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Research Article

Semilinear Volterra Integrodifferential Problems with Fractional Derivatives in the Nonlinearities

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A second-order semilinear Volterra integrodifferential equation involving fractional time derivatives is considered. We prove existence and uniqueness of mild solutions and classical solutions in appropriate spaces.

1. Introduction

In this work we discuss the following problem:ď

$$\begin{gathered} u''(t) = Au(t) + f(t) + \int_0^t g\Big(t, s, u(s), D^{\beta_1}u(s), \dots, D^{\beta_n}u(s)\Big)ds, \quad t > 0, \\ u(0) = u_0 \in X, \qquad u'(0) = u_1 \in X, \end{gathered} \tag{1.1}$$

where $0 < \beta_i \le 1$, $i = 1, \dots, n$. Here the prime denotes time differentiation and D^{β_i}, $i = 1, \dots, n$ denotes fractional time differentiation (in the sense of Riemann-Liouville or Caputo). The operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \ge 0$ of bounded linear operators in the Banach space X, f and g are nonlinear functions from $\mathbf{R}^+$ to X and $\mathbf{R}^+ \times \mathbf{R}^+ \times X \times \cdots \times X$ to X, respectively, u_0 and u_1 are given initial data in X. The problem with $\beta_1 = \cdots = \beta_n = 0$ or 1 has been investigated by several authors (see [1–7] and references therein, to cite a few). Well-posedness has been proved using fixed point theorems and the theory of strongly continuous cosine families in Banach spaces developed in [8, 9]. This theory allows us to treat a more general integral or integrodifferential equation, the solutions of which are called "mild" solutions. In case of regularity (of the initial data and the nonlinearities), the mild solutions are shown to be classical. In case $\beta_1 = \beta_2 = \cdots = \beta_n = 1$, the underlying space is the space of continuously differentiable functions.

In this work, when $0 < \beta_i < 1$, $i = 1, \ldots, n$, we will see that mild solutions need not be that regular (especially when dealing with Riemann-Liouville fractional derivatives). It is the objective of this paper to find the appropriate space and norm where the problem is solvable. We first consider the problem with a fractional derivative in the sense of Caputo and look for a mild solution in C^1. Under certain conditions on the data it is shown that this mild solution is classical. Then we consider the case of fractional derivatives in the sense of Riemann-Liouville. We prove existence and uniqueness of mild solution under much weaker regularity conditions than the expected ones. Indeed, when the nonlinearity involves a term of the form

$$\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u'(s)ds}{(t-s)^{\beta}}, \quad 0 < \beta < 1, \tag{1.2}$$

then one is attracted by $u'(s)$ in the integral and therefore it is natural to seek mild solutions in the space of continuously differentiable functions. This is somewhat surprising if instead of this expression one is given ${}^C D^{\beta} u(t)$ (the latter is exactly the definition of the former). However, this is not the case when we deal with the Riemann-Liouville fractional derivative. Solutions are only *β-differentiable* and not necessarily once continuously differentiable. It will be therefore wise to look for solutions in an appropriate "fractional" space. We will consider the new spaces E_β and FS_β (see (4.1)) instead of the classical ones E and C^1 (see [1–7]).

To simplify our task we will treat the following simpler problem

$$\begin{gathered} u''(t) = Au(t) + f(t) + \int_0^t g\Big(t, s, u(s), D^{\beta} u(s)\Big) ds, \quad t > 0, \\ u(0) = u_0 \in X, \qquad u'(0) = u_1 \in X, \end{gathered} \tag{1.3}$$

with $0 < \beta < 1$. The general case can be derived easily.

The rest of the paper is divided into three sections. In the second section we prepare some material consisting of notation and preliminary results needed in our proofs. The next section treats well-posedness when the fractional derivative is taken in the sense of Caputo. Section 4 is devoted to the Riemann-Liouville fractional derivative case.

2. Preliminaries

In this section we present some assumptions and results needed in our proofs later. This will fix also the notation used in this paper.

Definition 2.1. The integral

$$(I^{\alpha} h)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{h(t)dt}{(x-t)^{1-\alpha}}, \quad x > a \tag{2.1}$$

is called the Riemann-Liouville fractional integral of h of order $\alpha > 0$ when the right side exists.

Here Γ is the usual Gamma function

$$\Gamma(z) := \int_0^\infty e^{-s} s^{z-1} ds, \qquad z > 0. \tag{2.2}$$

Definition 2.2. The (left hand) Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by

$$(D_a^\alpha h)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{h(t)dt}{(x-t)^\alpha}, \qquad x > a, \tag{2.3}$$

whenever the right side is pointwise defined.

Definition 2.3. The fractional derivative of order $0 < \alpha < 1$ in the sense of Caputo is given by

$$\left({}^C D_a^\alpha h\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{h'(t)dt}{(x-t)^\alpha}, \qquad x > a. \tag{2.4}$$

Remark 2.4. The fractional integral of order α is well defined on L^p, $p \geq 1$ (see [10]). Further, from Definition 2.2, it is clear that the Riemann-Liouville fractional derivative is defined for any function $h \in L^p$, $p \geq 1$ for which $k_{1-\alpha} * h$ is differentiable (where $k_{1-\alpha}(t) := t^{-\alpha}/\Gamma(1-\alpha)$ and $*$ is the incomplete convolution). In fact, as domain of $D_0^\alpha = D^\alpha$ we can take

$$D(D^\alpha) = \left\{ h \in L^p(0,T) : k_{1-\alpha} * h \in W^{1,p}(0,T) \right\}, \tag{2.5}$$

where

$$W^{1,p}(0,T) := \left\{ u : \exists \varphi \in L^p(0,T) : u(t) = C + \int_0^t \varphi(s) ds \right\}. \tag{2.6}$$

In particular, we know that the absolutely continuous functions ($p = 1$) are differentiable almost everywhere and therefore the Riemann-Liouville fractional derivative exists a.e. In this case (for an absolutely continuous function) the derivative is summable [10, Lemma 2.2] and the fractional derivative in the sense of Caputo exists. Moreover, we have the following relationship between the two types of fractional derivatives:

$$\begin{aligned} (D_a^\alpha h)(x) &= \frac{1}{\Gamma(1-\alpha)} \left[\frac{h(a)}{(t-a)^\alpha} + \int_a^x \frac{h'(t)dt}{(x-t)^\alpha} \right] \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{h(a)}{(t-a)^\alpha} + \left({}^C D_a^\alpha h\right)(x), \qquad x > a. \end{aligned} \tag{2.7}$$

See [10–15] for more on fractional derivatives.

We will assume the following.

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbf{R}$, of bounded linear operators in the Banach space X.

The associated sine family $S(t)$, $t \in \mathbf{R}$ is defined by

$$S(t)x := \int_0^t C(s)x\,ds, \qquad t \in \mathbf{R},\ x \in X. \tag{2.8}$$

It is known (see [9, 16]) that there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$|C(t)| \leq Me^{\omega|t|}, \qquad t \in \mathbf{R}, \quad |S(t) - S(t_0)| \leq M\left|\int_{t_0}^t e^{\omega|s|}ds\right|, \qquad t, t_0 \in \mathbf{R}. \tag{2.9}$$

If we define

$$E := \{x \in X : C(t)x \text{ is once continuously differentiable on } \mathbf{R}\} \tag{2.10}$$

then we have the following.

Lemma 2.5 (see [9, 16]). *Assume that (H1) is satisfied. Then*

(i) $S(t)X \subset E$, $t \in \mathbf{R}$,

(ii) $S(t)E \subset D(A)$, $t \in \mathbf{R}$,

(iii) $(d/dt)C(t)x = AS(t)x$, $x \in E$, $t \in \mathbf{R}$,

(iv) $(d^2/dt^2)C(t)x = AC(t)x = C(t)Ax$, $x \in D(A)$, $t \in \mathbf{R}$.

Lemma 2.6 (see [9, 16]). *Suppose that (H1) holds, $v : \mathbf{R} \to X$ a continuously differentiable function and $q(t) = \int_0^t S(t-s)v(s)ds$. Then, $q(t) \in D(A)$, $q'(t) = \int_0^t C(t-s)v(s)ds$ and $q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t)$.*

Definition 2.7. A function $u(\cdot) \in C^2(I, X)$ is called a classical solution of (1.3) if $u(t) \in D(A)$, satisfies the equation in (1.3) and the initial conditions are verified.

In case of Riemann-Liouville fractional derivative then we require additionally that $D^\beta u(t)$ be continuous.

Definition 2.8. A continuously differentiable solution of the integrodifferential equation

$$\begin{aligned} u(t) = C(t)u_0 + S(t)u_1 &+ \int_0^t S(t-s)f(s)ds \\ &+ \int_0^t S(t-s)\int_0^s g\left(s, \tau, u(\tau), {}^C D^\beta u(\tau)\right)d\tau\,ds \end{aligned} \tag{2.11}$$

is called mild solution of problem (1.3).

In case of Riemann-Liouville fractional derivative the (continuous) solution is merely *β-differentiable* (i.e., $D^{\beta}u(t)$ exists and is continuous).

It follows from [8] that, in case of continuity of the nonlinearities, solutions of (1.3) are solutions of the more general problem (2.11).

3. Well-Posedness in $C^1([0,T])$

For the sake of comparison with the results in the next section we prove here existence and uniqueness of solutions in the space $C^1([0,T])$. This is the space where we usually look for mild solutions in case the first-order derivative of u appears in the nonlinearity (see [1–7]). We consider fractional derivatives in the sense of Caputo. In case of Riemann-Liouville fractional derivatives we can pass to Caputo fractional derivatives through the formula (2.7) provided that solutions are in $C^1([0,T])$ (in theory, absolute continuity is enough).

Let $X_A = D(A)$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$. We need the following assumptions on f and g:

(H2) $f : \mathbf{R}^+ \to X$ is continuously differentiable,

(H3) $g : \mathbf{R}^+ \times \mathbf{R}^+ \times X_A \times X \to X$ is continuous and continuously differentiable with respect to its first variable,

(H4) g and g_1 (the derivative of g with respect to its first variable) are Lipschitz continuous with respect to the last two variables, that is

$$\begin{aligned}\|g(t,s,x_1,y_1) - g(t,s,x_2,y_2)\| &\le A_g(\|x_1 - x_2\|_A + \|y_1 - y_2\|),\\ \|g_1(t,s,x_1,y_1) - g_1(t,s,x_2,y_2)\| &\le A_{g_1}(\|x_1 - x_2\|_A + \|y_1 - y_2\|),\end{aligned} \tag{3.1}$$

for some positive constants A_g and A_{g_1}.

Theorem 3.1. *Assume that (H1)–(H4) hold. If $u_0 \in D(A)$ and $u_1 \in E$ then there exists $T > 0$ and a unique function $u : [0,T] \to X$, $u \in C([0,T];X_A) \cap C^2([0,T];X)$ which satisfies* (1.3) *with Caputo fractional derivative ${}^{C}D^{\beta}u$.*

Proof. We start by proving existence and uniqueness of mild solutions in the space of continuously differentiable functions $C^1([0,T])$. To this end we consider for $t \in [0,T]$

$$\begin{aligned}(Ku)(t) : = C(t)u_0 + S(t)u_1 &+ \int_0^t S(t-s)f(s)ds\\ &+ \int_0^t S(t-s)\int_0^s g\Big(s,\tau,u(\tau),{}^{C}D^{\beta}u(\tau)\Big)d\tau\, ds.\end{aligned} \tag{3.2}$$

Notice that $C(t)u_0 \in D(A)$ because $u_0 \in D(A)$ and we have $AC(t)u_0 = C(t)Au_0$. Also from the facts that $u_1 \in E$ and $S(t)E \subset D(A)$ (see (ii) of Lemma 2.5) it is clear that $S(t)u_1 \in D(A)$.

Moreover, it follows from Lemma 2.6, (H2) and (H3) that both integral terms in (3.2) are in $D(A)$. Therefore, $Ku \in C([0,T];D(A))$. In addition to that we have from Lemma 2.6,

$$\begin{aligned}(AKu)(t) &= C(t)Au_0 + AS(t)u_1 + \int_0^t C(t-s)f'(s)ds + C(t)f(0) - f(t)\\ &\quad + \int_0^t C(t-s)\left[g\left(s,s,u(s),{}^CD^\beta u(s)\right) + \int_0^s g_1\left(s,\tau,u(\tau),{}^CD^\beta u(\tau)\right)d\tau\right]ds \\ &\quad - \int_0^t g\left(t,\tau,u(\tau),{}^CD^\beta u(\tau)\right)d\tau, \qquad t\in[0,T].\end{aligned} \tag{3.3}$$

Next, a differentiation of (3.2) yields

$$\begin{aligned}(Ku)'(t) &= S(t)Au_0 + C(t)u_1 + \int_0^t C(t-s)f(s)ds\\ &\quad + \int_0^t C(t-s)\int_0^s g\left(s,\tau,u(\tau),{}^CD^\beta u(\tau)\right)d\tau\, ds, \qquad t\in[0,T].\end{aligned} \tag{3.4}$$

Therefore, $Ku \in C^1([0,T];X)$ (remember that $u \in C^1([0,T];X)$) and K maps C^1 into C^1.

Now we want to prove that K is a contraction on C^1 endowed with the metric

$$\rho(u,v) := \sup_{0\le t\le T}\left(\|u(t)-v(t)\| + \|A(u(t)-v(t))\| + \left\|u'(t)-v'(t)\right\|\right). \tag{3.5}$$

For u, v in C^1, we can write

$$\begin{aligned}&\|(Ku)(t)-(Kv)(t)\|\\ &\quad \le \int_0^t\left(\int_0^{t-s} Me^{\omega\tau}d\tau\right)A_g\int_0^s\left(\|u(\tau)-v(\tau)\|_A + \left\|{}^CD^\beta u(\tau) - {}^CD^\beta v(\tau)\right\|\right)d\tau\, ds,\end{aligned} \tag{3.6}$$

and since

$$\begin{aligned}\left\|{}^CD^\beta u(\tau) - {}^CD^\beta v(\tau)\right\| &\le \frac{1}{\Gamma(1-\beta)}\int_0^\tau (\tau-\sigma)^{-\beta}\left\|u'(\sigma)-v'(\sigma)\right\|d\sigma\\ &\le \frac{\tau^{1-\beta}}{\Gamma(2-\beta)}\sup_{0\le t\le T}\left\|u'(t)-v'(t)\right\|,\end{aligned} \tag{3.7}$$

it appears that

$$\|(Ku)(t)-(Kv)(t)\| \le \frac{MA_gT^2}{2}\max\left(1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\right)\left(\int_0^T e^{\omega\tau}d\tau\right)\rho(u,v). \tag{3.8}$$

Moreover,

$$\begin{aligned}
&\|(AKu)(t) - (AKv)(t)\| \\
&\quad \leq \int_0^t M e^{\omega(t-s)} A_g \left(\|u(s) - v(s)\|_A + \left\|{}^C D^\beta u(s) - {}^C D^\beta v(s)\right\|\right) ds \\
&\qquad + \int_0^t M e^{\omega(t-s)} A_{g_1} \int_0^s \left(\|u(\tau) - v(\tau)\|_A + \left\|{}^C D^\beta u(\tau) - {}^C D^\beta v(\tau)\right\|\right) d\tau\, ds \\
&\qquad + \int_0^t A_g \left(\|u(s) - v(s)\|_A + \left\|{}^C D^\beta u(s) - {}^C D^\beta v(s)\right\|\right) ds
\end{aligned} \tag{3.9}$$

implies that

$$\begin{aligned}
&\|(AKu)(t) - (AKv)(t)\| \\
&\quad \leq \max\left(1, \frac{T^{1-\beta}}{\Gamma(2-\beta)}\right) \left[A_g T + M\left(A_g + A_{g_1} T\right)\right] \left(\int_0^T e^{\omega(T-s)} ds\right) \rho(u, v).
\end{aligned} \tag{3.10}$$

In addition to that, we see that

$$\begin{aligned}
&\left\|(Ku)'(t) - (Kv)'(t)\right\| \\
&\quad \leq \int_0^t M e^{\omega(t-s)} A_g \int_0^s \left(\|u(\tau) - v(\tau)\|_A + \left\|{}^C D^\beta u(\tau) - {}^C D^\beta v(\tau)\right\|\right) d\tau\, ds \\
&\quad \leq M A_g \int_0^t e^{\omega(t-s)} \int_0^s \left(\|u(\tau) - v(\tau)\|_A + \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sup_{0 \leq \sigma \leq \tau} \left\|u'(\sigma) - v'(\sigma)\right\|\right) d\tau\, ds \\
&\quad \leq \max\left(1, \frac{T^{1-\beta}}{\Gamma(2-\beta)}\right) M A_g T \left(\int_0^T e^{\omega(T-s)} ds\right) \rho(u, v).
\end{aligned} \tag{3.11}$$

These three relations (3.8), (3.10), and (3.11) show that, for T small enough, K is indeed a contraction on C^1, and hence there exists a unique mild solution $u \in C^1$. Furthermore, it is clear (from (3.4), Lemmas 1, and 2) that $u \in C^2([0, T]; X)$ and satisfies the problem (1.3). $\square$

4. Existence of Mild Solutions in Case of R-L Derivative

In the previous section we proved existence and uniqueness of classical solutions provided that $(u_0, u_1) \in D(A) \times E$. From the proof of Theorem 3.1 it can be seen that existence and uniqueness of mild solutions hold when $(u_0, u_1) \in E \times X$. In case of Riemann-Liouville fractional derivative one can still prove well-posedness in C^1 by passing to the Caputo fractional derivative with the help of (2.7) (with a problem of singularity at zero which may be solved through a multiplication by an appropriate term of the form t^γ). This also will require $(u_0, u_1) \in E \times X$. Moreover, from the integrofractional-differential equation (2.11) it is clear that the mild solutions do not have to be continuously differentiable. In this section we will prove existence and uniqueness of mild solutions for the case of Riemann-Liouville fractional derivative for a less regular space than $E \times X$. Namely, for $0 < \beta < 1$, we consider

$$\begin{gathered} E_\beta := \left\{ x \in X : D^\beta C(t)x \text{ is continuous on } \mathbf{R}^+ \right\} \\ FS_\beta := \left\{ v \in C([0,T]) : D^\beta v \in C([0,T]) \right\} \end{gathered} \tag{4.1}$$

equipped with the norm $\|v\|_\beta := \|v\|_C + \|D^\beta v\|_C$ where $\|\cdot\|_C$ is the uniform norm in $C([0,T])$.

We will use the following assumptions:

(H5) $f : \mathbf{R}^+ \to X$ is continuous,

(H6) $g : \mathbf{R}^+ \times \mathbf{R}^+ \times X \times X \to X$ is continuous and Lipschitzian, that is

$$\|g(t,s,x_1,y_1) - g(t,s,x_2,y_2)\| \le A_g(\|x_1 - x_2\| + \|y_1 - y_2\|), \tag{4.2}$$

for some positive constant A_g.

The result below is mentioned in [15, Lemma 2.10] (see also [15]) for functions. Here we state it and prove it for Bochner integral.

Lemma 4.1. *If $I^{1-\alpha}R(t)x \in C^1([0,T])$, $T > 0$, then one has*

$$D^\alpha \int_0^t R(t-s)x\,ds = \int_0^t D^\alpha R(t-s)x\,ds + \lim_{t\to 0^+} I^{1-\alpha}R(t)x, \quad x \in X,\ t \in [0,T]. \tag{4.3}$$

Proof. By Definition 2.2 and Fubini's theorem we have

$$\begin{aligned} D^\alpha \int_0^t R(t-s)x\,ds &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{d\tau}{(t-\tau)^\alpha} \int_0^\tau R(\tau - s)x\,ds \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t ds \int_s^t \frac{R(\tau-s)x}{(t-\tau)^\alpha} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t ds \frac{\partial}{\partial t} \int_s^t \frac{R(\tau-s)x}{(t-\tau)^\alpha} d\tau + \frac{1}{\Gamma(1-\alpha)} \lim_{s\to t^-} \int_s^t \frac{R(\tau-s)x}{(t-\tau)^\alpha} d\tau. \end{aligned} \tag{4.4}$$

These steps are justified by the assumption $I^{1-\alpha}R(t)x \in C^1([0,T])$. Moreover, a change of variable $\sigma = \tau - s$ leads to

$$\begin{aligned} D^\alpha \int_0^t R(t-s)x\,ds = {}& \frac{1}{\Gamma(1-\alpha)} \int_0^t ds \frac{\partial}{\partial t} \int_0^{t-s} \frac{R(\sigma)x}{(t-s-\sigma)^\alpha} d\sigma \\ & + \frac{1}{\Gamma(1-\alpha)} \lim_{t\to 0^+} \int_0^t \frac{R(\sigma)x}{(t-\sigma)^\alpha} d\sigma. \end{aligned} \tag{4.5}$$

This is exactly the formula stated in the lemma. □

Corollary 4.2. *For the sine family $S(t)$ associated with the cosine family $C(t)$ one has, for $x \in X$ and $t \in [0,T]$*

$$D^{\alpha}\int_0^t S(t-s)x\,ds = \int_0^t D^{\alpha}S(t-s)x\,ds = \int_0^t I^{1-\alpha}C(t-s)x\,ds. \tag{4.6}$$

Proof. First, from (2.7), we have

$$\begin{aligned}\frac{d}{dt}I^{1-\alpha}S(t)x = D^{\alpha}S(t)x &= \frac{1}{\Gamma(1-\alpha)}\left[\frac{S(0)x}{t^{\alpha}} + \int_0^t (t-s)^{-\alpha}\frac{dS(s)}{ds}x\,ds\right] \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}C(s)x\,ds = I^{1-\alpha}C(t)x.\end{aligned} \tag{4.7}$$

Notice that this means that $(d/dt)I^{1-\alpha}S(t)x = I^{1-\alpha}C(t)x$ which is in accordance with a general permutation property valid when the function is 0 at 0 (see [10, 15]). It also shows that in this case the Riemann-Liouville derivative and the Caputo derivative are equal. Now from the continuity of $C(t)$ it is clear that $I^{1-\alpha}C(t)x$ is continuous on $[0,T]$ (actually, the operator I^{α} has several smoothing properties, see [11]) and therefore $I^{1-\alpha}S(t)x \in C^1([0,T])$. We can therefore apply Lemma 4.1 to obtain

$$D^{\alpha}\int_0^t S(t-s)x\,ds = \int_0^t D^{\alpha}S(t-s)x\,ds + \lim_{t\to 0^+} I^{1-\alpha}S(t)x, \qquad x \in X,\ \ t \in [0,T]. \tag{4.8}$$

Next, we claim that $\lim_{t\to 0^+} I^{1-\alpha}S(t)x = 0$. This follows easily from the definition of $S(t)$ and $I^{1-\alpha}$. Indeed, we have

$$\left|I^{1-\alpha}S(t)x\right| \le \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}|S(s)x|ds \le \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}|S(t)x|. \tag{4.9}$$

□

We are now ready to state and prove our main result of this section.

Theorem 4.3. *Assume that (H1), (H5), and (H6) hold. If $(u_0,u_1) \in E_{\beta}\times X$, then there exists $T > 0$ and a unique mild solution $u \in FS_{\beta}$ of problem* (1.3) *with Riemann-Liouville fractional derivative.*

Proof. For $t \in [0,T]$, consider the operator

$$\begin{aligned}(Ku)(t) :&= C(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)ds \\ &\quad + \int_0^t S(t-s)\int_0^s g\Big(s,\tau,u(\tau),D^{\beta}u(\tau)\Big)d\tau\,ds.\end{aligned} \tag{4.10}$$

It is clear that $Ku \in C([0,T];X)$ when $u \in FS_\beta$. From Corollary 4.2, we see that

$$\begin{aligned} D^\beta(Ku)(t) &= D^\beta C(t)u_0 + D^\beta S(t)u_1 + \int_0^t I^{1-\beta}C(t-s)f(s)ds \\ &\quad + \int_0^t I^{1-\beta}C(t-s)\int_0^s g\Big(s,\tau,u(\tau),D^\beta u(\tau)\Big)d\tau\, ds. \end{aligned} \tag{4.11}$$

Therefore $Ku \in \mathrm{F}S_\beta$ and maps FS_β to FS_β because $u_0 \in E_\beta$,

$$D^\beta S(t)u_1 = \frac{d}{dt}I^{1-\beta}S(t)u_1 = {}^C D^\beta S(t)u_1 = I^{1-\beta}C(t)u_1, \tag{4.12}$$

and the integral terms are obviously continuous. For $u, v \in FS_\beta$, we find

$$\begin{aligned} &\|(Ku)(t) - (Kv)(t)\| \\ &\quad \le \int_0^t \left(\int_0^{t-s} Me^{\omega\tau}d\tau\right)A_g\int_0^s \Big(\|u(\tau)-v(\tau)\| + \Big\|D^\beta u(\tau) - D^\beta v(\tau)\Big\|\Big)d\tau\, ds \\ &\quad \le \frac{MA_gT^2}{2}\left(\int_0^T e^{\omega\tau}d\tau\right)\left(\sup_{0\le t\le T}\|u(t)-v(t)\| + \sup_{0\le t\le T}\Big\|D^\beta u(t) - D^\beta v(t)\Big\|\right) \\ &\quad \le \frac{MA_gT^2}{2}\left(\int_0^T e^{\omega\tau}d\tau\right)\|u(t)-v(t)\|_\beta. \end{aligned} \tag{4.13}$$

Further,

$$\begin{aligned} &\Big\|\Big(D^\beta Ku\Big)(t) - \Big(D^\beta Kv\Big)(t)\Big\| \\ &\quad \le \int_0^t \left\| I^{1-\beta}C(t-s)\int_0^s \Big[g\Big(s,\tau,u(\tau),D^\beta u(\tau)\Big) - g\Big(s,\tau,v(\tau),D^\beta v(\tau)\Big)\Big]d\tau\, ds\right\| \\ &\quad \le MA_g\int_0^t \frac{(t-s)^{1-\beta}e^{\omega(t-s)}}{\Gamma(2-\beta)}s\, ds \sup_{0\le t\le T}\Big(\|u(t)-v(t)\| + \Big\|D^\beta u(t) - D^\beta v(t)\Big\|\Big) \\ &\quad \le \frac{MA_gT^{2-\beta}}{\Gamma(2-\beta)}\left(\int_0^T e^{\omega(T-s)}ds\right)\|u(t)-v(t)\|_\beta. \end{aligned} \tag{4.14}$$

Thus, for T sufficiently small, K is a contraction on the complete metric space FS_β and hence there exists a unique mild solution to (1.3). □

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