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Research Article

Weighted Asymptotically Periodic Solutions of Linear Volterra Difference Equations

Josef Diblík,[1,2] Miroslava Růžičková,[3] Ewa Schmeidel,[4] and Małgorzata Zbąszyniak[4]

[1] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 66237 Brno, Czech Republic*
[2] *Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, 61600 Brno, Czech Republic*
[3] *Department of Mathematics, University of Žilina, 01026 Žilina, Slovakia*
[4] *Faculty of Electrical Engineering, Institute of Mathematics, Poznań University of Technology, 60965 Poznań, Poland*

Correspondence should be addressed to Ewa Schmeidel, ewa.schmeidel@put.poznan.pl

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A linear Volterra difference equation of the form $x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i)$, where $x : \mathbb{N}_0 \to \mathbb{R}$, $a : \mathbb{N}_0 \to \mathbb{R}$, $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ and $b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ is ω-periodic, is considered. Sufficient conditions for the existence of weighted asymptotically periodic solutions of this equation are obtained. Unlike previous investigations, no restriction on $\prod_{j=0}^{\omega-1} b(j)$ is assumed. The results generalize some of the recent results.

1. Introduction

In the paper, we study a linear Volterra difference equation

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i), \tag{1.1}$$

where $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $a : \mathbb{N}_0 \to \mathbb{R}$, $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$, and $b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ is ω-periodic, $\omega \in \mathbb{N} := \{1, 2, \ldots\}$. We will also adopt the customary notations

$$\sum_{i=k+s}^{k} \mathcal{O}(i) = 0, \qquad \prod_{i=k+s}^{k} \mathcal{O}(i) = 1, \tag{1.2}$$

where k is an integer, s is a positive integer, and "$\mathcal{O}$" denotes the function considered independently of whether it is defined for the arguments indicated or not.

In [1], the authors considered (1.1) under the assumption

$$\prod_{j=0}^{\omega-1} b(j) = 1, \tag{1.3}$$

and gave sufficient conditions for the existence of asymptotically ω-periodic solutions of (1.1) where the notion for an asymptotically ω-periodic function has been given by the following definition.

Definition 1.1. Let ω be a positive integer. The sequence $y : \mathbb{N}_0 \to \mathbb{R}$ is called ω-periodic if $y(n+\omega) = y(n)$ for all $n \in \mathbb{N}_0$. The sequence y is called asymptotically ω-periodic if there exist two sequences $u, v : \mathbb{N}_0 \to \mathbb{R}$ such that u is ω-periodic, $\lim_{n\to\infty} v(n) = 0$, and

$$y(n) = u(n) + v(n) \tag{1.4}$$

for all $n \in \mathbb{N}_0$.

In this paper, in general, we do not assume that (1.3) holds. Then, we are able to derive sufficient conditions for the existence of a weighted asymptotically ω-periodic solution of (1.1). We give a definition of a weighted asymptotically ω-periodic function.

Definition 1.2. Let ω be a positive integer. The sequence $y : \mathbb{N}_0 \to \mathbb{R}$ is called weighted asymptotically ω-periodic if there exist two sequences $u, v : \mathbb{N}_0 \to \mathbb{R}$ such that u is ω-periodic and $\lim_{n\to\infty} v(n) = 0$, and, moreover, if there exists a sequence $w : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ such that

$$\frac{y(n)}{w(n)} = u(n) + v(n), \tag{1.5}$$

for all $n \in \mathbb{N}_0$.

Apart from this, when we assume

$$\prod_{k=0}^{\omega-1} b(k) = -1, \tag{1.6}$$

then, as a consequence of our main result (Theorem 2.2), the existence of an asymptotically 2ω-periodic solution of (1.1) is obtained.

For the reader's convenience, we note that the background for discrete Volterra equations can be found, for example, in the well-known monograph by Agarwal [2], as well as by Elaydi [3] or Kocić and Ladas [4]. Volterra difference equations were studied by many others, for example, by Appleby et al. [5], by Elaydi and Murakami [6], by Győri and Horváth [7], by Győri and Reynolds [8], and by Song and Baker [9]. For some results on periodic solutions of difference equations, see, for example, [2–4, 10–13] and the related references therein.

2. Weighted Asymptotically Periodic Solutions

In this section, sufficient conditions for the existence of weighted asymptotically ω-periodic solutions of (1.1) will be derived. The following version of Schauder's fixed point theorem given in [14] will serve as a tool used in the proof.

Lemma 2.1. *Let Ω be a Banach space and S its nonempty, closed, and convex subset and let T be a continuous mapping such that $T(S)$ is contained in S and the closure $\overline{T(S)}$ is compact. Then, T has a fixed point in S.*

We set

$$\beta(n) := \prod_{j=0}^{n-1} b(j), \quad n \in \mathbb{N}_0, \tag{2.1}$$

$$\mathcal{B} := \beta(\omega). \tag{2.2}$$

Moreover, we define

$$n^* := n - 1 - \omega \left\lfloor \frac{n-1}{\omega} \right\rfloor, \tag{2.3}$$

where $\lfloor \cdot \rfloor$ is the floor function (the greatest-integer function) and n^* is the "remainder" of dividing $n-1$ by ω. Obviously, $\{\beta(n^*)\}$, $n \in \mathbb{N}$ is an ω-periodic sequence.

Now, we derive sufficient conditions for the existence of a weighted asymptotically ω-periodic solution of (1.1).

Theorem 2.2 (Main result). *Let ω be a positive integer, $b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ be ω-periodic, $a : \mathbb{N}_0 \to \mathbb{R}$, and $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$. Assume that*

$$\begin{gathered} \sum_{i=0}^{\infty} \left| \frac{a(i)}{\beta(i+1)} \right| < \infty, \\ \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left| \frac{K(j,i)\beta(i)}{\beta(j+1)} \right| < 1, \end{gathered} \tag{2.4}$$

and that at least one of the real numbers in the left-hand sides of inequalities (2.4) is positive.

Then, for any nonzero constant c, there exists a weighted asymptotically ω-periodic solution $x : \mathbb{N}_0 \to \mathbb{R}$ of (1.1) with $u, v : \mathbb{N}_0 \to \mathbb{R}$ and $w : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ in representation (1.5) such that

$$w(n) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor}, \qquad u(n) := c\beta(n^*+1), \qquad \lim_{n\to\infty} v(n) = 0, \tag{2.5}$$

that is,

$$\frac{x(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c\beta(n^*+1) + v(n), \quad n \in \mathbb{N}_0. \tag{2.6}$$

Proof. We will use a notation

$$M := \sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right|, \tag{2.7}$$

whenever this is useful.

Case 1. First assume $c > 0$. We will define an auxiliary sequence of positive numbers $\{\alpha(n)\}$, $n \in \mathbb{N}_0$. We set

$$\alpha(0) := \frac{\sum_{i=0}^{\infty}|a(i)/(\beta(i+1))| + c\sum_{j=0}^{\infty}\sum_{i=0}^{j}|(K(j,i)\beta(i))/(\beta(j+1))|}{1-\sum_{j=0}^{\infty}\sum_{i=0}^{j}|(K(j,i)\beta(i))/(\beta(j+1))|}, \tag{2.8}$$

where the expression on the right-hand side is well defined due to (2.4). Moreover, we define

$$\alpha(n) := \sum_{i=n}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right|, \tag{2.9}$$

for $n \geq 1$. It is easy to see that

$$\lim_{n\to\infty}\alpha(n) = 0. \tag{2.10}$$

We show, moreover, that

$$\alpha(n) \leq \alpha(0), \tag{2.11}$$

for any $n \in \mathbb{N}$. Let us first remark that

$$\alpha(0) = \sum_{i=0}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right|. \tag{2.12}$$

Then, due to the convergence of both series (see (2.4)), the inequality

$$\begin{aligned}\alpha(0) &= \sum_{i=0}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| \\ &\geq \sum_{i=n}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| = \alpha(n)\end{aligned} \tag{2.13}$$

obviously holds for every $n \in \mathbb{N}$ and (2.11) is proved.

Let B be the Banach space of all real bounded sequences $z : \mathbb{N}_0 \to \mathbb{R}$ equipped with the usual supremum norm $\|z\| = \sup_{n\in\mathbb{N}_0}|z(n)|$ for $z \in B$. We define a subset $S \subset B$ as

$$S := \{z \in B : c - \alpha(0) \le z(n) \le c + \alpha(0),\ n \in \mathbb{N}_0\}. \tag{2.14}$$

It is not difficult to prove that S is a nonempty, bounded, convex, and closed subset of B.

Let us define a mapping $T : S \to B$ as follows:

$$(Tz)(n) = c - \sum_{i=n}^{\infty}\frac{a(i)}{\beta(i+1)} - \sum_{j=n}^{\infty}\sum_{i=0}^{j}\frac{K(j,i)\beta(i)}{\beta(j+1)}z(i), \tag{2.15}$$

for any $n \in \mathbb{N}_0$.

We will prove that the mapping T has a fixed point in S.

We first show that $T(S) \subset S$. Indeed, if $z \in S$, then $|z(n) - c| \le \alpha(0)$ for $n \in \mathbb{N}_0$ and, by (2.11) and (2.15), we have

$$|(Tz)(n) - c| \le \sum_{i=n}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c + \alpha(0))\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| = \alpha(n) \le \alpha(0). \tag{2.16}$$

Next, we prove that T is continuous. Let $z^{(p)}$ be a sequence in S such that $z^{(p)} \to z$ as $p \to \infty$. Because S is closed, $z \in S$. Now, utilizing (2.15), we get

$$\begin{aligned}\left|\left(Tz^{(p)}\right)(n) - (Tz)(n)\right| &= \left|\sum_{j=n}^{\infty}\sum_{i=0}^{j}\frac{K(j,i)\beta(i)}{\beta(j+1)}\left(z^{(p)}(i) - z(i)\right)\right| \\ &\le M\sup_{i\ge 0}\left|z^{(p)}(i) - z(i)\right| = M\left\|z^{(p)} - z\right\|, \quad n \in \mathbb{N}_0.\end{aligned} \tag{2.17}$$

Therefore,

$$\begin{gathered}\left\|Tz^{(p)} - Tz\right\| \le M\left\|z^{(p)} - z\right\|, \\ \lim_{p\to\infty}\left\|Tz^{(p)} - Tz\right\| = 0.\end{gathered} \tag{2.18}$$

This means that T is continuous.

Now, we show that $\overline{T(S)}$ is compact. As is generally known, it is enough to verify that every ε-open covering of $\overline{T(S)}$ contains a finite ε-subcover of $\overline{T(S)}$, that is, finitely many of these open sets already cover $\overline{T(S)}$ ([15], page 756 (12)). Thus, to prove that $\overline{T(S)}$ is compact, we take an arbitrary $\varepsilon > 0$ and assume that an open ε-cover $\mathcal{C}_\varepsilon$ of $\overline{T(S)}$ is given. Then, from (2.10), we conclude that there exists an $n_\varepsilon \in \mathbb{N}$ such that $\alpha(n) < \varepsilon/4$ for $n \ge n_\varepsilon$.

Suppose that $x_T^1 \in \overline{T(S)}$ is one of the elements generating the ε-cover $\mathcal{C}_\varepsilon$ of $\overline{T(S)}$. Then (as follows from (2.16)), for an arbitrary $x_T \in \overline{T(\mathrm{S})}$,

$$\left| x_T^1(n) - x_T(n) \right| < \varepsilon \tag{2.19}$$

if $n \geq n_\varepsilon$. In other words, the ε-neighborhood of $x_T^1 - c^*$:

$$\left\| x_T^1 - c^* \right\| < \varepsilon, \tag{2.20}$$

where $c^* = \{c, c, \ldots\} \in S$ covers the set $\overline{T(S)}$ on an infinite interval $n \geq n_\varepsilon$. It remains to cover the rest of $\overline{T(S)}$ on a finite interval for $n \in \{0, 1 \ldots, n_\varepsilon - 1\}$ by a finite number of ε-neighborhoods of elements generating ε-cover $\mathcal{C}_\varepsilon$. Supposing that x_T^1 itself is not able to generate such cover, we fix $n \in \{0, 1, \ldots, n_\varepsilon - 1\}$ and split the interval

$$[c - \alpha(n), c + \alpha(n)] \tag{2.21}$$

into a finite number $h(\varepsilon, n)$ of closed subintervals

$$I_1(n), I_2(n), \ldots, I_{h(\varepsilon,n)}(n) \tag{2.22}$$

each with a length not greater then $\varepsilon/2$ such that

$$\begin{gathered} \bigcup_{i=1}^{h(\varepsilon,n)} I_i(n) = [c - \alpha(n), c + \alpha(n)], \\ \operatorname{int} I_i(n) \cap \operatorname{int} I_j(n) = \emptyset, \quad i, j = 1, 2, \ldots, h(\varepsilon, n),\ i \neq j. \end{gathered} \tag{2.23}$$

Finally, the set

$$\bigcup_{n=0}^{n_\varepsilon - 1} [c - \alpha(n), c + \alpha(n)] \tag{2.24}$$

equals

$$\bigcup_{n=0}^{n_\varepsilon - 1} \bigcup_{i=1}^{h(\varepsilon,n)} I_i(n) \tag{2.25}$$

and can be divided into a finite number

$$M_\varepsilon := \sum_{n=0}^{n_\varepsilon - 1} h(\varepsilon, n) \tag{2.26}$$

of different subintervals (2.22). This means that, at most, M_ε of elements generating the cover $\mathcal{C}_\varepsilon$ are sufficient to generate a finite ε-subcover of $\overline{T(S)}$ for $n \in \{0, 1, \ldots, n_\varepsilon - 1\}$. We remark that each of such elements simultaneously plays the same role as $x_T^1(n)$ for $n \geq n_\varepsilon$. Since $\varepsilon > 0$ can be chosen as arbitrarily small, $\overline{T(S)}$ is compact.

By Schauder's fixed point theorem, there exists a $z \in S$ such that $z(n) = (Tz)(n)$ for $n \in \mathbb{N}_0$. Thus,

$$z(n) = c - \sum_{i=n}^{\infty} \frac{a(i)}{\beta(i+1)} - \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{\beta(i)}{\beta(j+1)} K(j,i) z(i), \tag{2.27}$$

for any $n \in \mathbb{N}_0$.

Due to (2.10) and (2.16), for fixed point $z \in S$ of T, we have

$$\lim_{n \to \infty} |z(n) - c| = \lim_{n \to \infty} |(Tz)(n) - c| \leq \lim_{n \to \infty} \alpha(n) = 0, \tag{2.28}$$

or, equivalently,

$$\lim_{n \to \infty} z(n) = c. \tag{2.29}$$

Finally, we will show that there exists a connection between the fixed point $z \in S$ and the existence of a solution of (1.1) which divided by $\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}$ provides an asymptotically ω-periodic sequence. Considering (2.27) for $z(n+1)$ and $z(n)$, we get

$$\Delta z(n) = \frac{a(n)}{\beta(n+1)} + \sum_{i=0}^{n} \frac{\beta(i)}{\beta(n+1)} K(n,i) z(i), \tag{2.30}$$

where $n \in \mathbb{N}_0$. Hence, we have

$$z(n+1) - z(n) = \frac{a(n)}{\beta(n+1)} + \frac{1}{\beta(n+1)} \sum_{i=0}^{n} \beta(i) K(n,i) z(i), \quad n \in \mathbb{N}_0. \tag{2.31}$$

Putting

$$z(n) = \frac{x(n)}{\beta(n)}, \quad n \in \mathbb{N}_0 \tag{2.32}$$

in (2.31), we get (1.1) since

$$\frac{x(n+1)}{\beta(n+1)} - \frac{x(n)}{\beta(n)} = \frac{a(n)}{\beta(n+1)} + \frac{1}{\beta(n+1)} \sum_{i=0}^{n} K(n,i) x(i), \quad n \in \mathbb{N}_0 \tag{2.33}$$

yields

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i), \quad n \in \mathbb{N}_0. \tag{2.34}$$

Consequently, x defined by (2.32) is a solution of (1.1). From (2.29) and (2.32), we obtain

$$\frac{x(n)}{\beta(n)} = z(n) = c + o(1), \tag{2.35}$$

for $n \to \infty$ (where $o(1)$ is the Landau order symbol). Hence,

$$x(n) = \beta(n)(c + o(1)), \quad n \longrightarrow \infty. \tag{2.36}$$

It is easy to show that the function β defined by (2.1) can be expressed in the form

$$\beta(n) = \prod_{j=0}^{n-1} b(j) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor} \cdot \beta(n^* + 1), \tag{2.37}$$

for $n \in \mathbb{N}_0$. Then, as follows from (2.36),

$$x(n) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor} \cdot \beta(n^* + 1)(c + o(1)), \quad n \longrightarrow \infty, \tag{2.38}$$

or

$$\frac{x(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c\beta(n^* + 1) + \beta(n^* + 1)o(1), \quad n \longrightarrow \infty. \tag{2.39}$$

The proof is completed since the sequence $\{\beta(n^* + 1)\}$ is ω-periodic, hence bounded and, due to the properties of Landau order symbols, we have

$$\beta(n^* + 1)o(1) = o(1), \quad n \longrightarrow \infty, \tag{2.40}$$

and it is easy to see that the choice

$$u(n) := c\beta(n^* + 1), \qquad w(n) := \mathcal{B}^{\lfloor (n-1)/\omega \rfloor}, \quad n \in \mathbb{N}_0, \tag{2.41}$$

and an appropriate function $v : \mathbb{N}_0 \to \mathbb{R}$ such that

$$\lim_{n \to \infty} v(n) = 0 \tag{2.42}$$

finishes this part of the proof. Although for $n = 0$, there is no correspondence between formula (2.36) and the definitions of functions u and w, we assume that function v makes up for this.

Case 2. If $c < 0$, we can proceed as follows. It is easy to see that arbitrary solution $y = y(n)$ of the equation

$$y(n+1) = -a(n) + b(n)y(n) + \sum_{i=0}^{n} K(n,i)y(i) \tag{2.43}$$

defines a solution $x = x(n)$ of (1.1) since a substitution $y(n) = -x(n)$ in (2.43) turns (2.43) into (1.1). If the assumptions of Theorem 2.2 hold for (1.1), then, obviously, Theorem 2.2 holds for (2.43) as well. So, for an arbitrary $c > 0$, (2.43) has a solution that can be represented by formula (2.6), that is,

$$\frac{y(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c\beta(n^* + 1) + v(n), \quad n \in \mathbb{N}_0. \tag{2.44}$$

Or, in other words, (1.1) has a solution that can be represented by formula (2.44) as

$$\frac{x(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c_0\beta(n^* + 1) + v^*(n), \quad n \in \mathbb{N}_0, \tag{2.45}$$

with $c_0 = -c$ and $v^*(n) = -v(n)$. In (2.45), $c_0 < 0$ and the function $v^*(n)$ has the same properties as the function $v(n)$. Therefore, formula (2.6) is valid for an arbitrary negative c as well. □

Now, we give an example which illustrates the case where there exists a solution to equation of the type (1.1) which is weighted asymptotically periodic, but is not asymptotically periodic.

Example 2.3. We consider (1.1) with

$$\begin{gathered} a(n) = (-1)^{n+1}\left(1 - \frac{1}{3^{n+1}}\right), \\ b(n) = 3(-1)^n, \\ K(n,i) = (-1)^{n+(i(i-1))/2}\frac{1}{3^{2i}}, \end{gathered} \tag{2.46}$$

that is, the equation

$$x(n+1) = (-1)^{n+1}\left(1 - \frac{1}{3^{n+1}}\right) + 3(-1)^n x(n) + \sum_{i=0}^{n} (-1)^{n+(i(i-1))/2}\frac{1}{3^{2i}}x(i). \tag{2.47}$$

The sequence $b(n)$ is 2-periodic and

$$\begin{gathered}
\beta(n) = \prod_{j=0}^{n-1} b(j) = (-1)^{n(n-1)/2} 3^n , \\
\mathcal{B} = \beta(\omega) = \beta(2) = -9, \\
\beta(n^* + 1) = -3 + 6(-1)^{n+1}, \\
\frac{a(n)}{\beta(n+1)} = (-1)^{(-n^2+n+2)/2}\left(\frac{1}{3^{n+1}} - \frac{1}{3^{2(n+1)}}\right), \\
\sum_{i=0}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| < \infty, \\
\begin{aligned}
\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| &< \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{1}{3^{i+j+1}} \\
&= \frac{1}{3}\left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right)\left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right) = \frac{1}{3}\cdot\frac{1}{1-1/3}\cdot\frac{1}{1-1/3} \\
&= \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{3}{2} = \frac{3}{4} < 1.
\end{aligned}
\end{gathered} \tag{2.48}$$

By virtue of Theorem 2.2, for any nonzero constant c, there exists a solution $x : \mathbb{N}_0 \to \mathbb{R}$ of (1.1) which is weighed asymptotically 2-periodic. Let, for example, $c = 2/3$. Then,

$$\begin{gathered}
w(n) = (-9)^{\lfloor (n-1)/2 \rfloor}, \\
u(n) = c\beta(n^* + 1) = \frac{2}{3}\left(-3 + 6(-1)^{n+1}\right) = -2 + 4(-1)^{n+1},
\end{gathered} \tag{2.49}$$

and the sequence $x(n)$ given by

$$\frac{x(n)}{(-9)^{\lfloor (n-1)/2 \rfloor}} = -2 + 4(-1)^{n+1} + v(n), \quad n \in \mathbb{N}_0, \tag{2.50}$$

or, equivalently,

$$x(n) = (-9)^{\lfloor (n-1)/2 \rfloor}\left(-2 + 4(-1)^{n+1}\right) + v(n), \quad n \in \mathbb{N}_0 \tag{2.51}$$

is such a solution. We remark that such solution is not asymptotically 2-periodic in the meaning of Definition 1.1.

It is easy to verify that the sequence $x^*(n)$ obtained from (2.51) if $v(n) = 0$, $n \in \mathbb{N}_0$, that is,

$$x^*(n) = (-9)^{\lfloor (n-1)/2 \rfloor}\left(-2 + 4(-1)^{n+1}\right) = \frac{2}{3} \cdot (-1)^{n(n-1)/2} \cdot 3^n, \quad n \in \mathbb{N}_0 \tag{2.52}$$

is a true solution of (2.47).

3. Concluding Remarks and Open Problems

It is easy to prove the following corollary.

Corollary 3.1. *Let Theorem 2.2 be valid. If, moreover, $|\mathcal{B}| < 1$, then every solution $x = x(n)$ of* (1.1) *described by formula* (2.6) *satisfies*

$$\lim_{n \to \infty} x(n) = 0. \tag{3.1}$$

If $|\mathcal{B}| > 1$, then, for every solution $x = x(n)$ of (1.1) *described by formula* (2.6), *one has*

$$\liminf_{n \to \infty} x(n) = -\infty \tag{3.2}$$

or/and

$$\limsup_{n \to \infty} x(n) = \infty. \tag{3.3}$$

Finally, if $\mathcal{B} > 1$, then, for every solution $x = x(n)$ of (1.1) *described by formula* (2.6), *one has*

$$\lim_{n \to \infty} x(n) = \infty, \tag{3.4}$$

and if $\mathcal{B} < -1$, then, for every solution $x = x(n)$ of (1.1) *described by formula* (2.6), *one has*

$$\lim_{n \to \infty} x(n) = -\infty. \tag{3.5}$$

Now, let us discuss the case when (1.6) holds, that is, when

$$\mathcal{B} = \prod_{j=0}^{\omega-1} b(j) = -1. \tag{3.6}$$

Corollary 3.2. *Let Theorem 2.2 be valid. Assume that $\mathcal{B} = -1$. Then, for any nonzero constant c, there exists an asymptotically 2ω-periodic solution $x = x(n)$, $n \in \mathbb{N}_0$ of* (1.1) *such that*

$$x(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) + z(n), \quad n \in \mathbb{N}_0, \tag{3.7}$$

with

$$u(n) := c\beta(n^* + 1), \qquad \lim_{n\to\infty} z(n) = 0. \tag{3.8}$$

Proof. Putting $\mathcal{B} = -1$ in Theorem 2.2, we get

$$x(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) + (-1)^{\lfloor (n-1)/\omega \rfloor} v(n), \tag{3.9}$$

with

$$u(n) := c\beta(n^* + 1), \qquad \lim_{n\to\infty} v(n) = 0. \tag{3.10}$$

Due to the definition of n^*, we see that the sequence

$$\{\beta(n^* + 1)\} = \{\beta(\omega), \beta(1), \beta(2), \dots, \beta(\omega), \beta(1), \beta(2), \dots, \beta(\omega), \dots\}, \tag{3.11}$$

is an ω-periodic sequence. Since

$$\left\{\left\lfloor \frac{n-1}{\omega} \right\rfloor\right\} = \left\{-1, \underbrace{0, \dots, 0}_{\omega}, \underbrace{1, \dots, 1}_{\omega}, 2, \dots\right\}, \tag{3.12}$$

for $n \in \mathbb{N}_0$, we have

$$\left\{(-1)^{\lfloor (n-1)/\omega \rfloor}\right\} = \left\{-1, \underbrace{1, \dots, 1}_{\omega}, \underbrace{-1, \dots, -1}_{\omega}, 1, \dots\right\}. \tag{3.13}$$

Therefore, the sequence

$$\left\{(-1)^{\lfloor (n-1)/\omega \rfloor} u(n)\right\} = c\{-\beta(\omega), \beta(1), \beta(2), \dots, \beta(\omega), -\beta(1), -\beta(2), \dots, -\beta(\omega), \dots\} \tag{3.14}$$

is a 2ω-periodic sequence. Set

$$z(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} v(n). \tag{3.15}$$

Then,

$$\lim_{n\to\infty} z(n) = 0. \tag{3.16}$$

The proof is completed. □

Remark 3.3. From the proof, we see that Theorem 2.2 remains valid even in the case of $c = 0$. Then, there exists an "asymptotically weighted ω-periodic solution" $x = x(n)$ of (1.1) as well. The formula (2.6) reduces to

$$x(n) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor} v(n) = o(1), \quad n \in \mathbb{N}_0, \tag{3.17}$$

since $u(n) = 0$. In the light of Definition 1.2, we can treat this case as follows. We set (as a singular case) $u \equiv 0$ with an arbitrary (possibly other than "ω") period and with $v = o(1)$, $n \to \infty$.

Remark 3.4. The assumptions of Theorem 2.2 [1] are substantially different from those of the present Theorem 2.2. However, it is easy to see that Theorem 2.2 [1] is a particular case of the present Theorem 2.2 if (1.3) holds, that is, if $\mathcal{B} = 1$. Therefore, our results can be viewed as a generalization of some results in [1].

In connection with the above investigations, some open problems arise.

Open Problem 1. The results of [1] are extended to systems of linear Volterra discrete equations in [16, 17]. It is an open question if the results presented can be extended to systems of linear Volterra discrete equations.

Open Problem 2. Unlike the result of Theorem 2.2 [1] where a parameter c can be arbitrary, the assumptions of the results in [16, 17] are more restrictive since the related parameters should satisfy certain inequalities as well. Different results on the existence of asymptotically periodic solutions were recently proved in [8]. Using an example, it is shown that the results in [8] can be less restrictive. Therefore, an additional open problem arises if the results in [16, 17] can be improved in such a way that the related parameters can be arbitrary and if the expected extension of the results suggested in Open Problem 1 can be given in such a way that the related parameters can be arbitrary as well.

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