Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 392025, 14 pages
doi:10.1155/2011/392025

Research Article

On the q-Bernoulli Numbers and Polynomials with Weight α

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Received 13 May 2011; Accepted 26 July 2011

Academic Editor: Elena Litsyn

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We present a systemic study of some families of higher-order q-Bernoulli numbers and polynomials with weight α. From these studies, we derive some interesting identities on the q-Bernoulli numbers and polynomials with weight α.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm of $\mathbb{C}_p$ is defined as $|x|_p = p^{-r}$, where $x = p^r m/n$ with $(p,m) = (p,n) = 1$, $r \in \mathbb{Q}$ and $m, n \in \mathbb{Z}$. Let $\mathbb{N}$ and $\mathbb{Z}$ be the set of natural numbers and integers, respectively, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. The notation of q-number is defined by $[x]_w = (1-w^x)/(1-w)$ and $[x]_q = (1-q^x)/(1-q)$, (see [1–13]).

As the well known definition, the Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^x = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.1}$$

In the special case, $x = 0$, $B_n(0) = B_n$ are called the nth Bernoulli numbers. That is, the recurrence formula for the Bernoulli numbers is given by

$$B_0 = 1, \qquad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.2}$$

with the usual convention about replacing B^i with B_i.

In [1, 2], q-extension of Bernoulli numbers are defined by Carlitz as follows:

$$\beta_{0,q} = 1, \quad q(q\beta+1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.3}$$

with the usual convention about replacing β^i with $\beta_{i,q}$.

By (1.2) and (1.3), we get $\lim_{q\to 1}\beta_{i,q} = B_i$. In this paper, we assume that $\alpha \in \mathbb{N}$.

In [7], the q-Bernoulli numbers with weight α are defined by Kim as follows:

$$\widetilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q\left(q^\alpha \widetilde{\beta}^\alpha + 1\right)^n - \widetilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.4}$$

with the usual convention about replacing $(\widetilde{\beta}^{(\alpha)})^i$ with $\widetilde{\beta}_{i,q}^{(\alpha)}$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \tag{1.5}$$

(see[4, 5]). From (1.5), we note that

$$q^n I_q(f_n) = I_q(f) + (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{1.6}$$

where $f_n(x) = f(x+n)$ and $f'(l) = (df(x)/dx)|_{x=l}$.

By (1.4), (1.5), and (1.6), we set

$$\widetilde{\beta}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x), \quad \text{where } n \in \mathbb{Z}_+, \tag{1.7}$$

(see[7]). The q-Bernoulli polynomials are also given by

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \widetilde{\beta}_{l,q}^{(\alpha)}. \tag{1.8}$$

The purpose of this paper is to derive a new concept of higher-order q-Bernoulli numbers and polynomials with weight α from the fermionic p-adic q-integral on $\mathbb{Z}_p$. Finally, we present a systemic study of some families of higher-order q-Bernoulli numbers and polynomials with weight α.

2. Higher Order q-Bernoulli Numbers with Weight α

Let $\beta \in \mathbb{Z}$ and $\alpha \in \mathbb{N}$ in this paper. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we consider the expansion of higher-order q-Bernoulli polynomials with weight α as follows:

$$\widetilde{\beta}_{n,q}^{(\beta,k|\alpha)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{x_1(\beta-1)+\cdots+x_k(\beta-k)} d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned}\widetilde{\beta}_{n,q}^{(\beta,k|\alpha)}(x) &= \frac{(1-q)^{k-n}}{[\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\prod_{i=0}^{k-1} (\alpha l + \beta - i)}{\prod_{i=0}^{k-1} (1 - q^{\alpha l + \beta - i})} \\ &= \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l+\beta}{k} k!}{\binom{\alpha l+\beta}{k}_q [k]_q!},\end{aligned} \tag{2.2}$$

where $\binom{\alpha}{l}_q = ((1-q^\alpha)(1-q^{\alpha-1}) \cdots (1-q^{\alpha-l+1}))/((1-q)(1-q^2)\cdots(1-q^l))$ and $[k]_q! = [k]_q \cdots [2]_q [1]_q$.

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have*

$$\widetilde{\beta}_{n,q}^{(\beta,k|\alpha)}(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l+\beta}{k} k!}{\binom{\alpha l+\beta}{k}_q [k]_q!}. \tag{2.3}$$

In the special case, $x = 0$, $\widetilde{\beta}_{n,q}^{(\beta,k|\alpha)}(0) = \widetilde{\beta}_{n,q}^{(\beta,k|\alpha)}$ are called the nth higher order q-Bernoulli numbers with weight α.

From (2.1) and (2.2), we can derive

$$\widetilde{\beta}_{n,q}^{(\beta,k|\alpha)} = (q^{\alpha}-1)\widetilde{\beta}_{n+1,q}^{(\beta-\alpha,k|\alpha)} + \widetilde{\beta}_{n,q}^{(\beta-\alpha,k|\alpha)}. \tag{2.4}$$

By Theorem 2.1 and (2.4), we get

$$\begin{aligned}
\widetilde{\beta}_{0,q}^{(m\alpha,k|\alpha)} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k}(\alpha m-j)x_j}\, d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{l=0}^{m}\binom{m}{l}(q^{\alpha}-1)^{l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{l}\, q^{-\sum_{j=1}^{k} jx_j}\, d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{l=0}^{m}\binom{m}{l}(q^{\alpha}-1)^{l}\widetilde{\beta}_{l,q}^{(0,k|\alpha)} \\
&= \frac{(1-q)^{k}}{\prod_{i=0}^{k-1}(1-q^{\alpha m-k+1+i})} = (1-q)^{k}\sum_{l=0}^{k}\binom{k+l-1}{l}_{q} q^{(\alpha m-k+1)l}.
\end{aligned} \tag{2.5}$$

From (2.1), we have

$$\begin{aligned}
&\sum_{j=0}^{i}\binom{i}{j}(q^{\alpha}-1)^{j}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{n-i+j}\, q^{(\beta-\alpha-1)x_1+\cdots+(\beta-\alpha-k)x_k}\, d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{n-i}\, q^{(\beta-1)x_1+\cdots+(\beta-k)x_k}\, q^{\alpha(x_1+\cdots+x_k)(i-1)}\, d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad= \sum_{j=0}^{i-1}\binom{i-1}{j}(q^{\alpha}-1)^{j}\widetilde{\beta}_{n-i+j,q}^{(\beta,k|\alpha)}.
\end{aligned} \tag{2.6}$$

Thus, we obtain the following theorem.

Theorem 2.2. *For $i \in \mathbb{N}$, we have*

$$\sum_{j=0}^{i}\binom{i}{j}(q^{\alpha}-1)^{j}\widetilde{\beta}_{n-i+j,q}^{(\beta-\alpha,k|\alpha)} = \sum_{j=0}^{i-1}\binom{i-1}{j}(q^{\alpha}-1)^{j}\widetilde{\beta}_{n-i+j,q}^{(\beta,k|\alpha)}. \tag{2.7}$$

It is easy to show that

$$\begin{aligned}
\sum_{j=0}^{m}\binom{m}{j}(q^{\alpha}-1)^{j}\widetilde{\beta}_{j,q}^{(0,k|\alpha)} &= (1-q)^{k}\sum_{l=0}^{k}\binom{k+l-1}{l}_{q} q^{(\alpha m-k+1)l} \\
&= \frac{(1-q)^{k}}{\prod_{i=0}^{k-1}(1-q^{\alpha m-k+1+i})}.
\end{aligned}$$

3. Polynomials $\widetilde{\beta}_{n,q}^{(0,k|\alpha)}(x)$

In this section, we consider the polynomials $\widetilde{\beta}_{n,q}^{(0,k|\alpha)}(x)$ as follows:

$$\widetilde{\beta}_{n,q}^{(0,k|\alpha)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k+x]_{q^\alpha}^n q^{-\sum_{j=1}^k jx_j}d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{3.1}$$

From (3.1), we can easily derive the following equation:

$$\begin{aligned}\widetilde{\beta}_{n,q}^{(0,k|\alpha)}(x)&=\frac{(1-q)^k}{(1-q^\alpha)^n}\sum_{l=0}^n\binom{n}{l}(-1)^l q^{\alpha lx}\frac{\prod_{i=0}^k(\alpha l-i)}{\prod_{i=0}^{k-1}(1-q^{\alpha l-i})}\\&=\frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^n\binom{n}{l}(-1)^l q^{\alpha lx}\frac{\binom{\alpha l}{k}k!}{\binom{\alpha l}{k}_q[k]_q!}.\end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{j=1}^k(\alpha n-j)x_j+\alpha nx}d\mu_q(x_1)\cdots d\mu_q(x_k)=\sum_{l=0}^n\binom{n}{l}[\alpha]_q^l(q-1)^l\widetilde{\beta}_{l,q}^{(0,k|\alpha)}(x), \tag{3.3}$$

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{l=1}^k(\alpha n-l)x_l+\alpha nx}d\mu_q(x_1)\cdots d\mu_q(x_k)\\&\quad=\frac{q^{\alpha nx}(1-q)^k\left(\prod_{j=0}^{k-1}(\alpha n-j)\right)}{\prod_{j=0}^{k-1}(1-q^{\alpha n-j})}=\frac{q^{\alpha nx}\binom{\alpha n}{k}k!}{\binom{\alpha n}{k}_q[k]_q!}.\end{aligned} \tag{3.4}$$

Therefore, by (3.3) and (3.4), we obtain the following theorem.

Theorem 3.1. *For $n\in\mathbb{Z}_+$, we have*

$$(1-q)^n\widetilde{\beta}_{n,q}^{(0,k|\alpha)}(x)=\frac{1}{[\alpha]_q^n}\sum_{l=0}^n\binom{n}{l}(-1)^l q^{\alpha lx}\frac{\binom{\alpha l}{k}k!}{\binom{\alpha l}{k}_q[k]_q!}. \tag{3.5}$$

Moreover,

$$\sum_{l=0}^n\binom{n}{l}[\alpha]_q^l(q-1)^l\widetilde{\beta}_{l,q}^{(0,k|\alpha)}(x)=\frac{q^{\alpha nx}\binom{\alpha n}{k}k!}{\binom{\alpha n}{k}_q[k]_q!}. \tag{3.6}$$

Let $d \in \mathbb{N}$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{-\sum_{j=1}^k j x_j} d\mu_q(x_1) \cdots d\mu_q(x_k) \\
&\quad = \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{a_1,\cdots,a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1) a_j} \\
&\qquad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\frac{x + \sum_{j=1}^k a_j}{d} + \sum_{i=1}^k x_i \right]_{q^{\alpha d}}^n q^{-d \sum_{j=1}^k j x_j} d\mu_{q^d}(x_1) \cdots d\mu_{q^d}(x_k).
\end{aligned} \tag{3.7}$$

Thus, by (3.1) and (3.7), we obtain the following theorem.

Theorem 3.2. *For $d, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, we have*

$$\tilde{\beta}_{n,q}^{(0,k|\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{a_1,\ldots,a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1) a_j} \tilde{\beta}_{n,q^\alpha}^{(0,k|\alpha)} \left(\frac{x + a_1 + \cdots + a_k}{d} \right). \tag{3.8}$$

From (3.1), we note that

$$\begin{aligned}
\tilde{\beta}_{n,q}^{(0,k|\alpha)}(x) &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(0,k|\alpha)}, \\
\tilde{\beta}_{n,q}^{(0,k|\alpha)}(x+y) &= \sum_{l=0}^n \binom{n}{l} [y]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(0,k|\alpha)}(x).
\end{aligned} \tag{3.9}$$

4. Polynomials $\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x)$

For $h \in \mathbb{Z}$, let us define weighted (h, q)-Bernoulli polynomials $\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x)$ as follows:

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_q(x_1). \tag{4.1}$$

By (4.1), we easily see that

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \frac{1}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l + h}{[\alpha l + h]_q}. \tag{4.2}$$

Therefore, by (4.2), we obtain the following theorem.

Theorem 4.1. *For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have*

$$\widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \frac{1}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l + h}{[\alpha l + h]_q}. \tag{4.3}$$

From (4.1), we can derive the following equation:

$$\begin{aligned} q^{\alpha x} \int_{\mathbb{Z}_p} & [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_q(x_1) \\ &= (q^\alpha - 1) \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^{n+1} q^{x_1(h-\alpha-1)} d\mu_q(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-\alpha-1)} d\mu_q(x_1). \end{aligned} \tag{4.4}$$

By (4.4), we easily get

$$q^{\alpha x} \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = (q^\alpha - 1) \widetilde{\beta}_{n+1,q}^{(h-\alpha-1,1|\alpha)}(x) + \widetilde{\beta}_{n,q}^{(h-\alpha-1,1|\alpha)}(x). \tag{4.5}$$

From (4.1), we have

$$\widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_q(x_1) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \widetilde{\beta}_{l,q}^{(h,1|\alpha)}, \tag{4.6}$$

where $\widetilde{\beta}_{l,q}^{(h,1|\alpha)}(0) = \widetilde{\beta}_{l,q}^{(h,1|\alpha)}$.

By (4.6), we get the following recurrence formula:

$$\widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \left(q^{\alpha x} \widetilde{\beta}_q^{(h,1|\alpha)} + [x]_{q^\alpha} \right)^n, \quad \text{for } n \geq 1, \tag{4.7}$$

with the usual convention about replacing $(\widetilde{\beta}_q^{(h,1|\alpha)})^n$ with $\widetilde{\beta}_{n,q}^{(h,1|\alpha)}$.

From (1.6), we note that

$$q I_q(f_1) = I_q(f) + (q-1) f(0) + \frac{q-1}{\log q} f'(0). \tag{4.8}$$

For $h \in \mathbb{Z}_+$, by (4.8), we have

$$q^h \int_{\mathbb{Z}_p} f(x+1) q^{(h-1)x} d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + (q-1) h f(0) + \frac{q-1}{\log q} f'(0). \tag{4.9}$$

If $h \in \{-1, -2, -3, \ldots\}$, then we get

$$q^h \int_{\mathbb{Z}_p} f(x+1) q^{(h-1)x} d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + (1-q) h f(0) + \frac{q-1}{\log q} f'(0). \tag{4.10}$$

Let $h \in \mathbb{Z}_+$. By (4.9), we get

$$q^h \int_{\mathbb{Z}_p} [x + x_1 + 1]_{q^\alpha}^n q^{(h-1)x_1} d\mu_q(x_1) - \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{(h-1)x_1} d\mu_q(x_1) = (q-1)h[x]_{q^\alpha}^n + \frac{\alpha}{[\alpha]_q}[x]_{q^\alpha}^{n-1} q^{\alpha x}. \tag{4.11}$$

From (4.6) and (4.11), we note that

$$q^h \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x+1) - \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = (q-1)h[x]_{q^\alpha}^n + n\frac{\alpha}{[\alpha]_q}[x]_{q^\alpha}^{n-1} q^{\alpha x}. \tag{4.12}$$

If we take $x = 0$ in (4.12), then we have

$$\widetilde{\beta}_{0,q}^{(h,1|\alpha)} = \frac{h}{[h]_q}, \qquad q^h \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(1) - \widetilde{\beta}_{n,q}^{(h,1|\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{4.13}$$

Therefore, by (4.12) and (4.13), we obtain the following theorem.

Theorem 4.2. *For $h \in \mathbb{Z}_+$, we have*

$$\widetilde{\beta}_{0,q}^{(h,1|\alpha)} = \frac{h}{[h]_q}, \qquad q^h \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(1) - \widetilde{\beta}_{n,q}^{(h,1|\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \textit{if } n = 1, \\ 0 & \textit{if } n > 1. \end{cases} \tag{4.14}$$

By (4.7) and Theorem 4.2, we obtain the following corollary.

Corollary 4.3. *For $h \in \mathbb{Z}_+$, we have*

$$\widetilde{\beta}_{0,q}^{(h,1|\alpha)} = \frac{h}{[h]_q}, \qquad q^h \left(q^\alpha \widetilde{\beta}_{q}^{(h,1|\alpha)} + 1\right)^n - \widetilde{\beta}_{n,q}^{(h,1|\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \textit{if } n = 1, \\ 0 & \textit{if } n > 1, \end{cases} \tag{4.15}$$

with the usual convention about replacing $(\widetilde{\beta}_{q}^{(h,1|\alpha)})^n$ with $\widetilde{\beta}_{n,q}^{(h,1|\alpha)}$.

From (4.1), we have

$$\widetilde{\beta}_{0,q}^{(h,1|\alpha)} = \int_{\mathbb{Z}_p} q^{x_1(h-1)} d\mu_q(x_1) = \frac{h}{[h]_q}, \quad \text{if } h \in \mathbb{Z}_+. \tag{4.16}$$

It is not difficult to show that

$$\begin{aligned}\widetilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)}(1-x) &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^n q^{-x_1(h-1)} d\mu_{q^{-1}}(x_1) \\ &= (-1)^n q^{\alpha n+h-1} \int_{\mathbb{Z}_p} [x+x_1]_{q^\alpha} q^{x_1(h-1)} d\mu_q(x_1) \\ &= (-1)^n q^{\alpha n+h-1} \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x).\end{aligned} \tag{4.17}$$

Therefore, by (4.17), we obtain the following theorem.

Theorem 4.4. *For $h, n \in \mathbb{Z}_+$, we have*

$$\widetilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)}(1-x) = (-1)^n q^{\alpha n+h-1} \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x). \tag{4.18}$$

For $x = 1$ in Theorem 4.4, we get

$$\begin{aligned}\widetilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)} &= (-1)^n q^{\alpha n+h-1} \widetilde{\beta}_{n,q}^{(h,1|\alpha)}(1) \\ &= (-1)^n q^{\alpha n-1} \widetilde{\beta}_{n,q}^{(h,1|\alpha)} \quad \text{if } n > 1.\end{aligned} \tag{4.19}$$

Therefore, by (4.19), we obtain the following corollary.

Corollary 4.5. *For $h \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ with $n > 1$, we have*

$$\widetilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)} = (-1)^n q^{\alpha n-1} \widetilde{\beta}_{n,q}^{(h,1|\alpha)}. \tag{4.20}$$

Let $d \in \mathbb{N}$. By (4.1), we see that

$$\int_{\mathbb{Z}_p} q^{(h-1)x_1} [x+x_1]_{q^\alpha}^n d\mu_q(x_1) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} q^{ha} \int_{\mathbb{Z}_p} \left[\frac{x+a}{d} + x_1\right]_{q^{\alpha d}}^n q^{x_1(h-1)d} d\mu_{q^\alpha}(x_1). \tag{4.21}$$

By (4.1) and (4.21), we obtain the following equation:

$$\widetilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q}\sum_{a=0}^{d-1} q^{ha}\widetilde{\beta}_{n,q}^{(h,1|\alpha)}\left(\frac{x+a}{d}\right), \tag{4.22}$$

where $d \in \mathbb{N}$ and $h \in \mathbb{Z}_+$.

5. Polynomials $\widetilde{\beta}_{n,q}^{(h,k|\alpha)}(x)$ and $h = k$

From (2.1), we note that

$$\begin{aligned}\widetilde{\beta}_{n,q}^{(h,k|\alpha)}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{(h-1)x_1+\cdots+(h-k)x_k} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{(\alpha l+h)\cdots(\alpha l+h-k+1)}{[\alpha l+h]_q\cdots[\alpha l+h-k+1]_q},\end{aligned} \tag{5.1}$$

$$\begin{aligned} q^h\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}&[x+1+x_1+\cdots+x_k]_{q^\alpha}^n q^{(h-1)x_1+\cdots+(h-k)x_k} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{(h-1)x_1+\cdots+(h-k)x_k} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &\quad + (q-1)h\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_2+\cdots+x_k]_{q^\alpha}^n q^{(h-1)x_2+\cdots+(h-k)x_k} d\mu_q(x_2)\cdots d\mu_q(x_k) \\ &\quad + n\frac{\alpha}{[\alpha]_q}q^{\alpha x}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_2+\cdots+x_k]_{q^\alpha}^{n-1} \\ &\quad \times q^{\alpha(h-1)x_2+\cdots+(h-k+1)x_k} d\mu_q(x_2)\cdots d\mu_q(x_k).\end{aligned} \tag{5.2}$$

From (5.1), we have

$$q^h\widetilde{\beta}_{n,q}^{(h,k|\alpha)}(x+1) = \widetilde{\beta}_{n,q}^{(h,k|\alpha)}(x) + (q-1)h\widetilde{\beta}_{n,q}^{(h-1,k-1|\alpha)}(x) + q^{\alpha x}\frac{n\alpha}{[\alpha]_q}\widetilde{\beta}_{n,q}^{(h,k-1|\alpha)}(x). \tag{5.3}$$

Therefore, by (5.3), we obtain the following theorem.

Theorem 5.1. *For $h, n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have*

$$q^h\widetilde{\beta}_{n,q}^{(h,k|\alpha)}(x+1) - \widetilde{\beta}_{n,q}^{(h,k|\alpha)}(x) = (q-1)h\widetilde{\beta}_{n,q}^{(h-1,k-1|\alpha)}(x) + nq^{\alpha x}\frac{\alpha}{[\alpha]_q}\widetilde{\beta}_{n,q}^{(h,k-1|\alpha)}(x). \tag{5.4}$$

It is easy to show that

$$\begin{aligned}
&q^{\alpha x}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{hx_1+(h-1)x_2+\cdots+(h+1-k)x_k}d\mu_q(x_1)\cdots d\mu_q(x_k)\\
&\quad=(q^\alpha-1)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^{n+1} q^{(h-\alpha)x_1+\cdots+(h-\alpha+1-k)x_k}d\mu_q(x_1)\cdots d\mu_q(x_k)\\
&\qquad+\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^{n} q^{(h-\alpha)x_1+\cdots+(h-\alpha+1-k)x_k}d\mu_q(x_1)\cdots d\mu_q(x_k)\\
&\quad=(q^\alpha-1)\widetilde{\beta}_{n+1,q}^{(h+1-\alpha,k|\alpha)}(x)+\widetilde{\beta}_{n,q}^{(h+1-\alpha,k|\alpha)}(x).
\end{aligned}\tag{5.5}$$

Thus, by (5.5), we obtain the following proposition.

Proposition 5.2. *For $h, n \in \mathbb{Z}_+$, we have*

$$q^{\alpha x}\widetilde{\beta}_{n,q}^{(h+1,k|\alpha)}(x)=(q^\alpha-1)\widetilde{\beta}_{n+1,q}^{(h+1-\alpha,k|\alpha)}(x)+\widetilde{\beta}_{n,q}^{(h+1-\alpha,k|\alpha)}(x).\tag{5.6}$$

For $d \in \mathbb{N}$, we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[x+\sum_{j=1}^k x_j\right]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j}d\mu_q(x_1)\cdots d\mu_q(x_k)\\
&\quad=\frac{[d]_{q^\alpha}^n}{[d]_q^k}\sum_{a_1,\ldots,a_k=0}^{d-1} q^{h\sum_{j=1}^k a_j-\sum_{j=2}^k (j-1)a_j}\\
&\qquad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{x+\sum_{j=1}^k a_j}{d}+\sum_{j=1}^k x_j\right]_{q^{\alpha d}}^n q^{d\sum_{j=1}^k (h-j)x_j}d\mu_{q^d}(x_1)\cdots d\mu_{q^d}(x_k).
\end{aligned}\tag{5.7}$$

Thus, we obtain

$$\widetilde{\beta}_{n,q}^{(h,k|\alpha)}(x)=\frac{[d]_{q^\alpha}^n}{[d]_q^k}\sum_{a_1,\ldots,a_k=0}^{d-1} q^{h\sum_{j=1}^k a_j-\sum_{j=2}^k (j-1)a_j}\widetilde{\beta}_{n,q^d}^{(h,k|\alpha)}\left(\frac{x+a_1+\cdots+a_k}{d}\right).\tag{5.8}$$

Equation (5.8) is multiplication formula for the q-Bernoulli polynomials of order (h, k) with weight α.

Let us define $\widetilde{\beta}_{n,q}^{(k,k|\alpha)}(x) = \widetilde{\beta}_{n,q}^{(k|\alpha)}(x)$. Then we see that

$$\widetilde{\beta}_{n,q}^{(k|\alpha)}(x) = \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{(\alpha l+k)\cdots(\alpha l+1)}{[\alpha l+k]_q\cdots[\alpha l+1]_q}, \tag{5.9}$$

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[k-x+x_1+\cdots x_k]_{q^{-\alpha}}^n q^{-(k-1)x_1-\cdots-(k-k)x_k}d\mu_{q^{-1}}(x_1)\cdots d\mu_{q^{-1}}(x_k)\\&\quad=\frac{(-1)^n q^{\alpha n-k+\binom{k+1}{2}}}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{(\alpha l+k)\cdots(\alpha l+1)}{[\alpha l+k]_q\cdots[\alpha l+1]_q}\\&\quad=(-1)^n q^{\alpha n-k+\binom{k+1}{2}}\widetilde{\beta}_{n,q}^{(k|\alpha)}(x).\end{aligned} \tag{5.10}$$

Therefore, by (5.9) and (5.10), we obtain the following theorem.

Theorem 5.3. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have*

$$\widetilde{\beta}_{n,q^{-1}}^{(k|\alpha)}(k-x) = (-1)^n q^{\alpha n-k+\binom{k+1}{2}}\widetilde{\beta}_{n,q}^{(k|\alpha)}(x). \tag{5.11}$$

Let $x = k$ in Theorem 5.3. Then we see that

$$\widetilde{\beta}_{n,q^{-1}}^{(k|\alpha)} = (-1)^n q^{\alpha n-k+\binom{k+1}{2}}\widetilde{\beta}_{n,q}^{(k|\alpha)}(k). \tag{5.12}$$

From Theorem 5.1, we can derive the following equation:

$$q^k\widetilde{\beta}_{n,q}^{(k|\alpha)}(x+1) - \widetilde{\beta}_{n,q}^{(k|\alpha)}(x) = k(q-1)\widetilde{\beta}_{n,q}^{(k-1|\alpha)}(x) + nq^{\alpha x}\frac{\alpha}{[\alpha]_q}\widetilde{\beta}_{n,q}^{(k,k-1|\alpha)}(x). \tag{5.13}$$

By (5.9), we easily get

$$\widetilde{\beta}_{n,q}^{(k|\alpha)} = \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{(\alpha l+k)\cdots(\alpha l+1)}{[\alpha l+k]_q\cdots[\alpha l+1]_q}. \tag{5.14}$$

From the definition of p-adic q-integral on $\mathbb{Z}_p$, we note that

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}(q^\alpha-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^\alpha}^l q^{\sum_{l=0}^{k}(k-l)x_l}d\mu_q(x_1)\cdots d\mu_q(x_k)\\&\quad=\frac{(\alpha n+k)\cdots(\alpha n+1)}{[\alpha n+k]_q\cdots[\alpha n+1]_q}=\frac{\binom{\alpha n}{k}k!}{\binom{\alpha n}{k}_q[k]_q!}.\end{aligned} \tag{5.15}$$

Thus, by (5.15), we get

$$\sum_{l=0}^{n}\binom{n}{l}(q^{\alpha}-1)^{l}\tilde{\beta}_{n,q}^{(k|\alpha)}=\frac{\binom{\alpha n}{k}k!}{\binom{\alpha n}{k}_{q}[k]_{q}!}. \tag{5.16}$$

By the definition of polynomial $\tilde{\beta}_{n,q}^{(k|\alpha)}(x)$, we see that

$$\begin{aligned}\tilde{\beta}_{n,q}^{(k|\alpha)}(x)&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots x_k]_{q^{\alpha}}^{n}q^{(k-1)x_1+\cdots+(k-k)x_k}d\mu_q(x_1)\cdots d\mu_q(x_k)\\&=\sum_{l=0}^{n}\binom{n}{l}q^{\alpha l x}\tilde{\beta}_{l,q}^{(k|\alpha)}[x]_q^{n-l}=\left(q^{\alpha x}\tilde{\beta}_{q}^{(k|\alpha)}+[x]_{q^{\alpha}}\right)^{n},\quad \text{where } n\in\mathbb{Z}_{+},\end{aligned} \tag{5.17}$$

with the usual convention about replacing $(\tilde{\beta}_{q}^{(k|\alpha)})^{n}$ with $\tilde{\beta}_{n,q}^{(k|\alpha)}$.

Let $x=0$ in (5.13). Then we have

$$q^{k}\tilde{\beta}_{n,q}^{(k|\alpha)}(1)-\tilde{\beta}_{n,q}^{(k|\alpha)}=k(q-1)\tilde{\beta}_{n,q}^{(k-1|\alpha)}+\frac{n\alpha}{[\alpha]_q}\tilde{\beta}_{n,q}^{(k,k-1|\alpha)}. \tag{5.18}$$

Acknowledgments

The present research has been conducted by the Research Grant of Kwangwoon University in 2011. The authors would like to express their gratitude to referees for their valuable suggestions and comments.

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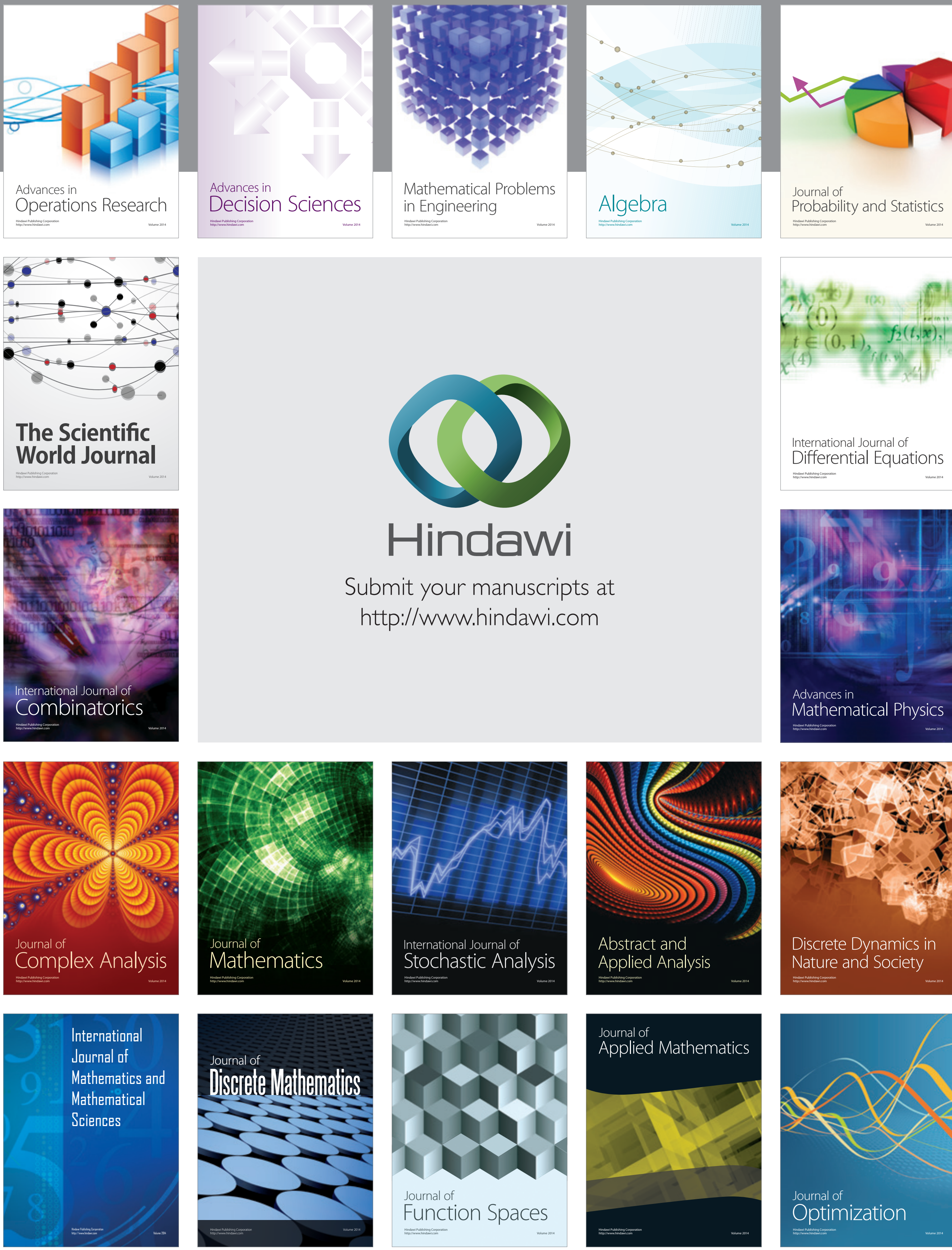
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