Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 489148, 8 pages
doi:10.1155/2011/489148

Research Article

The Lagrangian Stability for a Class of Second-Order Quasi-Periodic Reversible Systems

Yanling Shi and Jia Li

Department of Mathematics, Southeast University, Nanjing 211189, China

Correspondence should be addressed to Yanling Shi, shiyanling96998@163.com

Received 3 July 2011; Accepted 22 September 2011

Academic Editor: Martin D. Schechter

We study the following two-order differential equation, $(\Phi_p(x'))' + f(x,t)\Phi_p(x') + g(x,t) = 0$, where $\Phi_p(s) = |s|^{(p-2)}s$, $p > 0$. $f(x,t)$ and $g(x,t)$ are real analytic functions in x and t, $2a\pi_p-$ periodic in x, and quasi-periodic in t with frequencies $(\omega_1, \ldots, \omega_m)$. Under some odd-even property of $f(x,t)$ and $g(x,t)$, we obtain the existence of invariant curves for the above equations by a variant of small twist theorem. Then all solutions for the above equations are bounded in the sense of $\sup_{t\in R}|x'(t)| < +\infty$.

1. Introduction

The Kolmogorov-Arnold-Moser (KAM) theory was developed for conservative (Hamiltonian) dynamical systems that are nearly integrable. Integrable systems in their phase space contain lots of invariant tori, and KAM theory establishes persistence of such tori, which carry quasi-periodic motions, whereas parallel results exist for other classes of dynamical systems as well. In particular, the reversible KAM theory (starting with Moser's paper [1]) is to a great extent parallel to the Hamiltonian ones, see [2–6] and references therein. In the case of reversible diffeomorphisms, however, some special effects are exhibited [7], and the weakly reversible KAM theory has been developed in [8, 9].

In this paper, we will consider the P-Laplace equations with quasi-periodic reversible structure. Firstly, we give some concepts of reversible system, a mechanical system of s particles with the interaction forces which are independent of velocities or even functions in velocities; such a system ruled by the Newton equation, $d^2r/dt^2 = F(r,v)$, $r \in R^{3s}$, $v = dr/dt$, $F(r,-v) \equiv F(r,v)$, is time reversible, that is, reversing all the velocities v reverses all the trajectories in the configuration space R^{3s}. More generally, an autonomous differential equation $du/dt = V(u)$ and the corresponding vector field V are said to be reversible if there exists a phase space involution G (a mapping which $G^2 = Id$) that reverses the direction of time: $TG \circ V = -V \circ G$, where TG is the differential of G; that is, G transforms the field V into the opposite field $-V$. This $u(t)$, in addition to $G(u(-t))$, is a solution of the equation.

In the above example, $G : (r, v) \mapsto (r, -v)$. A system

$$\frac{dx}{dt} = f(x,t), \qquad \frac{dy}{dt} = g(x,t), \quad x \in R^p, \; y \in R^q, \tag{1.1}$$

is reversible with respect to the involution

$$G : (x,y) \longrightarrow (-x,y) \tag{1.2}$$

if and only if f is even in x and g is odd in x.
We can refer to [2, 8–11] for more detailed concepts of reversibility. Side by side with reversible vector fields, there are reversible diffeomorphisms. A mapping A is said to be reversible if there exists an involution G that conjugates A with its inverse A^{-1}, that is, $AGA = G$. The flow map of a reversible vector field for each fixed time is reversible with respect to the same involution, and vice versa.

In [12] Liu considered the following quasi-periodic mappings:

$$A : (x,y) \longrightarrow (x + \omega + y + f(x,y), y + g(x,y)), \tag{1.3}$$

where f and g are quasi-periodic in x with frequencies $\mu_1, \ldots, \mu_m$ and real analytic in x and y, the variable y ranges in a neighborhood of the origin of the real line R, and ω is a positive constant. He supposes that the mapping A is reversible with the involution $R : (x,y) \mapsto (-x,y)$, that is, $RAR = A^{-1}$. Such a map is often met when the vector field is quasi-periodic in time and reversible with respect to the involution R. In fact, the phase flow induces such a map on a cross-section transversal to the vector field.

The invariant curve theorem of reversible systems was first obtained by Moser [1] who then developed it [13] (for continuous systems), it was also developed by Sevryuk [9] (for both continuous and discrete system). In [1], the author also studied the existence of invariant tori of a reversible system depending quasi-periodically on time. In [12] Liu obtained an invariant curve theorem for reversible quasi-periodic mappings, as application, he studies the existence of quasi-periodic solutions and the boundedness of solutions for a pendulum-type equation

$$x'' + f(x,t)x' + g(x,t) = 0 \tag{1.4}$$

and an asymmetric oscillator depending quasi-periodically on time. By some theorems in [12], in this paper, we consider two-order differential equations

$$(\Phi_p(x'))' + f(x,t)\Phi_p(x') + g(x,t) = 0, \tag{1.5}$$

where $\Phi_p(s) = |s|^{(p-2)}s, \; p > 0, \; \omega > 0;$ if $p = 2$, it becomes (1.4).

Our main result is the following theorem.

2. Main Result

We first give some definitions [12].

Definition 1. A function $f : R \to R$ is called real analytic quasi-periodic with frequencies $\mu_1, \dots, \mu_m$ if it can be represented as a Fourier series of the type

$$f(t) = \sum_k f_k e^{\langle k,\mu\rangle t}, \tag{2.1}$$

where $k = (k_1, \dots, k_m)$, $\mu = (\mu_1, \dots, \mu_m)$, $\langle k, \mu\rangle = \sum k_j \mu_j \neq 0$; if $k \neq 0$, the coefficients f_k decay exponentially with $|k| = |k_1| + \cdots + |k_m|$.

Definition 2. The vector $\mu = (\mu_1, \dots, \mu_m)$ satisfies the Diophantine condition if:

$$\left|\langle k, \mu\rangle\right| \geq \frac{c}{|k|^{\sigma}}, \quad c, \sigma > 0 \tag{2.2}$$

for all integer vector $k \neq 0$.

For our study of (1.5), where $f(x,t)$ and $g(x,t)$ are real analytic in x and t, $2a\pi_p$ periodic in x, and quasi-periodic in t with frequencies $(\omega_1, \dots, \omega_m)$, where the number π_p is defined by

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{ds}{\left[1 - s^p/(p-1)\right]^{1/p}}. \tag{2.3}$$

Moreover, we assume

$$f(-x,-t) = f(x,t), \qquad g(-x,-t) = g(x,t). \tag{2.4}$$

Theorem 2.1. *Suppose that $(\omega_1, \dots, \omega_m)$ satisfy the Diophantine condition*

$$\left|\langle k, \mu\rangle\right| \geq \frac{c_0}{|k|^{\sigma_0}}, \quad for\ k \in Z^m \setminus \{0\}, \tag{2.5}$$

where c_0, σ_0 are positive constants. Then there are infinitely many quasi-periodic solutions with large amplitude, and the solutions of (1.5) *satisfy*

$$\sup_{t \in R}\left|x'(t)\right| < +\infty. \tag{2.6}$$

3. Coordination Transformation

In this section we first make a coordination transformation then study the boundedness of all solutions of the new system.

Equation (1.5) is equivalent to the planar system:

$$x' = \Phi_q(y),$$
$$y' = -f(x,t)y - g(x,t). \tag{3.1}$$

It is easy to verify that planar system (3.1) is reversible with respect to the involution $R : (x,y) \mapsto (-x,y)$.

Since we are concerned with the boundedness of solutions and the existence of quasi-periodic solutions with large amplitude, we may assume $|y| \geq 1$. Instead of considering planar system (3.1), we are concerned with the following system:

$$\frac{dt}{dx} = \frac{1}{\Phi_q(y)},$$
$$\frac{dy}{dx} = -\frac{f(x,t)y}{\Phi_q(y)} - \frac{g(x,t)}{\Phi_q(y)}. \tag{3.2}$$

We will prove that if $(t(x), y(x))$ is a solution of system (3.2), then $|y|$ is bounded.

4. Poincaré Map

In this section, we first introduce new action variable then give an expression for the Poincaré map of the new system.

Introduce a new action variable v and a small parameter ε as follows:

$$y = \frac{1}{\Phi_p(\varepsilon)v}, \quad v \in \left[\frac{1}{\gamma}, \gamma\right], \ \gamma > 1. \tag{4.1}$$

So system (3.2) is changed into the following form:

$$\frac{dt}{dx} = \varepsilon\Phi_q(v),$$
$$\frac{dv}{dx} = \varepsilon|v|^q f(x,t) + |\varepsilon|^p v^{q+1} g(x,t). \tag{4.2}$$

It is easy to verify that system (4.2) is reversible with respect to the involution $(t,v) \mapsto (-t,v)$. We make the ansatz that the solution $(t(x,v_0,t_0,\varepsilon),\ \rho(x,v_0,t_0,\varepsilon))$ has the following form:

$$t = t_0 + \varepsilon T(x,v_0,t_0;\varepsilon), \qquad v = V_0 + \varepsilon V(x,v_0,t_0;\varepsilon). \tag{4.3}$$

The functions T and V satisfy

$$\begin{aligned} \frac{dT}{dx} &= \Phi_q(V_0+\varepsilon V) = \Phi_q(V_0) + O(\varepsilon), \\ \frac{dV}{dx} &= f(x,t_0+\varepsilon T)|V_0+\varepsilon V|^q + \Phi_p(\varepsilon)|V_0+\varepsilon V|^{q+1} g(x,t_0+\varepsilon T) \\ &= f(x,t_0)V_0^q + O(\varepsilon). \end{aligned} \tag{4.4}$$

Denote by P the Poincaré map of (4.2); then, from the above equations, it is easy to see that

$$P(t_0,V_0) = \left(t_0 + 2a\pi_p \varepsilon \Phi_q(v_0) + O\left(\varepsilon^2\right),\ V_0 + \varepsilon V_0^q \int_0^{2a\pi_p} f(x,t_0)\,dx + O\left(\varepsilon^2\right) \right). \tag{4.5}$$

5. The Proof of Theorem 2.1

In this part, we will prove the map P has an invariant curve, and then boundedness of solutions of (1.5) follows from the standard arguments [14–18]. In the following, we will apply the invariant curves of quasi-periodic reversible mapping theorem to prove our conclusion.

Now we state Liu's result [12].

Consider the map

$$M_\delta : (x,y) \longrightarrow (x + \alpha + \delta L(x,y) + \delta f(x,y,\delta),\ y + \delta M(x,y) + \delta g(x,y,\delta)), \tag{5.1}$$

where L, M, f, g are quasi-periodic in x with the frequencies $\mu_1, \ldots, \mu_m$, $f(x,y,0) = g(x,y,0) = 0$.

We also assume that these functions are real analytic in a complex neighborhood of the domain $R \times [a_1, b_1]$. The functions L and M can be represented in the form

$$\begin{aligned} L(x,y) &:= \widetilde{L}(x,y) + \overline{L}(x,y) = \sum_{k\in Z^m\setminus K} L_k(y) e^{i\langle k,\mu\rangle x} + \sum_{k\in K} L_k(y) e^{i\langle k,\mu\rangle x}, \\ M(x,y) &:= \widetilde{M}(x,y) + \overline{M}(x,y) = \sum_{k\in Z^m\setminus K} M_k(y) e^{i\langle k,\mu\rangle x} + \sum_{k\in K} M_k(y) e^{i\langle k,\mu\rangle x}. \end{aligned} \tag{5.2}$$

Note that

$$\begin{gathered} e^{i\langle k,\mu\rangle \alpha} - 1 \neq 0, \quad \text{for } k \in Z^m \setminus K, \\ \overline{L}(x+\alpha) \equiv \overline{L}(x), \quad \overline{M}(x+\alpha) \equiv \overline{M}(x). \end{gathered} \tag{5.3}$$

Lemma 5.1 ([12, Theorem 4]). *Suppose the function L satisfies*

$$\overline{L}(x,y) > 0, \qquad \frac{\partial \overline{L}}{\partial y} > 0, \tag{5.4}$$

and there is a real analytic function $I(x,y) = I(x+\alpha, y)$ *satisfying*

$$\begin{gathered}\frac{\partial I}{\partial y} > 0,\\ \overline{L}(x,y)\frac{\partial I}{\partial x}(x,y) + \overline{M}(x,y)\frac{\partial I}{\partial y}(x,y) \equiv 0.\end{gathered} \tag{5.5}$$

Moreover, suppose that there are two numbers $\widetilde{a}$ and $\widetilde{b}$ such that $a_1 < \widetilde{a} < \widetilde{b} < b_1$ and

$$I_M(a_1) < I_m(\widetilde{a}) \le I_M(\widetilde{a}) < I_m\left(\widetilde{b}\right) \le I_M\left(\widetilde{b}\right) < I_m(b_1), \tag{5.6}$$

where

$$I_M(y) = \max_{x\in R} I(x,y), \qquad I_m(y) = \min_{x\in R} I(x,y). \tag{5.7}$$

Then there exist $\varepsilon > 0$ and $\Delta > 0$ such that if $\delta < \Delta$ and

$$\|f(\cdot,\cdot,\delta)\| + \|g(\cdot,\cdot,\delta)\| < \varepsilon, \tag{5.8}$$

the mapping M_δ has an invariant curve which is of the form $y = \phi(x)$, and ϕ is quasi-periodic in x with frequencies $\mu_1, \dots, \mu_m$. The constants ε and Δ depend on $a,\ \widetilde{a},\ \widetilde{b},\ b,\ L,\ M$, and I. In particular, ε is independent of δ. If $\alpha = 0$, the conclusion also holds; in this case, $K = Z^m,\ \widetilde{L} = \widetilde{M} = 0, \overline{L} = L,$ *and* $\overline{M} = M$.

For the Poincaré map $P(t_0, V_0)$ of (4.2), let

$$I(t_0,\ V_0) = V_0 e^{-(1/2a\pi_p)\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)\,dx\,dt}. \tag{5.9}$$

Since $f(-t,-x) = -f(t,x)$, we know that $I(t_0, V_0)$ is even and quasi-periodic in t_0 with frequencies $\mu_1, \dots, \mu_m$. And it is easy to see that

$$\begin{aligned}&\frac{\partial I}{\partial V_0} = e^{-(1/2a\pi_p)\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)\,dx\,dt} > 0,\\ &\overline{L}(V_0,t_0)\frac{\partial I}{\partial t_0}(V_0,t_0) + \overline{M}(V_0,t_0)\frac{\partial I}{\partial V_0}(V_0,t_0)\\ &\quad = 2a\pi_p\Phi_q(v_0)\cdot V_0\left(-\frac{1}{2a\pi_p}\right)\int_0^{2a\pi_p} f(x,t_0)\,dx\cdot e^{-(1/2a\pi_p)\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)\,dx\,dt}\\ &\qquad + V_0^q\int_0^{2a\pi_p} f(x,t_0)\,dx\cdot e^{-(1/2a\pi_p)\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)\,dx\,dt}\end{aligned}$$

$$
\begin{aligned}
&= V_0^q \int_0^{2a\pi_p} f(x,t_0)dx \cdot e^{-(1/2a\pi_p)\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt} \\
&\quad - V_0^q \int_0^{2a\pi_p} f(x,t_0)dx \cdot e^{-(1/2a\pi_p)\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt} = 0.
\end{aligned}
\tag{5.10}
$$

Moreover, if we define α and β by

$$
\begin{gathered}
I(t_0,V_0) = V_0 e^{-(1/2a\pi_p)\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt}, \\
\alpha = \min_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right), \quad \beta = \max_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right),
\end{gathered}
\tag{5.11}
$$

then $\beta \gg \alpha > 0$. Now we choose the constants $\gamma,\ \gamma_1,\ \gamma_2$ as

$$
\gamma = \left(2\frac{\beta}{\alpha}\right)^3 > 1, \qquad \gamma_1 = \left(2\frac{\beta}{\alpha}\right)^{-1}, \qquad \gamma_2 = \left(2\frac{\beta}{\alpha}\right). \tag{5.12}
$$

Then

$$
\begin{aligned}
I_M\left(\frac{1}{\gamma}\right) &= \frac{1}{\gamma}\max_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right) = \frac{\alpha^3}{8\beta^2}, \\
I_m(\gamma) &= \gamma \min_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right) = \frac{8\beta^3}{\alpha^2}, \\
I_m(\gamma_1) &= \gamma_1 \min_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right) = \frac{\alpha^2}{2\beta}, \\
I_M(\gamma_1) &= \gamma_1 \max_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right) = \frac{\alpha}{2}, \\
I_m(\gamma_2) &= \gamma_2 \min_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right) = 2\beta, \\
I_M(\gamma_2) &= \gamma_2 \max_{t_0\in R} \exp\left(-\frac{1}{2a\pi_p}\int_0^{t_0}\int_0^{2a\pi_p} f(x,t)dx\,dt\right) = -\frac{2\beta^2}{\alpha}.
\end{aligned}
\tag{5.13}
$$

We have already demonstrated that the map P satisfies all the conditions in Lemma 5.1; hence, P has an invariant curve; thus, all solutions of (1.5) are bounded.

Acknowledgments

The authors would like to thank the referees, anonymous editors, and reviewers very much for their valuable comments and suggestions.

References

[1] J. Moser, “Combination tones for Duffing’s equation,” *Communications on Pure and Applied Mathematics*, vol. 18, pp. 167–181, 1965.

[2] M. B. Sevryuk, “Partial preservation of frequencies in KAM theory,” *Nonlinearity*, vol. 19, no. 5, pp. 1099–1140, 2006.

[3] M. B. Sevryuk, “Partial preservation of frequencies and Floquet exponents in KAM theory,” *Trudy Matematicheskogo Instituta Imeni V. A. Steklova*, vol. 259, no. 2, pp. 174–202, 2007, translation in *Proceedings of the Steklov Institute of Mathematics*, vol. 259, no. 1, pp. 167–195, 2007.

[4] M. B. Sevryuk, “The finite-dimensional reversible KAM theory,” *Physica D*, vol. 112, no. 1-2, pp. 132–147, 1998.

[5] M. B. Sevryuk and A. Lahiri, “Bifurcations of families of invariant curves in four-dimensional reversible mappings,” *Physics Letters A*, vol. 154, no. 3-4, pp. 104–110, 1991.

[6] M. B. Sevryuk, “Lower-dimensional tori in reversible systems,” *Chaos*, vol. 1, no. 2, pp. 160–167, 1991.

[7] G. R. W. Quispel and M. B. Sevryuk, “KAM theorems for the product of two involutions of different types,” *Chaos*, vol. 3, no. 4, pp. 757–769, 1993.

[8] V. I. Arnol’d and M. B. Sevryuk, “Oscillations and bifurcations in reversible systems,” in *Nonlinear Phenomena in Plasma Physics and Hydrodynamics*, R. Z. Sagdeev, Ed., pp. 31–64, Mir, Moscow, Russia, 1986.

[9] M. B. Sevryuk, *Reversible Systems*, vol. 1211 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1986.

[10] R. L. Devaney, “Reversible diffeomorphisms and flows,” *Transactions of the American Mathematical Society*, vol. 218, pp. 89–113, 1976.

[11] V. I. Arnol’d, “Reversible Systems,” in *Nonlinear and Turbulent Processes in Physics, Vol. 3 (Kiev, 1983)*, pp. 1161–1174, Harwood Academic Publ., Chur, Switzerland, 1984.

[12] B. Liu, “Invariant curves of quasi-periodic reversible mappings,” *Nonlinearity*, vol. 18, no. 2, pp. 685–701, 2005.

[13] J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, USA, 2001, with special emphasis on celestial mechanics, Reprint of the 1973 original, with a foreword by P. J. Holme.

[14] G. R. Morris, “A case of boundedness in Littlewood’s problem on oscillatory differential equations,” *Bulletin of the Australian Mathematical Society*, vol. 14, no. 1, pp. 71–93, 1976.

[15] B. Liu, “Quasi-periodic solutions of a semilinear Liénard equation at resonance,” *Science in China. Series A*, vol. 48, no. 9, pp. 1234–1244, 2005.

[16] M. Levi, “Quasiperiodic motions in superquadratic time-periodic potentials,” *Communications in Mathematical Physics*, vol. 143, no. 1, pp. 43–83, 1991.

[17] R. Dieckerhoff and E. Zehnder, “Boundedness of solutions via the twist-theorem,” *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze*, vol. 14, no. 1, pp. 79–95, 1987.

[18] R. Ortega, “Boundedness in a piecewise linear oscillator and a variant of the small twist theorem,” *Proceedings of the London Mathematical Society*, vol. 79, no. 2, pp. 381–413, 1999.