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Research Article

Multiplicity of Solutions for Nonlocal Elliptic System of (p, q)-Kirchhoff Type

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This paper is concerned with the following nonlocal elliptic system of (p,q)-Kirchhoff type $-[M_1(\int_\Omega |\nabla u|^p)]^{p-1}\Delta_p u = \lambda F_u(x,u,v)$, in Ω, $-[M_2(\int_\Omega |\nabla v|^q)]^{q-1}\Delta_q v = \lambda F_v(x,u,v)$, in Ω, $u = v = 0$, on $\partial\Omega$. Under bounded condition on M and some novel and periodic condition on F, some new results of the existence of two solutions and three solutions of the above mentioned nonlocal elliptic system are obtained by means of Bonanno's multiple critical points theorems without the Palais-Smale condition and Ricceri's three critical points theorem, respectively.

1. Introduction and Preliminaries

We are concerned with the following nonlocal elliptic system of (p, q)-Kirchhoff type:

$$\begin{gathered}
-\left[M_1\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1}\Delta_p u = \lambda F_u(x,u,v), \quad \text{in } \Omega,\\
-\left[M_2\left(\int_\Omega |\nabla v|^q\right)\right]^{q-1}\Delta_q v = \lambda F_v(x,u,v), \quad \text{in } \Omega,\\
u = v = 0; \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where $\Omega \subset R^N$ $(N \geq 1)$ is a bounded smooth domain, $\lambda \in (0,+\infty)$, $p > N$, $q > N$, Δ_p is the p-Laplacian operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \tag{1.2}$$

and $M_i: \ R^+ \to R$, $i = 1, 2$, are continuous functions with bounded conditions.

(M) There are two positive constants m_0, m_1 such that

$$m_0 \leq M_i(t) \leq m_1, \quad \forall t \geq 0, \quad i = 1, 2. \tag{1.3}$$

Furthermore, $F : \Omega \times R \times R \rightarrow R$ is a function such that $F(x, s, t)$ is measurable in x for all $(s, t) \in R \times R$ and $F(x, s, t)$ is C^1 in (s, t) for a.e. $x \in \Omega$, and F_u denotes the partial derivative of F with respect to u. Moreover, $F(x, s, t)$ satisfies the following.

$(F1)$ $F(x, 0, 0) = 0$ for a.e. $x \in \Omega$.

$(F2)$ There exist two positive constants $\gamma < p$, $\beta < q$ and a positive real function $\alpha(x) \in L^{\infty}(\Omega)$ such that

$$|F(x, s, t)| \leq \alpha(x)\left(1 + |s|^{\gamma} + |t|^{\beta}\right), \quad \text{for a.e. } x \in \Omega \text{ and all } (s, t) \in R \times R. \tag{1.4}$$

The system (1.1) is related to a model given by the equation of elastic strings

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.5}$$

which was proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, where the parameters in (1.5) have the following meanings: ρ is the mass density, P_0 is the initial tension, h is the area of the cross-section, E is the Young modulus of the material, and L is the length of the string. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations.

Later, (1.5) was developed to the form

$$u_{tt} - M\left(\int_{\Omega} |\nabla u|^2\right)\Delta u = f(x, u) \quad \text{in } \Omega, \tag{1.6}$$

where $M : R^{+} \rightarrow R$ is a given function. After that, many people studied the nonlocal elliptic boundary value problem

$$-M\left(\int_{\Omega} |\nabla u|^2\right)\Delta u = f(x, u) \quad \text{in } \Omega, \ u = 0 \text{ on } \partial\Omega, \tag{1.7}$$

which is the stationary counterpart of (1.6). It is pointed out in [2] that (1.7) models several physical and biological systems, where u describes a process which depends on the average of itself (e.g., population density). By using the methods of sub and supersolutions, variational methods, and other techniques, many results of (1.7) were obtained, we can refer to [2–12] and the references therein. In particular, Alves et al. [2, Theorem 4] supposes that M satisfies bounded condition (M) and $f(x, t)$ satisfies condition (AR), that is, for some $\nu > 2$ and $R > 0$ such that

$$0 < \nu\ F(x, t) \leq f(x, t)t, \quad \forall |t| \geq R, \ x \in \Omega, \tag{AR}$$

where $F(x,t) = \int_0^t f(x,s)ds$; one positive solutions for (1.7) was obtained. It is well known that condition (AR) plays an important role for showing the boundedness of Palais-Smale sequences. More recently, Corrêa and Nascimento in [13] studied a nonlocal elliptic system of p-Kirchhoff type

$$\begin{gathered}
-\left[M_1\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1}\Delta_p u = f(u,v) + \rho_1(x), \quad \text{in } \Omega, \\
-\left[M_2\left(\int_\Omega |\nabla v|^p\right)\right]^{p-1}\Delta_p v = g(u,v) + \rho_2(x), \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{P}$$

where η is the unit exterior vector on $\partial\Omega$, and M_i, ρ_i $(i = 1,2)$, f, g satisfy suitable assumptions, where a special and important condition: periodic condition on nonlinearity was assumed. They obtained the existence of a weak solution for the nonlocal elliptic system of p-Kirchhoff type (P) under Neumann boundary condition via Ekeland's Variational Principle.

In the present paper, our objective is to consider the nonlocal elliptic system of $(p,\ q)$-Kirchhoff-type (1.1), instead of the nonlocal elliptic system of p-Kirchhoff type and single Kirchhoff type equation. Under bounded condition on M and some novel conditions without PS condition and periodic condition on F, we will prove the existence of two solutions and three solutions of system (1.1) by means of one multiple critical points theorem without the Palais-Smale condition of Bonanno in [14] and an equivalent formulation [15, Theorem 2.3] of Ricceri's three critical points theorem [16, Theorem 1], respectively.

In order to state our main results, we need the following preliminaries.

Let $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ be the Cartesian product of two Sobolev spaces, which is a reflexive real Banach space endowed with the norm

$$\|(u,v)\| = \|u\|_p + \|v\|_q, \tag{1.8}$$

where $\|\cdot\|_p$ and $\|\cdot\|_q$ denote the norms of $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively. That is,

$$\|u\|_p = \left(\int_\Omega |\nabla u|^p\right)^{1/p}, \qquad \|v\|_q = \left(\int_\Omega |\nabla v|^q\right)^{1/q} \tag{1.9}$$

for all $u \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,q}(\Omega)$.

Since $p > N$ and $q > N$, $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$ are compactly embedded in $C^0(\overline{\Omega})$. Let

$$C = \max\left\{ \sup_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}\{|u(x)|^p\}}{\|u\|_p^p}, \sup_{v\in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}\{|v(x)|^q\}}{\|v\|_q^q} \right\}; \tag{1.10}$$

then we have $C < +\infty$. Furthermore, it is known from [17] that

$$\sup_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}\{|u(x)|^p\}}{\|u\|_p} \leq \frac{N^{-1/P}}{\sqrt{\pi}}\left(\Gamma\left(1+\frac{N}{2}\right)\right)^{1/N}\left(\frac{p-1}{p-N}\right)^{1-1/P}|\Omega|^{(1/N)-(1/P)}, \tag{1.11}$$

where Γ denotes the Gamma function and $|\Omega|$ is the Lebesgue measure of Ω. Additionally, (1.11) is an equality when Ω is a ball.

Recall that $(u, v) \in X$ is called a weak solution of system (1.1) if

$$\begin{aligned} &\left[M_1\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi + \left[M_2\left(\int_\Omega |\nabla v|^q\right)\right]^{q-1}\int_\Omega |\nabla v|^{q-2}\nabla v\nabla\psi \\ &\quad - \lambda\int_\Omega F_u(x,u,v)\varphi(x)dx - \lambda\int_\Omega F_v(x,u,v)\psi(x)dx = 0, \end{aligned} \tag{1.12}$$

for all $(\varphi, \psi) \in X$. Define the functional $I : X \to R$ given by

$$I(u,v) = \frac{1}{p}\widehat{M_1}\left(\int_\Omega |\nabla u|^p\right) + \frac{1}{q}\widehat{M_2}\left(\int_\Omega |\nabla v|^q\right) - \lambda\int_\Omega F(x,u,v)dx \tag{1.13}$$

for all $(u, v) \in X$, and where

$$\widehat{M_1}(t) = \int_0^t [M_1(s)]^{p-1}ds, \qquad \widehat{M_2}(t) = \int_0^t [M_2(s)]^{q-1}ds, \qquad \forall t \geq 0. \tag{1.14}$$

By the conditions (M) and $(F2)$, it is easy to see that $I \in C^1(X, R)$ and a critical point of I corresponds to a weak solution of the system (1.1).

Now, giving $x_0 \in \Omega$ and choosing $R_2 > R_1 > 0$ such that $B(x_0, R_2) \subseteq \Omega$, where $B(x, R) = \{y \in R^N : |y - x| < R\}$. Next we give some notations.

$$\begin{aligned} \alpha_1 = \alpha_1(N,p,R_1,R_2) &= \frac{C^{1/P}\left(R_2^N - R_1^N\right)^{1/P}}{R_2 - R_1}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{1/P}, \\ \alpha_2 = \alpha_2(N,q,R_1,R_2) &= \frac{C^{1/q}\left(R_2^N - R_1^N\right)^{1/q}}{R_2 - R_1}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{1/q}. \end{aligned} \tag{1.15}$$

Moreover, let a, c be positive constants, denote

$$y(x) = \frac{a}{R_2 - R_1}\left(R_2 - \left\{\sum_{i=1}^{N}\left(x^i - x_0^i\right)^2\right\}^{1/2}\right), \quad \forall x \in B(x_0, R_2) \setminus B(x_0, R_1),$$

$$A(c) = \left\{(s,t) \in R \times R : |s|^p + |t|^q \le c\right\}, \qquad g(c) = \int_\Omega \sup_{(s,t)\in A(c)} F(x,s,t)dx,$$

$$k(a) = \int_{B(x_0,R_2)\setminus B(x_0,R_1)} F(x, y(x), y(x))dx + \int_{B(x_0,R_1)} F(x,a,a)dx,$$

$$h(c,a) = k(a) - g(c), \qquad M^+ = \max\left\{\frac{m_1^{p-1}}{p}, \frac{m_1^{q-1}}{q}\right\}, \qquad M_- = \min\left\{\frac{m_0^{p-1}}{p}, \frac{m_0^{q-1}}{q}\right\}. \tag{1.16}$$

Now we are ready to state our main results for the system (1.1)

Theorem 1.1. *Assume that* $(F1)$-$(F2)$ *hold and there are three positive constants* a, c_1, c_2 *with* $c_1 < (a\alpha_1)^p + (a\alpha_2)^q \le 1 < c_2$ *such that*

$$M^+ g(c_1) < M_- h(c_1, a), \qquad M^+ g(c_2) < M_- h(c_1, a). \tag{1.17}$$

Then, for each

$$\lambda \in \left(\frac{M^+\left[(a\alpha_1)^p + (a\alpha_2)^q\right]}{Ch(c_1,a)}, \frac{M_- c_1\left[(a\alpha_1)^p + (a\alpha_2)^q\right]}{C}\min\left\{\frac{1}{g(c_1)}, \frac{1}{g(c_2)}\right\}\right), \tag{1.18}$$

there exists a positive real number ρ *such that the system* (1.1) *has at least two weak solutions* $(u_i, v_i) \in X \quad (i = 1, 2)$ *whose norms in* $C^0(\Omega)$ *are less than some positive constant* ρ.

Theorem 1.2. *Assume that* $(F1)$-$(F2)$ *hold and there are two positive constants* a, b, *with* $(a\alpha_1)^p + (a\alpha_2)^q > bM^+/M_-$ *such that*

(F3) $F(x,s,t) \ge 0$ *for a.e.* $x \in \Omega \setminus B(x_0, R_1)$ *and all* $(s,t) \in [0,a] \times [0,a]$;

(F4) $[(a\alpha_1)^p + (a\alpha_2)^q]|\Omega|\sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)} F(x,s,t) < b\int_{B(x_0,R_1)} F(x,a,a)dx$.

Then there exist an open interval $\Lambda \subseteq [0, +\infty]$ *and a positive real number* ρ *such that, for each* $\lambda \in \Lambda$, *the system* (1.1) *has at least three weak solutions* $w_i = (u_i, v_i) \in X \quad (i = 1, 2, 3)$ *whose norms* $\|w_i\|$ *are less than* ρ.

2. Proofs of Main Results

Before proving the results, we state one multiple critical points theorem without the Palais-Smale condition of Bonanno in [13] and an equivalent formulation [14, Theorem 2.3] of Ricceri's three critical points theorem [15, Theorem 1], which are our main tools.

Theorem 2.1 (see [14, Theorem 2.1]). *Let X be a reflexive real Banach space, and let $\Psi, \Phi : X \to R$ be two sequentially weakly lower semicontinuous functions. Assume that Ψ is (strongly) continuous and satisfies $\lim_{\|u\| \to \infty} \Psi(u) = +\infty$. Assume also that there exist two constants r_1 and r_2 such that*

- (j) $\inf_X \Psi < r_1 < r_2$;
- (jj) $\varphi_1(r_1) < \varphi_2(r_1, r_2)$;
- (jjj) $\varphi_1(r_2) < \varphi_2(r_1, r_2)$;

where

$$\varphi_1(r) = \inf_{u \in \Psi^{-1}(-\infty, r)} \frac{\Phi(u) - \inf_{u \in \overline{\Psi^{-1}(-\infty, r)}^{w}} \Phi(u)}{r - \Psi(u)},$$
$$\varphi_2(r_1, r_2) = \inf_{u \in \Psi^{-1}(-\infty, r_1)} \sup_{v \in \Psi^{-1}[r_1, r_2)} \frac{\Phi(u) - \Phi(v)}{\Psi(v) - \Psi(u)}. \tag{2.1}$$

Then, for each

$$\lambda \in \left(\frac{1}{\varphi_2(r_1, r_2)}, \min\left\{ \frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_2(r_2)} \right\} \right), \tag{2.2}$$

the functional $\Psi + \lambda\Phi$ has two local minima which lie in $\Psi^{-1}(-\infty, r_1)$ and $\Psi^{-1}[r_1, r_2)$, respectively.

Theorem 2.2 (see [15, Theorem 2.3]). *Let X be a separable and reflexive real Banach space. $\Psi : X \to \mathbf{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Phi : X \to R$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Suppose that*

- (i) $\lim_{\|u\| \to \infty} (\Psi(u) + \lambda\Phi(u)) = +\infty$ *for each* $\lambda > 0$;
- (ii) *There are a real number r, and u_0, $u_1 \in X$ such that $\Psi(u_0) < r < \Psi(u_1)$;*
- (iii) $\inf_{u \in \Psi^{-1}(-\infty, r]} \Phi(u) > ((\Psi(u_1) - r)\Phi(u_0) + (r - \Psi(u_0))\Phi(u_1)) / (\Psi(u_1) - \Psi(u_0))$.

Then there exist an open interval $\Lambda \subseteq [0, +\infty]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the equation $\Psi'(u) + \lambda\Phi'(u) = 0$ has at least three weak solutions whose norms in X are less than ρ.

First, we give one basic lemma.

Lemma 2.3. *Assume that (M) and $(F2)$ hold; let*

$$\Psi(u, v) = \frac{1}{p} \widehat{M}_1 \left(\int_\Omega |\nabla u|^p \right) + \frac{1}{q} \widehat{M}_2 \left(\int_\Omega |\nabla v|^q \right), \qquad \Phi(u, v) = -\int_\Omega F(x, u, v) dx, \tag{2.3}$$

for all $(u, v) \in X$. Then Ψ and Φ are continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functionals. Moreover, the Gâteaux derivative of Ψ admits a continuous inverse on X^* and the Gâteaux derivative of Φ is compact.

Proof. By condition (M), it is easy to see that Ψ is continuously Gâteaux differentiable. Moreover, the Gâteaux derivative of Ψ admits a continuous inverse on X^*. Thanks to $p > N, q > N$, and $(F2)$, Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative is compact. Next We will prove that Ψ is a sequentially weakly lower semicontinuous functional. Indeed, for any $(u_n, v_n) \in X$ with $(u_n, v_n) \rightharpoonup (u, v)$ in X, then $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $v_n \rightharpoonup v$ in $W_0^{1,q}(\Omega)$. Therefore,

$$\liminf_{n\to\infty} \|u_n\|_p \geq \|u\|_p, \qquad \liminf_{n\to\infty} \|v_n\|_q \geq \|v\|_q \tag{2.4}$$

due to the weakly lower semicontinuity of norm. Hence by virtue of the continuity and monotonicity of $\widehat{M}_1$ and $\widehat{M}_2$, we conclude that

$$\begin{aligned} \widehat{M}_1\left(\int_\Omega |\nabla u|^p\right) &\leq \widehat{M}_1\left(\liminf_{n\to\infty}\int_\Omega |\nabla u_n|^p\right) \leq \liminf_{n\to\infty} \widehat{M}_1\left(\int_\Omega |\nabla u_n|^p\right), \\ \widehat{M}_2\left(\int_\Omega |\nabla v|^q\right) &\leq \widehat{M}_2\left(\liminf_{n\to\infty}\int_\Omega |\nabla v_n|^q\right) \leq \liminf_{n\to\infty} \widehat{M}_2\left(\int_\Omega |\nabla v_n|^q\right), \end{aligned} \tag{2.5}$$

Consequently, Ψ is a sequentially weakly lower semicontinuous functional. □

Proof of Theorem 1.1. Let

$$\Psi(u,v) = \frac{1}{p}\widehat{M}_1\left(\int_\Omega |\nabla u|^p\right) + \frac{1}{q}\widehat{M}_2\left(\int_\Omega |\nabla v|^q\right), \qquad \Phi(u,v) = -\int_\Omega F(x,u,v)dx \tag{2.6}$$

for all $(u, v) \in X$. Under condition (M), by a simple computation, we have

$$M_-\left(\|u\|_p^p + \|v\|_q^q\right) \leq \Psi(u,v) \leq M^+\left(\|u\|_p^p + \|v\|_q^q\right). \tag{2.7}$$

Therefore, (2.7) implies that

$$\lim_{\|(u,v)\|\to+\infty} \Psi(u,v) = +\infty. \tag{2.8}$$

Put

$$r_1 = \frac{M_- c_1}{C}\left[(a\alpha_1)^p + (a\alpha_2)^q\right], \qquad r_2 = \frac{M_- c_2}{C}\left[(a\alpha_1)^p + (a\alpha_2)^q\right]. \tag{2.9}$$

Denote

$$\begin{aligned} \varphi_1(r) &= \inf_{(u,v)\in\ \Psi^{-1}(-\infty,r)} \frac{\Phi(u,v) - \inf_{(u,v)\in\ \overline{\Psi^{-1}(-\infty,r)}^w}\Phi(u,v)}{r - \Psi(u,v)}, \\ \varphi_2(r_1,r_2) &= \inf_{(u,v)\in\ \Psi^{-1}(-\infty,r_1)}\ \sup_{(u_1,v_1)\in\ \Psi^{-1}[r_1,r_2)} \frac{\Phi(u,v) - \Phi(u_1,v_1)}{\Psi(u_1,v_1) - \Psi(u,v)}, \end{aligned} \tag{2.10}$$

and $\overline{\Psi^{-1}(-\infty,r)}^w$ is the closure of $\Psi^{-1}(-\infty,r)$ in the weak topology.

Set

$$w_0(x) = \begin{cases} 0, & x \in \overline{\Omega} \setminus B(x_0, R_2), \\ \dfrac{a}{R_2 - R_1}\left(R_2 - \left\{\displaystyle\sum_{i=1}^{N}(x^i - x_0^i)\right\}^{1/2}\right), & x \in B(x_0, R_2) \setminus B(x_0, R_1), \\ a, & x \in B(x_0, R_1). \end{cases} \tag{2.11}$$

Then $(u_0, v_0) \in X$, where $u_0(x) = v_0(x) = w_0(x)$ and

$$\|u_0\|_p^p = \|w_0\|_p^p = \frac{(a\alpha_1)^p}{C}, \qquad \|v_0\|_q^q = \|w_0\|_q^q = \frac{(a\alpha_1)^q}{C}. \tag{2.12}$$

Consequently, (2.7) and (2.12) imply that

$$r_1 < \Psi(u_0, v_0) < r_2. \tag{2.13}$$

Furthermore, (2.13) implies that

$$\begin{aligned} \varphi_2(r_1, r_2) &= \inf_{(u,v)\in\ \Psi^{-1}(-\infty, r_1)} \ \sup_{(u_1,v_1)\in\ \Psi^{-1}[r_1,r_2)} \frac{\Phi(u,v) - \Phi(u_1,v_1)}{\Psi(u_1,v_1) - \Psi(u,v)} \\ &\geq \inf_{(u,v)\in\Psi^{-1}(-\infty,r_1)} \frac{\Phi(u,v) - \Phi(u_0,v_0)}{\Psi(u_0,v_0) - \Psi(u,v)}. \end{aligned} \tag{2.14}$$

On the other hand, by $(F1)$, (1.17), and, (2.11), one has

$$\int_\Omega F(x, u_0, v_0)dx = k(a) > h(c_1, a) > \frac{M^+}{M_-} g(c_1) > g(c_1) = \int_\Omega \sup_{(s,t)\in A(c_1)} F(x,s,t)dx. \tag{2.15}$$

For each $(u, v) \in X$ with $\Psi(u, v) \leq r_1$, and $x \in \Omega$, by (2.7), we conclude

$$|u(x)|^p + |v(x)|^q \leq C\left(\|u\|_p^p + \|v\|_q^q\right) \leq \frac{Cr_1}{M_-} = c_1\left[(a\alpha_1)^p + (a\alpha_2)^q\right] \leq c_1. \tag{2.16}$$

Therefore, the combination of (2.15) and (2.16) implies

$$\begin{aligned}
\frac{\Phi(u,v)-\Phi(u_0,v_0)}{\Psi(u_0,v_0)-\Psi(u,v)} &= \frac{\int_\Omega F(x,u_0,v_0)dx - \int_\Omega F(x,u,v)dx}{\Psi(u_0,v_0)-\Psi(u,v)} \\
&\geq \frac{\int_\Omega F(x,u_0,v_0)dx - \int_\Omega \sup_{|u(x)|^p+|v(x)|^q\leq c_1} F(x,u,v)dx}{\Psi(u_0,v_0)-\Psi(u,v)} \\
&\geq \frac{\int_\Omega F(x,u_0,v_0)dx - \int_\Omega \sup_{|u(x)|^p+|v(x)|^q\leq c_1} F(x,u,v)dx}{\Psi(u_0,v_0)} \\
&\geq \frac{\int_\Omega F(x,u_0,v_0)dx - \int_\Omega \sup_{|u(x)|^p+|v(x)|^q\leq c_1} F(x,u,v)\ dx}{M^+\left(\|u_0\|_p^p+\|v_0\|_q^q\right)} \\
&= \frac{C}{M^+\left[(a\alpha_1)^p+(a\alpha_2)^q\right]}h(c_1,a).
\end{aligned} \tag{2.17}$$

By (2.14) and (2.17), we have

$$\varphi_2(r_1,r_2) \geq \frac{C}{M^+\left[(a\alpha_1)^p+(a\alpha_2)^q\right]}h(c_1,a). \tag{2.18}$$

Similarly, for every $(u,v)\in X$ such that $\Psi(u,v)\leq r$, where r is a positive real number, and $x\in\Omega$, one has

$$|u(x)|^p+|v(x)|^q \leq C\left(\|u\|_p^p+\|v\|_q^q\right) \leq \frac{Cr}{M_-}. \tag{2.19}$$

By virtue of Ψ being sequentially weakly lower semicontinuous, then $\overline{\Psi^{-1}(-\infty,r)}^w = \Psi^{-1}(-\infty,r]$. Consequently,

$$\begin{aligned}
\varphi_1(r) &= \inf_{(u,v)\in\ \Psi^{-1}(-\infty,r)} \frac{\Phi(u,v)-\inf_{(u,v)\in\ \overline{\Psi^{-1}(-\infty,r)}^w}\Phi(u,v)}{r-\Psi(u,v)} \\
&\leq \frac{\Phi(0,0)-\inf_{(u,v)\in\ \overline{\Psi^{-1}(-\infty,r)}^w}\Phi(u,v)}{r-\Psi(0,0)} \\
&\leq \frac{-\inf_{(u,v)\in\ \overline{\Psi^{-1}(-\infty,r)}^w}\Phi(u,v)}{r} \\
&\leq \frac{\int_\Omega \sup_{|u(x)|^p+|v(x)|^q\leq Cr/M_-} F(x,u,v)dx}{r}.
\end{aligned} \tag{2.20}$$

It implies that

$$\varphi_1(r_1) \le \frac{g(c_1)}{r_1} = \frac{C}{M_- c_1\left[(a\alpha_1)^p + (a\alpha_2)^q\right]} g(c_1) < \frac{C}{M^+\left[(a\alpha_1)^p + (a\alpha_2)^q\right]} h(c_1, a), \tag{2.21}$$

$$\varphi_1(r_2) \le \frac{g(c_2)}{r_2} = \frac{C}{M_- c_2\left[(a\alpha_1)^p + (a\alpha_2)^q\right]} g(c_2) < \frac{C}{M^+\left[(a\alpha_1)^p + (a\alpha_2)^q\right]} h(c_1, a). \tag{2.22}$$

By (2.18)–(2.22), we conclude

$$\varphi_1(r_1) \le \varphi_2(r_1, r_2), \qquad \varphi_1(r_2) \le \varphi_2(r_1, r_2). \tag{2.23}$$

Therefore, the conditions (j), (jj), and (jjj) in Theorem 2.1 are satisfied. Consequently, by Lemma 2.3 and above facts, the functional $\Psi + \lambda\Phi$ has two local minima $(u_1, v_1), (u_2, v_2) \in X$, which lie in $\Psi^{-1}(-\infty, r_1)$ and $\Psi^{-1}[r_1, r_2)$, respectively. Since $I = \Psi + \lambda\Phi \in C^1$, $(u_1, v_1), (u_2, v_2) \in X$ are the solutions of the equation

$$\Psi'(u, v) + \lambda\Phi'(u, v) = 0. \tag{2.24}$$

Then $(u_1, v_1), (u_2, v_2) \in X$ are the weak solutions of system (1.1).

Since $\Psi(u_i, v_i) < r_2$, $i = 1, 2$, by (1.10) and (2.7),

$$|u_i(x)|^p + |v_i(x)|^q \le \frac{Cr_2}{M_-} \le c_2, \qquad i = 1, 2; \tag{2.25}$$

which implies there exists a positive real number ρ such that the norms of $(u_i, v_i) \in X$ $(i = 1, 2)$ in $C^0(\Omega)$ are less than some positive constant ρ. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let

$$\Psi(u, v) = \frac{1}{p}\widehat{M}_1\left(\int_\Omega |\nabla u|^p\right) + \frac{1}{q}\widehat{M}_2\left(\int_\Omega |\nabla v|^q\right), \qquad \Phi(u, v) = -\int_\Omega F(x, u, v)dx \tag{2.26}$$

for all $(u, v) \in X$. By $(F2)$ and (2.7), we have

$$\begin{aligned}
\Psi(u, v) + \lambda\Phi(u, v) &= \frac{1}{p}\widehat{M}_1\left(\int_\Omega |\nabla u|^p\right) + \frac{1}{q}\widehat{M}_2\left(\int_\Omega |\nabla v|^q\right) - \lambda\int_\Omega F(x, u, v)dx \\
&\ge M_-\left(\|u\|_p^p + \|v\|_q^q\right) - \lambda\int_\Omega \alpha(x)\left(1 + |u(x)|^\gamma + |v(x)|^\beta\right)dx \\
&\ge M_-\left(\|u\|_p^p + \|v\|_q^q\right) - \lambda\|\alpha\|_\infty\left(|\Omega| + k_1\|u\|_p^\gamma + k_2\|v\|_q^\beta\right),
\end{aligned} \tag{2.27}$$

where $k_1,\ k_1$ are positive constants. Since $\gamma < p,\ \beta < q$, (2.27) implies that

$$\lim_{\|(u,v)\| \to +\infty} (\Psi(u,v) + \lambda\Phi(u,v)) = +\infty. \tag{2.28}$$

The same as in (2.11), defining a function $w_0(x)$, and letting $u_0(x) = v_0(x) = w_0(x)$, then (2.12) is also satisfied. Choosing $r = bM^+/C$, by (2.7), (2.12), and $(a\alpha_1)^p + (a\alpha_2)^q > bM^+/M_-$, we conclude

$$\Psi(u_0,v_0) \geq M_-\left(\|u_0\|_p^p + \|v_0\|_q^q\right) = \frac{M_-}{C}\left[(a\alpha_1)^p + (a\alpha_2)^q\right] > \frac{M_-}{C}\frac{bM^+}{M_-} = r. \tag{2.29}$$

By $(F3)$ and the definitions of u_0 and v_0, one has

$$\begin{aligned} |\Omega| \sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)} F(x,s,t) &< \frac{b}{(a\alpha_1)^p + (a\alpha_2)^q}\int_{B(x_0,R_1)} F(x,a,a)dx \\ &= \frac{bM^+}{C}\frac{\int_{B(x_0,R_1)} F(x,a,a)dx}{M^+\left((a\alpha_1)^p + (a\alpha_2)^q\right)/C} \\ &\leq \frac{bM^+}{C}\frac{\int_{\Omega\setminus B(x_0,R_1)} F(x,u_0,v_0)dx + \int_{B(x_0,R_1)} F(x,u_0,v_0)dx}{M^+\left(\|u_0\|_p^p + \|v_0\|_q^q\right)} \\ &\leq \frac{bM^+}{C}\frac{\int_\Omega F(x,u_0,v_0)dx}{\Psi(u_0,v_0)}. \end{aligned} \tag{2.30}$$

For every $(u,v) \in \mathrm{X}$ such that $\Psi(u,v) \leq r$, and $x \in \Omega$, one has

$$|u(x)|^p + |v(x)|^q \leq C\left(\|u\|_p^p + \|v\|_q^q\right) \leq \frac{Cr}{M_-} = \frac{C}{M_-}\frac{bM^+}{C} = \frac{bM^+}{M_-}. \tag{2.31}$$

By the combination of (2.30) and (2.31), we have

$$\begin{aligned} \sup_{(u,v)\in\Psi^{-1}(-\infty,r)} (-\Phi(u,v)) &= \sup_{\{(u,v)|\Psi(u,v)\leq r\}} \int_\Omega F(x,u,v)dx \\ &\leq \sup_{\{(u,v)||u(x)|^p+|v(x)|^q\leq bM^+/M_-\}} \int_\Omega F(x,u,v)dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\Omega} \sup_{(s,t)\in A(bM^+/M_-)} F(x,s,t)dx \\
&\leq |\Omega| \sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)} F(x,s,t) \\
&\leq \frac{bM^+}{C}\frac{\int_{\Omega} F(x,u_0,v_0)dx}{\Psi(u_0,v_0)} \\
&= r\frac{-\Phi(u_0,v_0)}{\Psi(u_0,v_0)}.
\end{aligned} \tag{2.32}$$

Therefore,

$$\inf_{(u,v)\in\Psi^{-1}(-\infty,r)} \Phi(u,v) > r\frac{\Phi(u_0,v_0)}{\Psi(u_0,v_0)}. \tag{2.33}$$

Note that $\Phi(0,0) = \Psi(0,0) = 0$, we conclude that

$$\inf_{(u,v)\in\Psi^{-1}(-\infty,r]} \Phi(u,v) > \frac{(\Psi(u_0,v_0) - r)\Phi(0,0) + (r - \Psi(0,0))\Phi(u_0,v_0)}{\Psi(u_0,v_0) - \Psi(0,0)}. \tag{2.34}$$

Hence, by Lemma 2.3 and above facts, Ψ and Φ satisfy all conditions of Theorem 2.2; then the conclusion directly follows from Theorem 2.2. □

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