclass of  $C(p, \alpha)$  consisting of functions of the form

$$f(z) = z^p - \frac{p(p - \alpha)\varepsilon}{(p + 1)(p + 1 - \alpha)} z^{p+1} - \sum_{k=2}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0, p \in \mathbb{N}, k \in \mathbb{N} - \{1\}; 0 \leq \alpha < p; 0 \leq \varepsilon < 1). \tag{1.12}$$

The classes  $T_\varepsilon^*(p, \alpha)$  and  $C_\varepsilon(p, \alpha)$  are studied by Aouf et al. [4].

The classes

$$T_\varepsilon^*(\alpha) := T_\varepsilon^*(1, \alpha), \quad C_\varepsilon(\alpha) := C_\varepsilon(1, \alpha) \tag{1.13}$$

were considered earlier by Silverman and Silvia [5].

Now, we give the following equalities for the functions  $f(z)$  belonging to the class  $A(p, n)$ :

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = z(D^0 f(z))' = z \left[ pz^{p-1} + \sum_{k=n}^{\infty} (p+k) a_{p+k} z^{p+k-1} \right] = pz^p + \sum_{k=n}^{\infty} (p+k) a_{p+k} z^{p+k}, \\ D^2 f(z) &= D(Df(z)) = z(D^1 f(z))' = z \left[ pz^p + \sum_{k=n}^{\infty} (p+k) a_{p+k} z^{p+k} \right]' = p^2 z^p + \sum_{k=n}^{\infty} (p+k)^2 a_{p+k} z^{p+k}, \\ &\vdots \\ D^\Omega f(z) &= D(D^{\Omega-1} f(z)) = p^\Omega z^p + \sum_{k=n}^{\infty} (k+p)^\Omega a_{p+k} z^{p+k}. \end{aligned} \tag{1.14}$$

We define  $\wp : A(p, n) \rightarrow A(p, n)$  such that

$$\wp(\Omega, \lambda, p) = \left( \frac{1}{p^\Omega} - \lambda \right) D^\Omega f(z) + \frac{\lambda}{p} z \left( D^\Omega f(z) \right)' \quad \left( 0 \leq \lambda \leq \frac{1}{p^\Omega}, \Omega \in \mathbb{N} \cup \{0\} \right). \quad (1.15)$$

A function  $f(z) \in A(p, n)$  is said to be in the class  $\mathfrak{J}(\Omega, \lambda, p, \alpha)$  if it satisfies the inequality

$$\operatorname{Re} \left\{ \frac{z(\wp(\Omega, \lambda, p))'}{\wp(\Omega, \lambda, p)} \right\} = \operatorname{Re} \left\{ z \frac{(1/p^\Omega + (1/p - 1)\lambda)(D^\Omega f(z))' + (\lambda/p)z(D^\Omega f(z))''}{(1/p^\Omega - \lambda)D^\Omega f(z) + (\lambda/p)z(D^\Omega f(z))'} \right\} > \alpha, \quad (1.16)$$

for some  $\alpha$  ( $0 \leq \alpha < p$ ),  $0 \leq \lambda \leq 1/p^\Omega$ ,  $\Omega \in \mathbb{N} \cup \{0\}$  and for all  $z \in U$ .

If  $\Omega = 0$  and  $\lambda = 0$ , we obtain the condition (1.2). Furthermore, we obtain the condition (1.3) for  $\Omega = 0$  and  $\lambda = 1$ .

We denote by  $T(p, n)$  the subclass of the class  $A(p, n)$  consisting of functions of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0; n, p \in \mathbb{N}), \quad (1.17)$$

and define the class  $\mathfrak{J}^*(\Omega, \lambda, p, \alpha)$  by

$$\mathfrak{J}^*(\Omega, \lambda, p, \alpha) = \mathfrak{J}(\Omega, \lambda, p, \alpha) \cap T(p, n). \quad (1.18)$$

Furthermore, we denote by  $\mathfrak{J}_\varepsilon^*(\Omega, \lambda, p, \alpha)$  the subclass of  $\mathfrak{J}^*(\Omega, \lambda, p, \alpha)$  consisting of functions of the form

$$f(z) = z^p - \frac{(p - \alpha)\varepsilon}{((p + n)/p)^\Omega (1 + \lambda n p^{\Omega-1})(n + p - \alpha)} z^{p+n} - \sum_{k=n+1}^{\infty} a_{p+k} z^{p+k} \quad (0 \leq \varepsilon < 1). \quad (1.19)$$

The main object of the present paper is to investigate interesting properties and characteristics of the classes  $\mathfrak{J}^*(\Omega, \lambda, p, \alpha)$  and  $\mathfrak{J}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ . Also, the partial sums is defined for  $f$  function defined by (1.19).

## 2. A Coefficient Inequality for the Class $\mathfrak{J}^*(\Omega, \lambda, p, \alpha)$ and Some Theorems for the Class $\mathfrak{J}_\varepsilon^*(\Omega, \lambda, p, \alpha)$

First, we give a coefficient inequality for the class  $\mathfrak{J}^*(\Omega, \lambda, p, \alpha)$ .

**Theorem 2.1.** *Let the function  $f$  be defined by (1.17). Then,  $f$  is in the class  $\mathfrak{J}^*(\Omega, \lambda, p, \alpha)$  if and only if*

$$\sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right)^\Omega (1 + \lambda k p^{\Omega-1})(k + p - \alpha) a_{p+k} \leq p - \alpha. \quad (2.1)$$

*Proof.* Suppose that  $f(z) \in \mathfrak{J}^*(\Omega, \lambda, p, \alpha)$ . Then, we find from (1.16) that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z(\hat{\rho}(\Omega, \lambda, p))'}{\hat{\rho}(\Omega, \lambda, p)} \right\} &= \operatorname{Re} \left\{ z \frac{(1/p^\Omega + (1/p - 1)\lambda)(D^\Omega f(z))' + (\lambda/p)z(D^\Omega f(z))''}{(1/p^\Omega - \lambda)D^\Omega f(z) + (\lambda/p)z(D^\Omega f(z))'} \right\} \\ &= \operatorname{Re} \left\{ \frac{pz^p - \sum_{k=n}^{\infty} (k+p)^{\Omega+1} (1/p^\Omega + \lambda k/p) a_{p+k} z^{p+k}}{z^p - \sum_{k=n}^{\infty} (k+p)^\Omega (1/p^\Omega + \lambda k/p) a_{p+k} z^{p+k}} \right\} > \alpha. \end{aligned} \quad (2.2)$$

If we choose  $z$  to be real and let  $z \rightarrow 1^-$ , we get

$$\left\{ \frac{p - \sum_{k=n}^{\infty} (k+p)^{\Omega+1} (1/p^\Omega + \lambda k/p) a_{p+k}}{1 - \sum_{k=n}^{\infty} (k+p)^\Omega (1/p^\Omega + \lambda k/p) a_{p+k}} \right\} \geq \alpha \quad (2.3)$$

or, equivalently,

$$p - \sum_{k=n}^{\infty} (k+p)^{\Omega+1} (1/p^\Omega + \lambda k/p) a_{p+k} \geq \alpha \left\{ 1 - \sum_{k=n}^{\infty} (k+p)^\Omega (1/p^\Omega + \lambda k/p) a_{p+k} \right\}. \quad (2.4)$$

Thus, we have

$$\sum_{k=n}^{\infty} \left\{ (k+p)^{\Omega+1} (1/p^\Omega + \lambda k/p) - (k+p)^\Omega (1/p^\Omega + \lambda k/p) \right\} a_{p+k} \leq p - \alpha \quad (2.5)$$

or

$$\sum_{k=n}^{\infty} \left\{ \left( \frac{k+p}{p} \right)^\Omega (1 + \lambda k p^{\Omega-1}) (k+p - \alpha) \right\} a_{p+k} \leq p - \alpha. \quad (2.6)$$

Conversely, suppose that the inequality (2.1) holds true and let

$$z \in \partial U = \{z : z \in \mathbb{C}, |z| = 1\}. \quad (2.7)$$

Then, we find from the definition (1.4) that

$$\begin{aligned} \left| \frac{z(\hat{\rho}(\Omega, \lambda, p))'}{\hat{\rho}(\Omega, \lambda, p)} - p \right| &= \left| z \frac{(1/p^\Omega + (1/p - 1)\lambda)(D^\Omega f(z))' + (\lambda/p)z(D^\Omega f(z))''}{(1/p^\Omega - \lambda)D^\Omega f(z) + (\lambda/p)z(D^\Omega f(z))'} - p \right| \\ &= \left| \frac{pz^p - \sum_{k=n}^{\infty} ((k+p)/p)^\Omega (k+p)(1 + \lambda k p^{\Omega-1}) a_{p+k} z^{p+k}}{z^p - \sum_{k=n}^{\infty} ((k+p)/p)^\Omega (1 + \lambda k p^{\Omega-1}) a_{p+k} z^{p+k}} - p \right| \\ &\leq \frac{\sum_{k=n}^{\infty} k((k+p)/p)^\Omega (1 + \lambda k p^{\Omega-1}) a_{p+k}}{1 - \sum_{k=n}^{\infty} ((k+p)/p)^\Omega (1 + \lambda k p^{\Omega-1}) a_{p+k}}. \end{aligned} \quad (2.8)$$

By means of inequality (2.1), we can write

$$\begin{aligned} & \sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) (k+p-\alpha) a_{p+k} \\ & \leq p - \alpha \implies \sum_{k=n}^{\infty} k \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) a_{p+k} \\ & \leq p - \alpha - \sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) (p - \alpha) a_{p+k} \end{aligned} \quad (2.9)$$

or

$$\sum_{k=n}^{\infty} k \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) a_{p+k} \leq p - \alpha - (p - \alpha) \sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) a_{p+k}. \quad (2.10)$$

Thus, we obtain

$$\begin{aligned} & \left| z \frac{(1/p^{\Omega} + (1/p - 1)\lambda)(D^{\Omega} f(z))' + (\lambda/p)z(D^{\Omega} f(z))''}{(1/p^{\Omega} - \lambda)D^{\Omega} f(z) + (\lambda/p)z(D^{\Omega} f(z))'} - p \right| \\ & = \left| \frac{\sum_{k=n}^{\infty} k \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) a_{p+k} z^k}{1 - \sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) a_{p+k} z^k} \right| \\ & \leq \frac{p - \alpha - (p - \alpha) \sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) a_{p+k}}{1 - \sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) a_{p+k}} \\ & = p - \alpha. \end{aligned} \quad (2.11)$$

This evidently completes the proof of Theorem 2.1.  $\square$

Now, we give a characterization theorem for the class  $\mathfrak{J}_{\varepsilon}^*(\Omega, \lambda, p, \alpha)$ .

**Theorem 2.2.** *Let the function  $f$  be defined by (1.19). Then,  $f$  is in the class  $\mathfrak{J}_{\varepsilon}^*(\Omega, \lambda, p, \alpha)$  if and only if*

$$\begin{aligned} & \sum_{k=n+1}^{\infty} \left( \frac{k+p}{p} \right)^{\Omega} (1 + \lambda k p^{\Omega-1}) (k+p-\alpha) a_{p+k} \\ & \leq (p - \alpha)(1 - \varepsilon) \quad \left( 0 \leq \alpha < p, 0 \leq \varepsilon < 1, 0 \leq \lambda \leq \frac{1}{p^{\Omega}} \right). \end{aligned} \quad (2.12)$$

The result is sharp for the function  $f$  given by

$$f(z) = z^p - \frac{(p - \alpha)\varepsilon}{\left(\frac{(p+n)}{p}\right)^\Omega (1 + \lambda np^{\Omega-1})(n+p-\alpha)} z^{p+n} - \frac{(p - \alpha)(1 - \varepsilon)}{\left(\frac{(p+k)}{p}\right)^\Omega (1 + \lambda kp^{\Omega-1})(k+p-\alpha)} z^{p+k} \quad (k = n + 1, n + 2, \dots; n \in \mathbb{N}). \tag{2.13}$$

*Proof.* Using inequality (2.1), we have

$$\left(\frac{n+p}{p}\right)^\Omega (1 + \lambda np^{\Omega-1})(n+p-\alpha)a_{p+n} + \sum_{k=n+1}^\infty \left(\frac{k+p}{p}\right)^\Omega (1 + \lambda kp^{\Omega-1})(k+p-\alpha)a_{p+k} \leq p - \alpha \tag{2.14}$$

or

$$\sum_{k=n+1}^\infty \left(\frac{k+p}{p}\right)^\Omega (1 + \lambda kp^{\Omega-1})(k+p-\alpha)a_{p+k} \leq p - \alpha - \left(\frac{n+p}{p}\right)^\Omega (1 + \lambda np^{\Omega-1})(n+p-\alpha)a_{p+n}. \tag{2.15}$$

Thus, by setting

$$a_{p+n} = \frac{(p - \alpha)\varepsilon}{\left(\frac{(p+n)}{p}\right)^\Omega (1 + \lambda np^{\Omega-1})(n+p-\alpha)}, \tag{2.16}$$

we obtain

$$\begin{aligned} & \sum_{k=n+1}^\infty \left(\frac{k+p}{p}\right)^\Omega (1 + \lambda kp^{\Omega-1})(k+p-\alpha)a_{p+k} \\ & \leq p - \alpha - \left(\frac{n+p}{p}\right)^\Omega (1 + \lambda np^{\Omega-1})(n+p-\alpha) \frac{(p - \alpha)\varepsilon}{\left(\frac{(p+n)}{p}\right)^\Omega (1 + \lambda np^{\Omega-1})(n+p-\alpha)} \end{aligned} \tag{2.17}$$

or

$$\sum_{k=n+1}^\infty \left(\frac{k+p}{p}\right)^\Omega (1 + \lambda kp^{\Omega-1})(k+p-\alpha)a_{p+k} \leq (p - \alpha)(1 - \varepsilon). \tag{2.18}$$

A closure theorem for the class  $\mathcal{T}_\varepsilon^*(\Omega, \lambda, p, \alpha)$  is given by the following. □

**Theorem 2.3.** *Let*

$$f_s(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^\Omega(1+\lambda np^{\Omega-1})(n+p-\alpha)} z^{p+n} - \sum_{k=n+1}^{\infty} a_{p+k,s} z^{p+k} \quad (a_{p+k,s} \geq 0; p, n \in \mathbb{N}; 0 \leq \varepsilon < 1; s = 1, 2, 3, \dots, m). \quad (2.19)$$

If  $f_s \in \mathfrak{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ , then the function  $g$  given by

$$g(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^\Omega(1+\lambda np^{\Omega-1})(n+p-\alpha)} z^{p+n} - \sum_{k=n+1}^{\infty} b_{p+k} z^{p+k}, \quad (2.20)$$

with

$$b_{p+k} := \frac{1}{m} \sum_{s=1}^m a_{p+k,s} \geq 0, \quad (2.21)$$

is also in the class  $\mathfrak{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ .

*Proof.* Since  $f_s \in \mathfrak{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$  for  $s = 1, 2, \dots, m$ , it follows from Theorem 2.2 that

$$\sum_{k=n+1}^{\infty} \left(\frac{k+p}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha) a_{p+k,s} \leq (p-\alpha)(1-\varepsilon) \quad (s = 1, 2, \dots, m). \quad (2.22)$$

By applying (2.22) and the definition (2.21), we write

$$\begin{aligned} & \sum_{k=n+1}^{\infty} \left(\frac{k+p}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha) b_{p+k} \\ &= \sum_{k=n+1}^{\infty} \left(\frac{k+p}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha) \left\{ \frac{1}{m} \sum_{s=1}^m a_{p+k,s} \right\} \\ &= \frac{1}{m} \sum_{s=1}^m \left\{ \sum_{k=n+1}^{\infty} \left(\frac{k+p}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha) a_{p+k,s} \right\} \\ &\leq \frac{1}{m} \sum_{s=1}^m (p-\alpha)(1-\varepsilon) = (p-\alpha)(1-\varepsilon). \end{aligned} \quad (2.23)$$

□



**Theorem 2.4.** *Let*

$$\begin{aligned}
 f_{p+n}(z) &= z^p - \frac{(p-\alpha)\varepsilon}{\left(\frac{(p+n)}{p}\right)^\Omega (1+\lambda np^{\Omega-1})(n+p-\alpha)} z^{p+n}, \\
 f_{p+k}(z) &= z^p - \frac{(p-\alpha)\varepsilon}{\left(\frac{(p+n)}{p}\right)^\Omega (1+\lambda np^{\Omega-1})(n+p-\alpha)} z^{p+n} \\
 &\quad - \frac{(p-\alpha)(1-\varepsilon)}{\left(\frac{(p+k)}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha)} z^{p+k},
 \end{aligned}
 \tag{2.24}$$

where  $k \in \{n+1, n+2, \dots\}$ .

Then,  $f$  is in the class  $\mathfrak{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$  if and only if it can be expressed in the form

$$f(z) = \sum_{k=n}^{\infty} \eta_{p+k} f_{p+k}(z) \quad \left( \eta_{p+k} \geq 0; \sum_{k=n}^{\infty} \eta_{p+k} = 1 \right).
 \tag{2.25}$$

*Proof.* Suppose that  $f$  is given by (2.25), so that we find from (2.24) that

$$\begin{aligned}
 f(z) &= z^p - \frac{(p-\alpha)\varepsilon}{\left(\frac{(p+n)}{p}\right)^\Omega (1+\lambda np^{\Omega-1})(n+p-\alpha)} z^{p+n} \\
 &\quad - \sum_{k=n+1}^{\infty} \frac{(p-\alpha)(1-\varepsilon)}{\left(\frac{(p+k)}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha)} \eta_{p+k} z^{p+k},
 \end{aligned}
 \tag{2.26}$$

where the coefficients  $\eta_{p+k}$  are given with  $\sum_{k=n}^{\infty} \eta_{p+k} = 1$ ,  $\eta_{p+k} \geq 0$ . Then, since,

$$\begin{aligned}
 &\sum_{k=n+1}^{\infty} \left(\frac{k+p}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha) \frac{(p-\alpha)(1-\varepsilon)}{\left(\frac{(k+p)}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha)} \eta_{p+k} \\
 &= (p-\alpha)(1-\varepsilon) \sum_{k=n+1}^{\infty} \eta_{p+k} \\
 &= (p-\alpha)(1-\varepsilon)(1-\eta_{p+n}) \\
 &\leq (p-\alpha)(1-\varepsilon),
 \end{aligned}
 \tag{2.27}$$

we conclude from Theorem 2.2 that  $f \in \mathfrak{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ .

Conversely, let us assume that the function  $f$  defined by (1.19) is in the class  $\mathfrak{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ . Then,

$$a_{p+k} \leq \frac{(p-\alpha)(1-\varepsilon)}{\left(\frac{(k+p)}{p}\right)^\Omega (1+\lambda kp^{\Omega-1})(k+p-\alpha)},
 \tag{2.28}$$

which follows readily from (2.12) for  $k \in \{n+1, n+2, \dots\}$ .

For  $k \in \{n+1, n+2, \dots\}$ , setting

$$\eta_{p+k} = \frac{((k+p)/p)^\Omega (1 + \lambda k p^{\Omega-1})(k+p-\alpha)}{(p-\alpha)(1-\varepsilon)} a_{p+k} \quad (2.29)$$

and  $\eta_{p+n} = 1 - \sum_{k=n+1}^{\infty} \eta_{p+k}$ , we thus arrive at (2.25). This completes the proof of Theorem 2.4.  $\square$

**Theorem 2.5.** Let  $f$  be given by (1.19) and define the partial sums  $s_m(z)$  by

$$s_m(z) = \begin{cases} z^p - \frac{p^\Omega(p-\alpha)\varepsilon}{(p+n)^\Omega(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n}, & m = n, \\ z^p - \frac{p^\Omega(p-\alpha)\varepsilon}{(p+n)^\Omega(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \sum_{k=n+1}^m a_{p+k} z^{p+k}, & m = n+1, n+2, \dots \end{cases} \quad (2.30)$$

Suppose also that

$$\sum_{k=n+1}^{\infty} d_{p+k} a_{p+k} \leq 1 - \frac{p^\Omega(p-\alpha)\varepsilon}{(p+n)^\Omega(1+\lambda n p^{\Omega-1})(n+p-\alpha)},$$

$$\left( \text{where } d_{p+k} = \frac{(p+k)^\Omega(1+\lambda k p^{\Omega-1})(k+p-\alpha)}{p^\Omega(p-\alpha)(1-\varepsilon)}; 0 \leq \alpha < p, 0 \leq \varepsilon < 1, 0 \leq \lambda \leq \frac{1}{p^\Omega} \right). \quad (2.31)$$

Then, for  $m \geq n+1$ , one has

$$\operatorname{Re} \left\{ \frac{f(z)}{s_m(z)} \right\} > 1 - \frac{p^\Omega(p-\alpha)(1-\varepsilon)}{(p+m+1)^\Omega [1 + \lambda(m+1)p^{\Omega-1}] (p+m+1-\alpha)}, \quad (2.32)$$

$$\operatorname{Re} \left( \frac{s_m(z)}{f(z)} \right) > \frac{(p+m+1)^\Omega \{1 + \lambda(m+1)p^{\Omega-1}\} (m+p+1-\alpha)}{p^\Omega(p-\alpha)(1-\varepsilon) + (p+m+1)^\Omega \{1 + \lambda(m+1)p^{\Omega-1}\} (m+p+1-\alpha)}. \quad (2.33)$$

Each of the bounds in (2.32) and (2.33) is the best possible.

*Proof.* From (2.31) and Theorem 2.2, we have that  $f \in \mathfrak{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ . By definition  $d_{p+k}$ , we can write

$$\begin{aligned} d_{p+k} &= \frac{((p+k)/p)^\Omega (1 + \lambda k p^{\Omega-1})(k+p-\alpha)}{(p-\alpha)(1-\varepsilon)} \\ &= \left(1 + \frac{k}{p}\right)^\Omega (1 + \lambda k p^{\Omega-1}) \left[ \frac{p-\alpha+k}{(p-\alpha)(1-\varepsilon)} \right] \\ &= \left(1 + \frac{k}{p}\right)^\Omega (1 + \lambda k p^{\Omega-1}) \left[ \frac{1}{1-\varepsilon} + \frac{k}{(p-\alpha)(1-\varepsilon)} \right] > 1. \end{aligned} \tag{2.34}$$

Under the hypothesis of this theorem, we can see from (2.31) that  $d_{p+k+1} > d_{p+k} > 1$ , for  $k = n+1, n+2, \dots$ . Therefore, we have

$$\sum_{k=n+1}^m a_{p+k} + d_{p+m+1} \sum_{k=m+1}^\infty a_{p+k} \leq \sum_{k=n+1}^\infty d_{p+k} a_{p+k} \leq 1 - \frac{p^\Omega (p-\alpha)\varepsilon}{(p+n)^\Omega (1 + \lambda n p^{\Omega-1})(p+n-\alpha)}. \tag{2.35}$$

Using (1.19) and (2.30), we can write

$$\begin{aligned} \frac{f(z)}{s_m(z)} &= \frac{z^p - \left(p^\Omega (p-\alpha)\varepsilon / (p+n)^\Omega (1 + \lambda n p^{\Omega-1})(n+p-\alpha)\right) z^{p+n} - \sum_{k=n+1}^\infty a_{p+k} z^{p+k}}{z^p - \left(p^\Omega (p-\alpha)\varepsilon / (p+n)^\Omega (1 + \lambda n p^{\Omega-1})(n+p-\alpha)\right) z^{p+n} - \sum_{k=n+1}^m a_{p+k} z^{p+k}} \\ &= 1 - \frac{\sum_{k=m+1}^\infty a_{p+k} z^{p+k}}{z^p - \left(p^\Omega (p-\alpha)\varepsilon / (p+n)^\Omega (1 + \lambda n p^{\Omega-1})(n+p-\alpha)\right) z^{p+n} - \sum_{k=n+1}^m a_{p+k} z^{p+k}}. \end{aligned} \tag{2.36}$$

Set

$$\begin{aligned} \Psi_1(z) &= d_{p+m+1} \left[ \frac{f(z)}{s_m(z)} - \left(1 - \frac{1}{d_{p+m+1}}\right) \right] \\ &= 1 - \frac{d_{p+m+1} \sum_{k=m+1}^\infty a_{p+k} z^k}{1 - \left(p^\Omega (p-\alpha)\varepsilon / (p+n)^\Omega (1 + \lambda n p^{\Omega-1})(n+p-\alpha)\right) z^n - \sum_{k=n+1}^m a_{p+k} z^k}. \end{aligned} \tag{2.37}$$

By applying (2.35) and (2.37), we find that

$$\begin{aligned}
 & \left| \frac{\psi_1(z) - 1}{\psi_1(z) + 1} \right| \\
 &= \left| \frac{-d_{p+m+1} \sum_{k=m+1}^{\infty} a_{p+k} z^k}{2 - \left( \frac{2p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)} \right) z^n - 2 \sum_{k=n+1}^m a_{p+k} z^k - d_{p+m+1} \sum_{k=m+1}^{\infty} a_{p+k} z^k} \right| \\
 &\leq \frac{d_{p+m+1} \sum_{k=m+1}^{\infty} a_{p+k} z^{p+k}}{2 \left[ 1 - \left( \frac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)} \right) - \sum_{k=n+1}^m a_{p+k} \right] - d_{p+m+1} \sum_{k=m+1}^{\infty} a_{p+k}} \\
 &\leq 1 \quad (z \in U; m \geq n+1),
 \end{aligned} \tag{2.38}$$

which shows that  $\operatorname{Re} \Psi_1(z) > 0$ . Thus, we obtain

$$\operatorname{Re} \Psi_1(z) = \operatorname{Re} \left\{ d_{p+m+1} \left[ \frac{f(z)}{s_m(z)} - \left( 1 - \frac{1}{d_{p+m+1}} \right) \right] \right\} > 0 \implies \operatorname{Re} \left\{ \frac{f(z)}{s_m(z)} \right\} > 1 - \frac{1}{d_{p+m+1}} \tag{2.39}$$

or

$$\operatorname{Re} \left\{ \frac{f(z)}{s_m(z)} \right\} > 1 - \frac{p^{\Omega}(p-\alpha)(1-\varepsilon)}{(p+m+1)^{\Omega} [1 + \lambda(m+1)p^{\Omega-1}] (p+m+1-\alpha)}. \tag{2.40}$$

Let

$$\begin{aligned}
 f(z) &= z^p - \frac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)} z^{p+n} \\
 &\quad - \frac{1 - p^{\Omega}(p-\alpha)\varepsilon / (p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)}{d_{p+m+1}} z^{p+m+1}.
 \end{aligned} \tag{2.41}$$

Then,  $f(z)$  satisfies the condition (2.31) and  $f \in \mathcal{I}_\varepsilon^*(\Omega, \lambda, p, \alpha)$ . Thus, we can write

$$\begin{aligned} \frac{f(z)}{s_m(z)} &= \frac{1 - \left(p^\Omega(p - \alpha)\varepsilon / (p + n)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^n}{1 - \left(p^\Omega(p - \alpha)\varepsilon / (p + n)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^n} \\ &\quad - \frac{\left(1 - \left(p^\Omega(p - \alpha)\varepsilon / (p + n)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)\right) / d_{p+m+1}z^{m+1}}{1 - \left(p^\Omega(p - \alpha)\varepsilon / (p + n)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^n} \\ &= 1 - \frac{1 - \left(\left(p^\Omega(p - \alpha)\varepsilon / (p + n)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)\right) / d_{p+m+1}z^{m+1}}{1 - \left(p^\Omega(p - \alpha)\varepsilon / (p + n)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^n} \end{aligned} \tag{2.42}$$

and taking as  $z \rightarrow 1^-$

$$\rightarrow 1 - \frac{1}{d_{p+m+1}} = 1 - \frac{p^\Omega(p - \alpha)(1 - \varepsilon)}{(p + m + 1)^\Omega [1 + \lambda(m + 1)p^{\Omega-1}](p + m + 1 - \alpha)}, \tag{2.43}$$

which shows that the bound in (2.32) is the best possible.

By using definitions of  $f(z)$  and  $s_m(z)$ , we can write

$$\begin{aligned} \frac{s_m(z)}{f(z)} &= \frac{z^p - \left((p - \alpha)\varepsilon / ((p + n)/p)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^{p+n} - \sum_{k=n+1}^m a_{p+k}z^{p+k}}{z^p - \left((p - \alpha)\varepsilon / ((p + n)/p)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^{p+n} - \sum_{k=n+1}^\infty a_{p+k}z^{p+k}} \\ &= \frac{z^p - \left((p - \alpha)\varepsilon / ((p + n)/p)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^{p+n}}{z^p - \left((p - \alpha)\varepsilon / ((p + n)/p)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^{p+n} - \sum_{k=n+1}^\infty a_{p+k}z^{p+k}} \\ &\quad - \frac{\sum_{k=n+1}^\infty a_{p+k}z^{p+k} + \sum_{k=m+1}^\infty a_{p+k}z^{p+k}}{z^p - \left((p - \alpha)\varepsilon / ((p + n)/p)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^{p+n} - \sum_{k=n+1}^\infty a_{p+k}z^{p+k}} \\ &= 1 + \frac{\sum_{k=m+1}^\infty a_{p+k}z^k}{1 - \left((p - \alpha)\varepsilon / ((p + n)/p)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^n - \sum_{k=n+1}^\infty a_{p+k}z^k}. \end{aligned} \tag{2.44}$$

If we put

$$\begin{aligned} \psi_2(z) &= (1 + d_{p+m+1}) \left\{ \frac{s_m(z)}{f(z)} - \frac{d_{p+m+1}}{1 + d_{p+m+1}} \right\} \\ &= 1 + \left( \frac{(1 + d_{p+m+1}) \sum_{k=m+1}^\infty a_{p+k}z^k}{1 - \left((p - \alpha)\varepsilon / ((p + n)/p)\right)^\Omega (1 + \lambda np^{\Omega-1})(n + p - \alpha)z^n - \sum_{k=n+1}^\infty a_{p+k}z^k} \right) \end{aligned} \tag{2.45}$$

and make use of (2.35), we can deduce that

$$\begin{aligned}
 & \left| \frac{\psi_2(z) - 1}{\psi_2(z) + 1} \right| \\
 &= \left| \frac{(1 + d_{p+m+1}) \sum_{k=m+1}^{\infty} a_{p+k} z^k}{2 - \left( \frac{2(p-\alpha)\varepsilon}{\left(\frac{p+n}{p}\right)^\Omega (1 + \lambda n p^{\Omega-1})(n+p-\alpha)} \right) z^{n-2} \sum_{k=n+1}^m a_{p+k} z^k + (d_{p+m+1} - 1) \sum_{k=m+1}^{\infty} a_{p+k} z^k} \right| \\
 &\leq \frac{(1 + d_{p+m+1}) \sum_{k=m+1}^{\infty} a_{p+k}}{2 - \left( \frac{2(p-\alpha)\varepsilon}{\left(\frac{p+n}{p}\right)^\Omega (1 + \lambda n p^{\Omega-1})(n+p-\alpha)} \right) - 2 \sum_{k=n+1}^m a_{p+k} - (d_{p+m+1} - 1) \sum_{k=m+1}^{\infty} a_{p+k}} \\
 &= \frac{(1 + d_{p+m+1}) \sum_{k=m+1}^{\infty} a_{p+k}}{2 \left\{ 1 - \left( \frac{(p-\alpha)\varepsilon}{\left(\frac{p+n}{p}\right)^\Omega (1 + \lambda n p^{\Omega-1})(n+p-\alpha)} \right) - \sum_{k=n+1}^m a_{p+k} \right\} - (d_{p+m+1} - 1) \sum_{k=m+1}^{\infty} a_{p+k}} \\
 &\leq \frac{(1 + d_{p+m+1}) \sum_{k=m+1}^{\infty} a_{p+k}}{2d_{p+m+1} \sum_{k=m+1}^{\infty} a_{p+k} - (d_{p+m+1} - 1) \sum_{k=m+1}^{\infty} a_{p+k}} = 1,
 \end{aligned} \tag{2.46}$$

$|\psi_2(z) - 1|/|\psi_2(z) + 1| \leq 1$  requires that  $\text{Re}\{\psi_2(z)\} > 0$ . Thus, we obtain

$$\text{Re}\{\Psi_2(z)\} = (1 + d_{p+m+1}) \text{Re} \left[ \frac{s_m(z)}{f(z)} - \left( \frac{d_{p+m+1}}{1 + d_{p+m+1}} \right) \right] > 0 \tag{2.47}$$

or

$$\text{Re} \left\{ \frac{s_m(z)}{f(z)} \right\} > \left( \frac{d_{p+m+1}}{1 + d_{p+m+1}} \right). \tag{2.48}$$

It follows from the last inequality that assertion (2.33) of the Theorem 2.5 holds.

The bound in (2.33) is sharp with the extremal function given by (2.41). The proof of the theorem is thus completed.  $\square$

If  $\Omega = 0$ ,  $\lambda = 0$ , and  $n = 1$  are taken in Theorem 2.5, the following result is obtained given by Liu [6].

**Corollary 2.6.** *Let  $f$  be given by (1.19) and define the partial sums  $s_m(z)$  by*

$$s_m(z) = \begin{cases} z^p - \frac{(p-\alpha)\varepsilon}{p+1-\alpha} z^{p+1}, & m = 1, \\ z^p - \frac{(p-\alpha)\varepsilon}{(n+p-\alpha)} z^{p+1} - \sum_{k=2}^m a_{p+k} z^{p+k}, & m = 2, 3, \dots \end{cases} \quad (2.49)$$

Suppose also that

$$\sum_{k=2}^{\infty} d_{p+k} a_{p+k} \leq 1 - \frac{(p-\alpha)\varepsilon}{p+1-\alpha} \quad \left( \text{where } d_{p+k} = \frac{k+p-\alpha}{(p-\alpha)(1-\varepsilon)}; 0 \leq \alpha < p, 0 \leq \varepsilon < 1 \right). \quad (2.50)$$

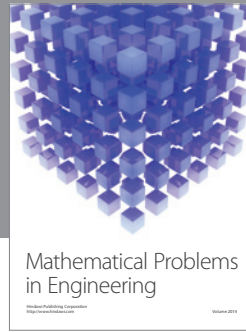
Then, for  $m \geq 2$ , one has

$$\begin{aligned} \operatorname{Re} \left\{ \frac{f(z)}{s_m(z)} \right\} &> \frac{(p-\alpha)\varepsilon + m + 1}{p + m + 1 - \alpha} \\ \operatorname{Re} \left( \frac{s_m(z)}{f(z)} \right) &> \frac{p + m + 1 - \alpha}{(p-\alpha)(2-\varepsilon) + m + 1}. \end{aligned} \quad (2.51)$$

Each of the bounds in (2.51) is the best possible.

## References

- [1] S. Owa, "Some properties of certain multivalent functions," *Applied Mathematics Letters*, vol. 4, no. 5, pp. 79–83, 1991.
- [2] D. A. Patil and N. K. Thakare, "On convex hulls and extreme points of  $p$ -valent starlike and convex classes with applications," *Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie*, vol. 27(75), no. 2, pp. 145–160, 1983.
- [3] S. Owa, "On certain classes of  $p$ -valent functions with negative coefficients," *Simon Stevin*, vol. 59, no. 4, pp. 385–402, 1985.
- [4] M. K. Aouf, H. M. Hossen, and H. M. Srivastava, "Some families of multivalent functions," *Computers & Mathematics with Applications*, vol. 39, no. 7-8, pp. 39–48, 2000.
- [5] H. Silverman and E. M. Silvia, "Fixed coefficients for subclasses of starlike functions," *Houston Journal of Mathematics*, vol. 7, no. 1, pp. 129–136, 1981.
- [6] J.-L. Liu, "Some further properties of certain class of multivalent analytic functions," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 25, no. 4, pp. 369–376, 2009.



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