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Research Article

On the Distance to a Root of Polynomials

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In 2002, Dierk Schleicher gave an explicit estimate of an upper bound for the number of iterations of Newton's method it takes to find all roots of polynomials with prescribed precision. In this paper, we provide a method to improve the upper bound given by D. Schleicher. We give here an iterative method for finding an upper bound for the distance between a fixed point z in an immediate basin of a root α to α, which leads to a better upper bound for the number of iterations of Newton's method.

1. Introduction

Let P be a polynomial of degree d, and let $N_p(z) = z - P(z)/P'(z)$ be the Newton map induced by P. Let $\mathbb{N}$ be the set of positive integers. For each $k \in \mathbb{N}$, let N_p^k denote the k-iterate of N_p, that is, $N_p^1 = N_p, N_p^2 = N_p \circ N_p$, and $N_p^k = N_p^{k-1} \circ N_p$. For a root α of P, we say that a set U is the *immediate basin* of α if U is the largest connected open set containing α and $N_p^k(z) \to \alpha$, as $k \to \infty$, for all $z \in U$. Every immediate basin U is forward invariant, that is, $N_p(U) = U$, and is simply connected (see [1, 2]). In 2002, Schleicher [3] provided an upper bound for the number of iterations of Newton's method for complex polynomials of fixed degree with a prescribed precision. More precisely, Schleicher proved that if all roots of P are inside the unit disc and $0 < \varepsilon < 1$, there is a constant $n(d, \varepsilon)$ such that for every root α of P, there is a point z with $|z| = 2$ such that $|N_p^n(z) - \alpha| < \varepsilon$ for all $n \geq n(d, \varepsilon)$. Schleicher also showed that $n(d, \varepsilon)$ can be chosen so that

$$n(d, \varepsilon) \leq \frac{9\pi d^4 f_d^2}{\varepsilon^2 \log 2} + \frac{|\log \varepsilon| + \log 13}{\log 2} + 1 \tag{1.1}$$

with

$$f_d := \frac{d^2(d-1)}{2(2d-1)}\binom{2d}{d}. \tag{1.2}$$

To obtain this estimate, Schleicher employed several rough estimates which cause the bound far from an efficient upper bound. The main point that causes the extremely inefficiency is the way Schleicher used to obtain f_d which arose when he estimated an upper bound for the distance of a point z to a root α. Schleicher showed that if z is in the immediate basin of α and $|N_p(z) - z| = \delta$, then the distance between z and α is at most δf_d.

In this paper, we give an algorithm to improve the value of f_d. Even though, it is not an explicit formula, it can be easily computed. The following is our main result.

Main Theorem. *Let $P(z)$ be a polynomial of degree $d \geq 3$, and let y be a positive number larger than $4d-3$. If z_0 is in an immediate basin of a root α and $|N_p(z_0) - z_0| = \varepsilon$, then $|z_0 - \alpha| \leq \varepsilon M(d, y)$, where $M(d, y) := \max\{y, A_d + y(d-1)/(y-1)\}$ and A_d can be derived from the following iterative algorithm.*

Let $b = y(y-d)/(y-1)$, and

$$A_2 = \frac{y(d-1)\left[2d(y-2d+3) - 3y - 1\right]}{(y-1)(y-4d+3)}. \tag{1.3}$$

For $k = 2, \ldots, d-1$, set $a_k = 1 + \sum_{j=2}^{k-1}(A_k/(A_k - A_j))$.

If $2A_k < b$ then let

$$A_{k+1} = A_k\left(\frac{(a_k + d - k)A_k + b(k+1-a_k-d)}{A_k(a_k+1) - ba_k}\right). \tag{1.4}$$

Otherwise let

$$A_{k+1} = A_k\frac{a_k + d - k}{a_k}. \tag{1.5}$$

Note that the value of $M(d, y)$ in the main theorem depends only on the constant y and the degree d. Hence if we select y appropriately the value $M(d, y)$ will be optimized under this method. However this estimate is still far away from the best possible one. We believe that this new upper bound $M(d, y)$ is less than $f_d/2^{d/2}$ for all $d \geq 10$ when $y = d^{1.5}2^{(4d/3)-2}$. We will discuss further about this matter in Section 4.

2. Preliminary Results

We will use $B(a, r)$ for the open ball $\{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{B}(a, r)$ for the closed ball $\{z \in \mathbb{C} : |z - a| \leq r\}$, where $\mathbb{C}$ is the set of complex numbers. If S is a subset of $\mathbb{C}$, we denote the boundary of S by ∂S.

Lemma 2.1. *Let P be a polynomial. Let β be a complex number and $r > 0$. Suppose that $\mathrm{Re}\{(z-\beta)P'(z)/P(z)\} \geq 1/2$ whenever $|z - \beta| = r$ and $P(z) \neq 0$. Let U be an immediate basin of a root α of P. If $U \cap \overline{B}(\beta, r) \neq \emptyset$, then α is in $B(\beta, r)$.*

Proof. For $|z-\beta| = r$ with $P(z) \neq 0$, we have

$$N_p(z) - \beta = (z-\beta)\left(1 - \frac{1}{g(z)}\right), \tag{2.1}$$

where $g(z) = (z-\beta)P'(z)/P(z)$. Hence, $|N_p(z)-\beta| \le |z-\beta|$ if and only if $|(g(z)-1)/g(z)| \le 1$ which holds if $\text{Re}\{g(z)\} \ge 1/2$. It means that if z is a point in $\partial B(\beta, r)$ and $\text{Re}\{g(z)\} \ge 1/2$, then the distance of $N_p(z)$ to β is at most the distance of z to β. In other words, the image of z under the map N_p also lies inside $\overline{B}(\beta, r)$.

Let α be a root of P and U be its immediate basin. Suppose that $\alpha \notin \overline{B}(\beta, r)$ and $z \in U \cap \overline{B}(\beta, r)$. Since U is forward invariant under N_p, $N_p(z)$ still stays in U. Since U is connected, there is a curve γ_0 connecting z to $N_p(z)$ and lying entirely in U. Since $N_p^k(\gamma_0)$ converges uniformly to α as $k \to \infty$, the set $\bigcup_{k=1}^{\infty} N_p^k(\gamma_0) \cup \{\alpha\}$ forms a continuous curve γ joining z and α. Note that γ is contained in U because $N_p^k(\gamma_0)$ lies inside U for all $k \in \mathbb{N}$.

Let w be the last intersection point of γ with $\partial B(\beta, r)$ (i.e., the part of the curve γ that connects w to α stays outside $\overline{B}(\beta, r)$ except at w). So N_p must send w to a point outside $\overline{B}(\beta, r)$, otherwise β is a fixed point of N_p, which is impossible because all fixed points of N_p are only the roots of P, and here $P(z) \neq 0$ on $|z-\beta| = r$. From the first paragraph, however, we also have $N_p(w) \in \overline{B}(\beta, r)$. Hence we get a contradiction. Therefore if $U \cap \overline{B}(\beta, r)$ is not empty, then α is in $B(\beta, r)$, as desired. □

Remark that, from the proof of Lemma 2.1, if β is a root of P and $\text{Re}\{(z-\beta)P'(z)/P(z)\} \ge 1/2$ for all $|z-\beta| \le r$, then the closed ball $\overline{B}(\beta, r)$ is contained in the immediate basin of β.

Lemma 2.2. *Let P be a polynomial of degree $d \ge 3$. Let α_1 be a root of P and α_2 the nearest root to α_1. Let $\beta = |\alpha_1 - \alpha_2|$, and let m be the multiplicity of α_1. Suppose that there is a root α of P such that $|\alpha_1 - \alpha| \ge b$ for some positive number $b \ge \beta$. Then the closed ball $\{z \in \mathbb{C} : |z - \alpha_1| \le \delta\}$ is contained entirely in the immediate basin of α_1, where*

$$\delta = \frac{1}{2(2d-1)}\left[(2m+1)\beta + (2d-3)b - \sqrt{\left[(2m+1)\beta + b(2d-3)\right]^2 - 4(2d-1)(2m-1)b\beta}\right]. \tag{2.2}$$

Proof. Without loss of generality, we assume that $\alpha_1 = 0$. From the previous remark, it suffices to show that $\text{Re}\{zP'(z)/P(z)\} \ge 1/2$ for all $|z| \le \delta$. Let $P(z) = z^m \prod_{k=2}^{d-m}(z-\alpha_k)$. We have

$$\frac{zP'(z)}{P(z)} = m + \sum_{k=2}^{d-m} \frac{z}{z-\alpha_k}. \tag{2.3}$$

Hence

$$\text{Re}\left\{\frac{zP'(z)}{P(z)}\right\} = m + \sum_{k=2}^{d-m} \text{Re}\left\{\frac{z}{z-\alpha_k}\right\} \ge m + \frac{r(d-m-1)}{r-\beta} + \frac{r}{r-b}, \tag{2.4}$$

where $r = |z|$. Note that $\beta \leq b$. For $r < \beta$, we have

$$m + \frac{r(d-m-1)}{r-\beta} + \frac{r}{r-b} \geq \frac{1}{2}, \tag{2.5}$$

if $r \leq \delta$. This shows that $\operatorname{Re}\{zP'(z)/P(z)\} \geq 1/2$ for all $|z| \leq \delta$, as needed. □

Note that if we set $b = \beta$ in Lemma 2.2, then the closed ball centered at α_1 of radius $\beta(2m-1)/(2d-1)$ is contained in the immediate basin of α_1. Furthermore, if $m = 1$, the radius of the ball is $\beta/(2d-1)$. (Schleicher [3, Lemma 4, page 938] made a small mistake about the radius of the ball. Indeed, he should get $\beta/(2d-1)$ instead of $\beta/2(d-1)$).

Lemma 2.3. *Let P be a polynomial of degree d. For any complex number z and any positive number $y > 1$, if $|N_p(z) - z| = \varepsilon$ and there is a root α_d of P with $|z - \alpha_d| \geq y\varepsilon$, then there is a root α of P such that $|z - \alpha| \leq y(d-1)\varepsilon/(y-1)$.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be all roots of P. Suppose that $|z - \alpha_d| \geq y\varepsilon$. If $|z - \alpha_j| > y(d-1)\varepsilon/y - 1$ for $1 \leq j \leq d-1$, then

$$\left|N_p(z) - z\right| \geq \left(\sum_{j=1}^{d} \frac{1}{\left|z - \alpha_j\right|}\right)^{-1} > \left(\frac{y-1}{y(d-1)\varepsilon}(d-1) + \frac{1}{y\varepsilon}\right)^{-1} = \varepsilon, \tag{2.6}$$

a contradiction. □

We are now ready to prove our main theorem.

3. Proof of Main Theorem

Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be all roots of P such that α_1 is the nearest root to z_0 and $|\alpha_1 - \alpha_k| \leq |\alpha_1 - \alpha_{k+1}|$ for $k = 2, \ldots, d-1$. Suppose that $|z_0 - \alpha_d| \geq y\varepsilon$. By Lemma 2.3, we have $|z_0 - \alpha_1| \leq y(d-1)\varepsilon/(y-1)$. Note that $|\alpha_1 - \alpha_d| \geq b\varepsilon$. If $\alpha = \alpha_1$, we are done. Otherwise, z is not in the immediate basin of α_1; thus by Lemma 2.2 with $m = 1$, we get that $|z_0 - \alpha_1| > \delta$, where δ is defined in Lemma 2.2, that is,

$$\delta = \frac{3r_2 + b\varepsilon(2d-3) - \sqrt{\left[3r_2 + b\varepsilon(2d-3)\right]^2 - 4(2d-1)b\varepsilon r_2}}{2(2d-1)}, \tag{3.1}$$

where $r_2 = |\alpha_1 - \alpha_2|$. Thus z_0 satisfies the inequalities

$$\delta < |z_0 - \alpha_1| \leq \frac{y(d-1)\varepsilon}{y-1}, \tag{3.2}$$

which holds if $|\alpha_1 - \alpha_2| < A_2\varepsilon$. If $\alpha = \alpha_2$, we are done. Suppose next that $\alpha \neq \alpha_2$.

Table 1: Examples of values of $M(d, y)$ compared to f_d when $y = d^{1.5}2^{4d/3-2}$.

$d =$	$M(d, y)$ is less than	f_d is greater than	$f_d/2^{d/2}M(d, y)$ is greater than
10	1.3385×10^{5}	4.3758×10^{6}	1.0216
20	1.0131×10^{10}	1.343×10^{13}	1.2946
30	4.4559×10^{14}	2.6158×10^{19}	1.7915
40	1.5878×10^{19}	4.2458×10^{25}	2.5502
50	5.0059×10^{23}	6.2420×10^{31}	3.7162
60	1.1486×10^{28}	8.6222×10^{37}	5.4054
70	4.2054×10^{32}	1.1410×10^{44}	7.8967
80	1.1429×10^{37}	1.4634×10^{50}	11.6467
90	3.0424×10^{41}	1.8327×10^{56}	17.1212
100	7.9376×10^{45}	2.2523×10^{62}	25.2027
110	2.0274×10^{50}	2.7262×10^{68}	37.3244
120	5.1302×10^{54}	3.2588×10^{74}	55.0978
130	1.2839×10^{59}	3.8546×10^{80}	81.3792
140	3.1697×10^{63}	4.5186×10^{86}	120.7511
150	7.7889×10^{67}	5.2563×10^{92}	178.6315
160	1.8954×10^{72}	6.0735×10^{98}	265.0635
170	4.5932×10^{76}	6.9764×10^{104}	392.6175
180	1.1074×10^{81}	7.9718×10^{110}	581.5469
190	2.6450×10^{85}	9.0669×10^{116}	863.7282
200	6.3268×10^{89}	1.0269×10^{123}	1280.4536

Now let $|\alpha_1 - \alpha_k| = \varepsilon r_k$. If $|z - \alpha_1| = A_2\varepsilon$ and $r_3 > A_3$, then

$$\begin{aligned}\operatorname{Re}\left\{\frac{(z-\alpha_1)P'(z)}{P(z)}\right\} &\geq 1 + \frac{A_2}{A_2 + r_2} + \frac{A_2(d-3)}{A_2 - r_3} + \frac{A_2}{A_2 - r_d} \\ &> 1 + \frac{1}{2} + \frac{A_2(d-3)}{A_2 - r_3} + \frac{A_2}{A_2 - b} > \frac{1}{2}.\end{aligned} \tag{3.3}$$

hence by Lemma 2.1 α must be either α_1 or α_2 which is not the case. Therefore $r_3 \leq A_3$, and if α is α_3 we are done. Otherwise, let $|z - \alpha_1| = A_3\varepsilon$ and suppose $r_4 > A_4$; then $\operatorname{Re}\{(z - \alpha_1)P'(z)/P(z)\} > 1/2$, and by Lemma 2.1 we get a contradiction. Thus we obtain $r_4 \leq A_4$, and if α is α_4 we are done. Continuing this process, finally we get $r_d \leq A_d$ which gives $|z_0 - \alpha_d| \leq \varepsilon(A_d + y(d-1)/(y-1))$.

Note that if $A_d < b$, it is a contradiction to the fact that $\varepsilon r_d = |\alpha_1 - \alpha_d| \geq b\varepsilon$, which implies that assumption $|z_0 - \alpha_d| \geq y\varepsilon$ is false. Hence in this case we have $|z_0 - \alpha_d| < y\varepsilon$. The proof is now complete.

4. Discussion

For a fixed d, $M(d, y)$ depends on only y. If we choose y too large (for instance, $y \geq f_d$), the value of $M(d, y)$ is useless when it is compared to f_d. So we have to choose y carefully so that $M(d, y)$ is minimal as possible. We do not know yet whether there is an explicit formula for the value y that minimizes $M(d, y)$. Table 1 below shows the values of $M(d, y)$ where we set $y = d^{1.5}2^{4d/3-2}$. It seems that this method can reduce upper bounds for the distance of

z_0 to the root it converges to at least $2^{d/2}$ times compared to f_d. If we replace f_d in (1.1) by $M(d, y)$, we derive a new upper bound for the number of iterations.

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