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Research Article

A New Iterative Algorithm for the Set of Fixed-Point Problems of Nonexpansive Mappings and the Set of Equilibrium Problem and Variational Inequality Problem

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We introduce a new iterative scheme and a new mapping generated by infinite family of nonexpansive mappings and infinite real number. By using both of these ideas, we obtain strong convergence theorem for finding a common element of the set of solution of equilibrium problem and the set of variational inequality and the set of fixed-point problems of infinite family of nonexpansive mappings. Moreover, we apply our main result to obtain strong convergence theorems for finding a common element of the set of solution of equilibrium problem and the set of variational inequality and the set of common fixed point of pseudocontractive mappings.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear mapping and let $F : C \times C \to \mathbb{R}$ be a bifunction. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient $\overline{\gamma}$ if there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2. \tag{1.1}$$

The equilibrium problem for F is to find $x \in C$, such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(F)$. Many problems in physics, optimization, and economics are seeking some elements of $\mathrm{EP}(F)$, see [2, 3]. Several iterative methods have been proposed to solve the equilibrium problem, see, for instance, [2–4]. In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem.

The variational inequality problem is to find a point $u \in C$, such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted by $\mathrm{VI}(C, A)$. Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games reduce to find element of (1.2) and (1.3).

A mapping A of C into H is called *inverse-strongly monotone*, see [5], if there exists a positive real number α, such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \| Ax - Ay \|^2 \tag{1.4}$$

for all $x, y \in C$.

The problem of finding a common fixed point of a family of nonexpansive mappings has been studied by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed-point sets of a family of nonexpansive mapping (see [6, 7]).

The problem of finding a common element of $\mathrm{EP}(F)$ and the set of all common fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and importance. Many iterative methods are purposed for finding a common element of the solutions of the equilibrium problem and fixed-point problem of nonexpansive mappings, see [8–10].

In 2007, S. Takahashi and W. Takahashi [10] introduced a general iterative method for finding a common element of $\mathrm{EP}(F, A)$ and $F(T)$. They defined $\{x_n\}$ in the following way:

$$\begin{gathered} u, x_1 \in C, \text{ arbitrarily;} \\ F(z_n, y) + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) S z_n, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{1.5}$$

where $\{\beta_n\} \subset [0, 1]$, and proved strong convergence of the scheme (1.5) to $z \in F(T) \bigcap \mathrm{EP}(F)$, where $z = P_{F(S) \bigcap \mathrm{EP}(F)} f(z)$ in the framework of a Hilbert space, under some suitable conditions on $\{\beta_n\}$, $\{\lambda_n\}$ and bifunction F.

In this paper, by motive of (1.5), we prove strong convergence theorem for finding a common element of the set of solution of equilibrium problem and the set of variational inequality and the set of fixed-point problems by using a new mapping generated by infinite family of nonexpansive mapping and infinite real number. Moreover, we apply our main result to obtain strong convergence theorems for finding a common element of the set of solution of equilibrium problem and the set of variational inequality and the set of common fixed point of pseudocontractive mappings.

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Let C be closed convex subset of a real Hilbert space H, and let P_C be the metric projection of H onto C, that is, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{2.1}$$

The following characterizes the projection P_C.

Lemma 2.1 (see [11]). *Given $x \in H$ and $y \in C$, then $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C. \tag{2.2}$$

Lemma 2.2 (see [12]). *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $S : C \to C$ be a nonexpansive mapping, then $I - S$ is demiclosed at zero.*

Lemma 2.3 (see [13]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n) s_n + \delta_n, \quad \forall n \geq 0, \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence, such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n \to \infty} \delta_n / \alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$

then $\lim_{n \to \infty} s_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$, (A3) for all $x, y, z \in C$,

$$\lim_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y), \tag{2.4}$$

(A4) for all $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 2.4 (see [2]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$, then there exists $z \in C$, such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle, \tag{2.5}$$

for all $x \in C$.

Lemma 2.5 (see [3]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}, \tag{2.6}$$

for all $z \in H$, then the following hold:

(1) *T_r is single valued,*

(2) *T_r is firmly nonexpansive, that is,*

$$\left\| T_r(x) - T_r(y) \right\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle \quad \forall x, y \in H, \tag{2.7}$$

(3) *$F(T_r) = \mathrm{EP}(F)$,*

(4) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 2.6 (see [14]). *Let H be a Hibert space, let C be a nonempty closed convex subset of H, and let A be a mapping of C into H. Let $u \in C$, then for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Longleftrightarrow u \in \mathrm{VI}(C, A), \tag{2.8}$$

where P_C is the metric projection of H onto C.

Definition 2.7. Let C be a nonempty convex subset of a real Hilbert space. Let $T_i,\ i = 1,2,\dots$ be mappings of C into itself. For each $j = 1,2,\dots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$, we define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I, \\
U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I, \\
&\ \vdots \\
U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I, \\
U_{n,k} &= \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I, \\
&\ \vdots \\
U_{n,2} &= \alpha_1^2 T_2 U_{n,3} + \alpha_2^2 U_{n,3} + \alpha_3^2 I, \\
S_n = U_{n,1} &= \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I.
\end{aligned} \tag{2.9}$$

This mapping is called *S-mapping* generated by $T_n, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$.

Lemma 2.8. *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{\infty}$ be nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0,1)$, for all $j = 1,2,\dots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$, then for every $x \in C$ and $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n,k} x$ exists.*

Proof. Let $x \in C$ and $y \in \bigcap_{i=1}^{\infty} F(T_i)$. Fix $k \in \mathbb{N}$, then for every $n \in \mathbb{N}$ with $n \geq k$, we have

$$\begin{aligned}
\|U_{n+1,k}x - U_{n,k}x\|^2 &= \Big\| \alpha_1^k T_k U_{n+1,k+1} x + \alpha_2^k U_{n+1,k+1} x + \alpha_3^k x \\
&\qquad - \alpha_1^k T_k U_{n,k+1} x - \alpha_2^k U_{n,k+1} x - \alpha_3^k x \Big\|^2 \\
&= \Big\| \alpha_1^k (T_k U_{n+1,k+1} x - T_k U_{n,k+1} x) + \alpha_2^k (U_{n+1,k+1} x - U_{n,k+1} x) \Big\|^2 \\
&\leq \alpha_1^k \|T_k U_{n+1,k+1} x - T_k U_{n,k+1} x\|^2 + \alpha_2^k \|U_{n+1,k+1} x - U_{n,k+1} x\|^2 \\
&\leq \alpha_1^k \|U_{n+1,k+1} x - U_{n,k+1} x\|^2 + \alpha_2^k \|U_{n+1,k+1} x - U_{n,k+1} x\|^2 \\
&\leq \left(1 - \alpha_3^k\right) \|U_{n+1,k+1} x - U_{n,k+1} x\|^2 \\
&\ \vdots
\end{aligned}$$

$$
\begin{aligned}
&\le \prod_{j=k}^{n}\left(1-\alpha_3^j\right)\|U_{n+1,n+1}x-U_{n,n+1}x\|^2\\
&= \prod_{j=k}^{n}\left(1-\alpha_3^j\right)\left\|\alpha_1^{n+1}T_{n+1}U_{n+1,n+2}x+\alpha_2^{n+1}U_{n+1,n+2}x+\alpha_3^{n+1}x-x\right\|^2\\
&= \prod_{j=k}^{n}\left(1-\alpha_3^j\right)\left\|\alpha_1^{n+1}T_{n+1}x+(1-\alpha_1^{n+1})x-x\right\|^2\\
&= \prod_{j=k}^{n}\left(1-\alpha_3^j\right)\left\|\alpha_1^{n+1}(T_{n+1}x-x)\right\|^2\\
&\le \prod_{j=k}^{n}\left(1-\alpha_3^j\right)\left(\|T_{n+1}x-y\|+\|y-x\|\right)^2\\
&\le \prod_{j=k}^{n}\left(1-\alpha_3^j\right)\left(\|x-y\|+\|y-x\|\right)^2\\
&\le \prod_{j=k}^{n}\left(1-\alpha_3^j\right)\left(2\|x-y\|\right)^2\\
&\le b^{n-(k-1)}\left(2\|x-y\|\right)^2.
\end{aligned}
\tag{2.10}
$$

It follows that

$$
\begin{aligned}
\|U_{n+1,k}x-U_{n,k}x\| &\le b^{(n-(k-1))/2}(2\|x-y\|)\\
&= \frac{b^{n/2}}{b^{(k-1)/2}}(2\|x-y\|)\\
&= \frac{a^n}{a^{k-1}}M,
\end{aligned}
\tag{2.11}
$$

where $a=b^{1/2}\in(0,1)$ and $M=2\|x-y\|$.

For any $k,n,p\in\mathbb{N},\ p>0,\ n\ge k$, we have

$$
\begin{aligned}
\|U_{n+p,k}x-U_{n,k}x\| &\le \|U_{n+p,k}x-U_{n+p-1,k}x\|+\|U_{n+p-1,k}x-U_{n+p-2,k}x\|\\
&\quad+\cdots+\|U_{n+1,k}x-U_{n,k}x\|\\
&= \sum_{j=n}^{n+p-1}\|U_{j+1,k}x-U_{j,k}x\|\\
&\le \sum_{j=n}^{n+p-1}\frac{a^j}{a^{k-1}}M\\
&\le \frac{a^n}{(1-a)a^{k-1}}M.
\end{aligned}
\tag{2.12}
$$

Since $a \in (0,1)$, we have $\lim_{n\to\infty} a^n = 0$. From (2.12), we have that $\{U_{n,k}x\}$ is a Cauchy sequence. Hence, $\lim_{n\to\infty} U_{n,k}x$ exists. □

For every $k \in \mathbb{N}$ and $x \in C$, we define mapping $U_{\infty,k}$ and $S : C \to C$ as follows:

$$\begin{aligned} \lim_{n\to\infty} U_{n,k}x &= U_{\infty,k}x, \\ \lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x &= Sx. \end{aligned} \tag{2.13}$$

Such a mapping S is called S-mapping generated by $T_n, T_{n-1}, \ldots$ and $\alpha_n, \alpha_{n-1}, \ldots$.

Remark 2.9. For each $n \in \mathbb{N}, S_n$ is nonexpansive and $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - Sx\| = 0$ for every bounded subset D of C. To show this, let $x, y \in C$ and D be a bounded subset of C, then we have

$$\begin{aligned} \|S_n x - S_n y\|^2 &= \left\|\alpha_1^1(T_1U_{n,2}x - T_1U_{n,2}y) + \alpha_2^1(U_{n,2}x - U_{n,2}y) + \alpha_3^1(x-y)\right\|^2 \\ &\le \alpha_1^1\|T_1U_{n,2}x - T_1U_{n,2}y\|^2 + \alpha_2^1\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2 \\ &\le \alpha_1^1\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_2^1\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2 \\ &= \left(1-\alpha_3^1\right)\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2 \\ &\le \left(1-\alpha_3^1\right)\left(\left(1-\alpha_3^2\right)\|U_{n,3}x - U_{n,3}y\|^2 + \alpha_3^2\|x-y\|^2\right) + \alpha_3^1\|x-y\|^2 \\ &= \left(1-\alpha_3^1\right)\left(1-\alpha_3^2\right)\|U_{n,3}x - U_{n,3}y\|^2 + \alpha_3^2\left(1-\alpha_3^1\right)\|x-y\|^2 + \alpha_3^1\|x-y\|^2 \\ &= \prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|U_{n,3}x - U_{n,3}y\|^2 + \left(1 - \prod_{j=1}^{2}\left(1-\alpha_3^j\right)\right)\|x-y\|^2 \\ &\;\vdots \\ &\le \prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|U_{n,n+1}x - U_{n,n+1}y\|^2 + \left(1 - \prod_{j=1}^{2}\left(1-\alpha_3^j\right)\right)\|x-y\|^2 \\ &= \|x-y\|^2. \end{aligned} \tag{2.14}$$

Then, we have that $S : C \to C$ is also nonexpansive, Indeed, observe that for each $x, y \in C$,

$$\|Sx - Sy\| = \lim_{n \to \infty} \|S_n x - S_n y\| \leq \|x - y\|. \tag{2.15}$$

By (2.11), we have

$$\begin{aligned} \|S_{n+1}x - S_n x\| &= \|U_{n+1,1}x - U_{n,1}x\| \\ &\leq a^n M. \end{aligned} \tag{2.16}$$

This implies that for $m > n$ and $x \in D$,

$$\begin{aligned} \|S_m x - S_n x\| &\leq \sum_{j=n}^{m-1} \|S_{j+1}x - S_j x\| \\ &\leq \sum_{j=n}^{m-1} a^j M \\ &\leq \frac{a^n}{1-a} M. \end{aligned} \tag{2.17}$$

By letting $m \to \infty$, for any $x \in D$, we have

$$\|Sx - S_n x\| \leq \frac{a^n}{1-a} M. \tag{2.18}$$

It follows that

$$\lim_{n \to \infty} \sup_{x \in D} \|S_n x - Sx\| = 0. \tag{2.19}$$

Lemma 2.10. *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{\infty}$ be nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S be S-mappings generated by $T_n, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, \ldots$, and $\alpha_n, \alpha_{n-1}, \ldots$, respectively, then $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. It is easy to see that $\bigcap_{i=1}^{\infty} F(T_i) \subseteq F(S)$. For every $n, k \in \mathbb{N}$, with $n \geq k$, let $x_0 \in F(S)$ and $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, then we have

$$\begin{aligned}
\|S_n x_0 - x^*\|^2 &= \left\| \alpha_1^1 (T_1 U_{n,2} x_0 - x^*) + \alpha_2^1 (U_{n,2} x_0 - x^*) + \alpha_3^1 (x_0 - x^*) \right\|^2 \\
&\leq \alpha_1^1 \|T_1 U_{n,2} x_0 - x^*\|^2 + \alpha_2^1 \|U_{n,2} x_0 - x^*\|^2 + \alpha_3^1 \|x_0 - x^*\|^2 \\
&\quad - \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
&\leq \alpha_1^1 \|U_{n,2} x_0 - x^*\|^2 + \alpha_2^1 \|U_{n,2} x_0 - x^*\|^2 \\
&\quad + \alpha_3^1 \|x_0 - x^*\|^2 - \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
&= \left(1 - \alpha_3^1\right) \|U_{n,2} x_0 - x^*\|^2 + \alpha_3^1 \|x_0 - x^*\|^2 \\
&\quad - \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
&\leq \left(1 - \alpha_3^1\right) \Big(\left(1 - \alpha_3^2\right) \|U_{n,3} x_0 - x^*\|^2 + \alpha_3^2 \|x_0 - x^*\|^2 \\
&\qquad\qquad - \alpha_1^2 \alpha_2^2 \|T_2 U_{n,3} x_0 - U_{n,3} x_0\|^2 - \alpha_2^2 \alpha_3^2 \|U_{n,3} x_0 - x_0\|^2 \Big) + \alpha_3^1 \|x_0 - x^*\|^2 \\
&\quad - \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
&= \left(1 - \alpha_3^1\right) \left(1 - \alpha_3^2\right) \|U_{n,3} x_0 - x^*\|^2 + \alpha_3^2 \left(1 - \alpha_3^1\right) \|x_0 - x^*\|^2 + \alpha_3^1 \|x_0 - x^*\|^2 \\
&\quad - \alpha_1^2 \alpha_2^2 \left(1 - \alpha_3^1\right) \|T_2 U_{n,3} x_0 - U_{n,3} x_0\|^2 - \alpha_2^2 \alpha_3^2 \left(1 - \alpha_3^1\right) \|U_{n,3} x_0 - x_0\|^2 \\
&\quad - \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
&= \prod_{j=1}^{2} \left(1 - \alpha_3^j\right) \|U_{n,3} x_0 - x^*\|^2 + \left(1 - \prod_{j=1}^{2} \left(1 - \alpha_3^j\right)\right) \|x_0 - x^*\|^2 \\
&\quad - \alpha_1^2 \alpha_2^2 \left(1 - \alpha_3^1\right) \|T_2 U_{n,3} x_0 - U_{n,3} x_0\|^2 - \alpha_2^2 \alpha_3^2 \left(1 - \alpha_3^1\right) \|U_{n,3} x_0 - x_0\|^2 \\
&\quad - \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
&\leq \prod_{j=1}^{2} \left(1 - \alpha_3^j\right) \Big(\left(1 - \alpha_3^3\right) \|U_{n,4} x_0 - x^*\|^2 + \alpha_3^3 \|x_0 - x^*\|^2 \\
&\qquad\qquad - \alpha_1^3 \alpha_2^3 \|T_3 U_{n,4} x_0 - U_{n,4} x_0\|^2 - \alpha_2^3 \alpha_3^3 \|U_{n,4} x_0 - x_0\|^2 \Big) \\
&\quad + \left(1 - \prod_{j=1}^{2} \left(1 - \alpha_3^j\right)\right) \|x_0 - x^*\|^2 - \alpha_1^2 \alpha_2^2 \left(1 - \alpha_3^1\right) \|T_2 U_{n,3} x_0 - U_{n,3} x_0\|^2 \\
&\quad - \alpha_2^2 \alpha_3^2 \left(1 - \alpha_3^1\right) \|U_{n,3} x_0 - x_0\|^2 - \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
&= \prod_{j=1}^{2} \left(1 - \alpha_3^j\right) \left(1 - \alpha_3^3\right) \|U_{n,4} x_0 - x^*\|^2 + \alpha_3^3 \prod_{j=1}^{2} \left(1 - \alpha_3^j\right) \|x_0 - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \alpha_1^3\alpha_2^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|T_3U_{n,4}x_0-U_{n,4}x_0\|^2 \\
&- \alpha_2^3\alpha_3^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|U_{n,4}x_0-x_0\|^2+\left(1-\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\right)\|x_0-x^*\|^2 \\
&- \alpha_1^2\alpha_2^2\left(1-\alpha_3^1\right)\|T_2U_{n,3}x_0-U_{n,3}x_0\|^2-\alpha_2^2\alpha_3^2\left(1-\alpha_3^1\right)\|U_{n,3}x_0-x_0\|^2 \\
&- \alpha_1^1\alpha_2^1\|T_1U_{n,2}x_0-U_{n,2}x_0\|^2-\alpha_2^1\alpha_3^1\|U_{n,2}x_0-x_0\|^2 \\
= &\prod_{j=1}^{3}\left(1-\alpha_3^j\right)\|U_{n,4}x_0-x^*\|^2+\left(1-\prod_{j=1}^{3}\left(1-\alpha_3^j\right)\right)\|x_0-x^*\|^2 \\
&- \alpha_1^3\alpha_2^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|T_3U_{n,4}x_0-U_{n,4}x_0\|^2-\alpha_2^3\alpha_3^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|U_{n,4}x_0-x_0\|^2 \\
&- \alpha_1^2\alpha_2^2\left(1-\alpha_3^1\right)\|T_2U_{n,3}x_0-U_{n,3}x_0\|^2-\alpha_2^2\alpha_3^2\left(1-\alpha_3^1\right)\|U_{n,3}x_0-x_0\|^2 \\
&- \alpha_1^1\alpha_2^1\|T_1U_{n,2}x_0-U_{n,2}x_0\|^2-\alpha_2^1\alpha_3^1\|U_{n,2}x_0-x_0\|^2 \\
&\vdots \\
\le &\prod_{j=1}^{k+1}\left(1-\alpha_3^j\right)\|U_{n,k+2}x_0-x^*\|^2+\left(1-\prod_{j=1}^{k+1}\left(1-\alpha_3^j\right)\right)\|x_0-x^*\|^2 \\
&- \alpha_1^{k+1}\alpha_2^{k+1}\prod_{j=1}^{k}\left(1-\alpha_3^j\right)\|T_{k+1}U_{n,k+2}x_0-U_{n,k+2}x_0\|^2 \\
&- \alpha_2^{k+1}\alpha_3^{k+1}\prod_{j=1}^{k}\left(1-\alpha_3^j\right)\|U_{n,k+2}x_0-x_0\|^2 \\
&- \alpha_1^{k}\alpha_2^{k}\prod_{j=1}^{k-1}\left(1-\alpha_3^j\right)\|T_kU_{n,k+1}x_0-U_{n,k+1}x_0\|^2 \\
&- \alpha_2^{k}\alpha_3^{k}\prod_{j=1}^{k-1}\left(1-\alpha_3^j\right)\|U_{n,k+1}x_0-x_0\|^2 \\
&- \alpha_1^{k-1}\alpha_2^{k-1}\prod_{j=1}^{k-2}\left(1-\alpha_3^j\right)\|T_{k-1}U_{n,k}x_0-U_{n,k}x_0\|^2 \\
&- \alpha_2^{k-1}\alpha_3^{k-1}\prod_{j=1}^{k-2}\left(1-\alpha_3^j\right)\|U_{n,k}x_0-x_0\|^2 \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha_1^3\alpha_2^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|T_3U_{n,4}x_0-U_{n,4}x_0\|^2 - \alpha_2^3\alpha_3^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|U_{n,4}x_0-x_0\|^2 \\
&- \alpha_1^2\alpha_2^2\left(1-\alpha_3^1\right)\|T_2U_{n,3}x_0-U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2\left(1-\alpha_3^1\right)\|U_{n,3}x_0-x_0\|^2 \\
&- \alpha_1^1\alpha_2^1\|T_1U_{n,2}x_0-U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0-x_0\|^2 \\
&\vdots \\
\le{}& \prod_{j=1}^{n}\left(1-\alpha_3^j\right)\|U_{n,n+1}x_0-x^*\|^2 + \left(1-\prod_{j=1}^{n}\left(1-\alpha_3^j\right)\right)\|x_0-x^*\|^2 \\
&- \alpha_1^n\alpha_2^n\prod_{j=1}^{n-1}\left(1-\alpha_3^j\right)\|T_nU_{n,n+1}x_0-U_{n,n+1}x_0\|^2 \\
&- \alpha_2^n\alpha_3^n\prod_{j=1}^{n-1}\left(1-\alpha_3^j\right)\|U_{n,n+1}x_0-x_0\|^2 \\
&\vdots \\
&- \alpha_1^{k+1}\alpha_2^{k+1}\prod_{j=1}^{k}\left(1-\alpha_3^j\right)\|T_{k+1}U_{n,k+2}x_0-U_{n,k+2}x_0\|^2 \\
&- \alpha_2^{k+1}\alpha_3^{k+1}\prod_{j=1}^{k}\left(1-\alpha_3^j\right)\|U_{n,k+2}x_0-x_0\|^2 \\
&- \alpha_1^{k}\alpha_2^{k}\prod_{j=1}^{k-1}\left(1-\alpha_3^j\right)\|T_{k}U_{n,k+1}x_0-U_{n,k+1}x_0\|^2 \\
&- \alpha_2^{k}\alpha_3^{k}\prod_{j=1}^{k-1}\left(1-\alpha_3^j\right)\|U_{n,k+1}x_0-x_0\|^2 \\
&- \alpha_1^{k-1}\alpha_2^{k-1}\prod_{j=1}^{k-2}\left(1-\alpha_3^j\right)\|T_{k-1}U_{n,k}x_0-U_{n,k}x_0\|^2 \\
&- \alpha_2^{k-1}\alpha_3^{k-1}\prod_{j=1}^{k-2}\left(1-\alpha_3^j\right)\|U_{n,k}x_0-x_0\|^2 \\
&\vdots \\
&- \alpha_1^3\alpha_2^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|T_3U_{n,4}x_0-U_{n,4}x_0\|^2 - \alpha_2^3\alpha_3^3\prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|U_{n,4}x_0-x_0\|^2 \\
&- \alpha_1^2\alpha_2^2\left(1-\alpha_3^1\right)\|T_2U_{n,3}x_0-U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2\left(1-\alpha_3^1\right)\|U_{n,3}x_0-x_0\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2 \\
= {}& \|x_0 - x^*\|^2 \\
&- \alpha_1^n \alpha_2^n \prod_{j=1}^{n-1} \left(1 - \alpha_3^j\right) \|T_n U_{n,n+1} x_0 - U_{n,n+1} x_0\|^2 \\
&\vdots \\
&- \alpha_1^{k+1} \alpha_2^{k+1} \prod_{j=1}^{k} \left(1 - \alpha_3^j\right) \|T_{k+1} U_{n,k+2} x_0 - U_{n,k+2} x_0\|^2 \\
&- \alpha_2^{k+1} \alpha_3^{k+1} \prod_{j=1}^{k} \left(1 - \alpha_3^j\right) \|U_{n,k+2} x_0 - x_0\|^2 \\
&- \alpha_1^{k} \alpha_2^{k} \prod_{j=1}^{k-1} \left(1 - \alpha_3^j\right) \|T_{k} U_{n,k+1} x_0 - U_{n,k+1} x_0\|^2 \\
&- \alpha_2^{k} \alpha_3^{k} \prod_{j=1}^{k-1} \left(1 - \alpha_3^j\right) \|U_{n,k+1} x_0 - x_0\|^2 \\
&- \alpha_1^{k-1} \alpha_2^{k-1} \prod_{j=1}^{k-2} \left(1 - \alpha_3^j\right) \|T_{k-1} U_{n,k} x_0 - U_{n,k} x_0\|^2 \\
&- \alpha_2^{k-1} \alpha_3^{k-1} \prod_{j=1}^{k-2} \left(1 - \alpha_3^j\right) \|U_{n,k} x_0 - x_0\|^2 \\
&\vdots \\
&- \alpha_1^3 \alpha_2^3 \prod_{j=1}^{2} \left(1 - \alpha_3^j\right) \|T_3 U_{n,4} x_0 - U_{n,4} x_0\|^2 - \alpha_2^3 \alpha_3^3 \prod_{j=1}^{2} \left(1 - \alpha_3^j\right) \|U_{n,4} x_0 - x_0\|^2 \\
&- \alpha_1^2 \alpha_2^2 \left(1 - \alpha_3^1\right) \|T_2 U_{n,3} x_0 - U_{n,3} x_0\|^2 - \alpha_2^2 \alpha_3^2 \left(1 - \alpha_3^1\right) \|U_{n,3} x_0 - x_0\|^2 \\
&- \alpha_1^1 \alpha_2^1 \|T_1 U_{n,2} x_0 - U_{n,2} x_0\|^2 - \alpha_2^1 \alpha_3^1 \|U_{n,2} x_0 - x_0\|^2.
\end{aligned}
\tag{2.20}
$$

For $k \in \mathbb{N}$ and (2.20), we have

$$
\alpha_2^{k-1} \alpha_3^{k-1} \prod_{j=1}^{k-2} \left(1 - \alpha_3^j\right) \|U_{n,k} x_0 - x_0\|^2 \leq \|x_0 - x^*\|^2 - \|S_n x_0 - x^*\|^2, \tag{2.21}
$$

as $n \to \infty$. This implies that $U_{\infty,k} x_0 = x_0$, for all $k \in \mathbb{N}$.

Again by (2.20), we have

$$\alpha_1^k \alpha_2^k \prod_{j=1}^{k-1}\left(1-\alpha_3^j\right)\|T_k U_{n,k+1}x_0 - U_{n,k+1}x_0\|^2 \le \|x_0 - x^*\|^2 - \|S_n x_0 - x^*\|^2, \tag{2.22}$$

as $n \to \infty$. Hence,

$$\alpha_1^k \alpha_2^k \prod_{j=1}^{k-1}\left(1-\alpha_3^j\right)\|T_k U_{\infty,k+1}x_0 - U_{\infty,k+1}x_0\|^2 \le 0. \tag{2.23}$$

From $U_{\infty,k}x_0 = x_0$, for all $k \in \mathbb{N}$, and (2.23), we obtain that $T_k x_0 = x_0$, for all $k \in \mathbb{N}$. This implies that $x_0 \in \bigcap_{i=1}^{\infty} F(T_i)$. □

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $A : C \to H$ be a α-inverse-strongly monotone mapping. Let $\{T_i\}_{i=1}^{\infty}$ be infinite family of nonexpansive mappings with $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \mathrm{EP}(F) \bigcap \mathrm{VI}(C, A)$, and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \le b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0,1)$ for all $j = 1,2,\ldots$. For every $n \in \mathbb{N}$, let S_n and S be S-mappings generated by $T_n, \ldots, T_1$ and $\rho_n, \rho_{n-1}, \ldots, \rho_1$ and $T_n, T_{n-1}, \ldots$, and $\rho_n, \rho_{n-1}, \ldots$, respectively. Let $\{x_n\}, \{u_n\}$ be sequences generated by $x_1, u \in C$ and*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda A)x_n + \gamma_n S_n P_C(I - \lambda A)u_n, \quad \forall n \ge 1, \end{gathered} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$, such that $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n \in [c,d] \subset (0,1)$ $r_n \in [a,b] \subset (0, 2\alpha)$, $\lambda \subset (0, 2\alpha)$. Assume that

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} \alpha_1^n < \infty$,

(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n|, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then the sequence $\{x_n\}, \{u_n\}$ converge strongly to $z = P_{\mathfrak{F}} u$.

Proof. First, we show that $(I-\lambda A)$ is nonexpansive. Let $x, y \in C$. Since A is α-inverse-strongly monotone and $\lambda < 2\alpha$, we have

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y-\lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\le \|x-y\|^2 - 2\alpha\lambda\|Ax-Ay\|^2 + \lambda^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2 \\ &\le \|x-y\|^2.\end{aligned} \tag{3.2}$$

Thus, $(I-\lambda A)$ is nonexpansive. We will divide our proof into 5 steps.

Step 1. We shall show that the sequence $\{x_n\}$ is bounded. Since

$$F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in C. \tag{3.3}$$

By Lemma 2.5, we have $u_n = T_{r_n}x_n$ and $\mathrm{EP}(F) = F(T_{r_n})$.

Let $z \in \mathfrak{F}$. By nonexpansiveness of $I-\lambda A$ and T_{r_n}, we have

$$\begin{aligned}\|x_{n+1}-z\| &= \|\alpha_n u + \beta_n P_C(I-\lambda A)x_n + \gamma_n S_n P_C(I-\lambda A)u_n - z\| \\ &= \|\alpha_n(u-z) + \beta_n(P_C(I-\lambda A)x_n - z) + \gamma_n(S_n P_C(I-\lambda A)u_n - z)\| \\ &\le \alpha_n\|u-z\| + \beta_n\|P_C(I-\lambda A)x_n - z\| + \gamma_n\|S_n P_C(I-\lambda A)u_n - z\| \\ &\le \alpha_n\|u-z\| + \beta_n\|x_n-z\| + \gamma_n\|u_n-z\| \\ &= \alpha_n\|u-z\| + \beta_n\|x_n-z\| + \gamma_n\|T_{r_n}x_n-z\| \\ &\le \alpha_n\|u-z\| + (1-\alpha_n)\|x_n-z\| \\ &\le \max\{\|u-z\|, \|x_n-z\|\}.\end{aligned} \tag{3.4}$$

By induction, we can prove that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By definition of x_n, we have

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \big\|\alpha_n u+\beta_n P_C(I-\lambda A)x_n+\gamma_n S_n P_C(I-\lambda A)u_n \\
&\qquad -\alpha_{n-1}u-\beta_{n-1}P_C(I-\lambda A)x_{n-1}-\gamma_{n-1}S_{n-1}P_C(I-\lambda A)u_{n-1}\big\| \\
&= \big\|(\alpha_n-\alpha_{n-1})u+\beta_n P_C(I-\lambda A)x_n-\beta_{n-1}P_C(I-\lambda A)x_n+\beta_{n-1}P_C(I-\lambda A)x_n \\
&\qquad +\gamma_n S_n P_C(I-\lambda A)u_n-\gamma_{n-1}S_n P_C(I-\lambda A)u_n+\gamma_{n-1}S_n P_C(I-\lambda A)u_n \\
&\qquad -\beta_{n-1}P_C(I-\lambda A)x_{n-1}-\gamma_{n-1}S_{n-1}P_C(I-\lambda A)u_{n-1}\big\| \\
&= \big\|(\alpha_n-\alpha_{n-1})u+\left(\beta_n-\beta_{n-1}\right)P_C(I-\lambda A)x_n \\
&\qquad +\beta_{n-1}(P_C(I-\lambda A)x_n-P_C(I-\lambda A)x_{n-1})+\left(\gamma_n-\gamma_{n-1}\right)S_n P_C(I-\lambda A)u_n \\
&\qquad +\gamma_{n-1}(S_n P_C(I-\lambda A)u_n-S_{n-1}P_C(I-\lambda A)u_{n-1})\big\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\left|\beta_n-\beta_{n-1}\right|\|P_C(I-\lambda A)x_n\| \\
&\quad +\beta_{n-1}\|P_C(I-\lambda A)x_n-P_C(I-\lambda A)x_{n-1}\|+\left|\gamma_n-\gamma_{n-1}\right|\|S_n P_C(I-\lambda A)u_n\| \\
&\quad +\gamma_{n-1}\|S_n P_C(I-\lambda A)u_n-S_{n-1}P_C(I-\lambda A)u_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\left|\beta_n-\beta_{n-1}\right|\|P_C(I-\lambda A)x_n\| \\
&\quad +\beta_{n-1}\|x_n-x_{n-1}\|+\left|\gamma_n-\gamma_{n-1}\right|\|S_n P_C(I-\lambda A)u_n\| \\
&\quad +\gamma_{n-1}(\|S_n P_C(I-\lambda A)u_n-S_n P_C(I-\lambda A)u_{n-1}\| \\
&\qquad\qquad +\|S_n P_C(I-\lambda A)u_{n-1}-S_{n-1}P_C(I-\lambda A)u_{n-1}\|) \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\left|\beta_n-\beta_{n-1}\right|\|P_C(I-\lambda A)x_n\| \\
&\quad +\beta_{n-1}\|x_n-x_{n-1}\|+\left|\gamma_n-\gamma_{n-1}\right|\|S_n P_C(I-\lambda A)u_n\| \\
&\quad +\gamma_{n-1}\|u_n-u_{n-1}\|+\gamma_{n-1}\|S_n P_C(I-\lambda A)u_{n-1}-S_{n-1}P_C(I-\lambda A)u_{n-1}\|.
\end{aligned}
\tag{3.5}
$$

Since $u_n=T_{r_n}x_n$, by definition of T_{r_n}, we have

$$
F\left(T_{r_n}x_n,y\right)+\frac{1}{r_n}\left\langle y-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle\ge 0,\quad \forall y\in C. \tag{3.6}
$$

Similarly,

$$
F\left(T_{r_{n+1}}x_{n+1},y\right)+\frac{1}{r_{n+1}}\left\langle y-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\ge 0,\quad \forall y\in C. \tag{3.7}
$$

From (3.6) and (3.7), we obtain

$$\begin{gathered}F(T_{r_n}x_n, T_{r_{n+1}}x_{n+1}) + \frac{1}{r_n}\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\rangle \geq 0,\\ F(T_{r_{n+1}}x_{n+1}, T_{r_n}x_n) + \frac{1}{r_{n+1}}\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \geq 0.\end{gathered} \tag{3.8}$$

By (3.8), we have

$$\frac{1}{r_n}\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\rangle + \frac{1}{r_{n+1}}\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.9}$$

It follows that

$$\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, \frac{T_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \geq 0. \tag{3.10}$$

This implies that

$$0 \leq \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle. \tag{3.11}$$

It follows that

$$\begin{aligned}\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|^2 &\leq \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle\\ &= \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle\\ &\leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left\|x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\|\\ &\leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|T_{r_{n+1}}x_{n+1} - x_{n+1}\|\right)\\ &= \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left(\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|T_{r_{n+1}}x_{n+1} - x_{n+1}\|\right)\\ &\leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left(\|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|T_{r_{n+1}}x_{n+1} - x_{n+1}\|\right).\end{aligned} \tag{3.12}$$

It follows that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|. \tag{3.13}$$

Putting $y_n = P_C(I-\lambda A)u_n$, then $\{y_n\}$ is bounded. By definition of S_n, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|S_n y_{n-1} - S_{n-1} y_{n-1}\| &= \|U_{n,1} y_{n-1} - U_{n-1,1} y_{n-1}\| \\
&= \|\alpha_1^1 T_1 U_{n,2} y_{n-1} + \alpha_2^1 U_{n,2} y_{n-1} + \alpha_3^1 y_{n-1} \\
&\quad - \alpha_1^1 T_1 U_{n-1,2} y_{n-1} - \alpha_2^1 U_{n-1,2} y_{n-1} - \alpha_3^1 y_{n-1}\| \\
&\le \left(1-\alpha_3^1\right)\|U_{n,2} y_{n-1} - U_{n-1,2} y_{n-1}\| \\
&\le \left(1-\alpha_3^1\right)\left(1-\alpha_3^2\right)\|U_{n,3} y_{n-1} - U_{n-1,3} y_{n-1}\| \\
&= \prod_{j=1}^{2}\left(1-\alpha_3^j\right)\|U_{n,3} y_{n-1} - U_{n-1,3} y_{n-1}\| \\
&\ \ \vdots \\
&\le \prod_{j=1}^{n-1}\left(1-\alpha_3^j\right)\|U_{n,n} y_{n-1} - U_{n-1,n} y_{n-1}\| \\
&\le \|U_{n,n} y_{n-1} - y_{n-1}\| \\
&= \|\alpha_1^n T_n y_{n-1} + (1-\alpha_1^n) y_{n-1} - y_{n-1}\| \\
&= \alpha_1^n \|T_n y_{n-1} - y_{n-1}\| \\
&\le \alpha_1^n 2\|y_{n-1} - z\|.
\end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.5), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le |\alpha_n - \alpha_{n-1}|\|u\| + \left|\beta_n - \beta_{n-1}\right|\|P_C(I-\lambda A)x_n\| \\
&\quad + \beta_{n-1}\|x_n - x_{n-1}\| + \left|\gamma_n - \gamma_{n-1}\right|\|S_n P_C(I-\lambda A)u_n\| \\
&\quad + \gamma_{n-1}\|u_n - u_{n-1}\| + \gamma_{n-1}\|S_n P_C(I-\lambda A)u_{n-1} - S_{n-1} P_C(I-\lambda A)u_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\|u\| + \left|\beta_n - \beta_{n-1}\right|\|P_C(I-\lambda A)x_n\| \\
&\quad + \beta_{n-1}\|x_n - x_{n-1}\| + \left|\gamma_n - \gamma_{n-1}\right|\|S_n P_C(I-\lambda A)u_n\| \\
&\quad + \gamma_{n-1}\left(\|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|\right) \\
&\quad + 2\gamma_{n-1}\alpha_1^n\|y_{n-1} - z\| \\
&\le |\alpha_n - \alpha_{n-1}|\|u\| + \left|\beta_n - \beta_{n-1}\right|\|P_C(I-\lambda A)x_n\| \\
&\quad + (1-\alpha_{n-1})\|x_n - x_{n-1}\| + \left|\gamma_n - \gamma_{n-1}\right|\|S_n P_C(I-\lambda A)u_n\| \\
&\quad + \frac{1}{a}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| + 2\gamma_{n-1}\alpha_1^n\|y_{n-1} - z\| \\
&\le (1-\alpha_{n-1})\|x_n - x_{n-1}\| + \left|\beta_n - \beta_{n-1}\right|M_1 \\
&\quad + |\alpha_n - \alpha_{n-1}|M_1 + \left|\gamma_n - \gamma_{n-1}\right|M_1 + \frac{1}{a}|r_{n+1} - r_n|M_1 + 2\alpha_1^n M_1,
\end{aligned} \tag{3.15}$$

where $M_1 = \max_{n\in\mathbb{N}}\{\|u\|, \|P_C(I-\lambda A)x_n\|,\ \|S_nP_C(I-\lambda A)u_n\|,\ \|u_n - x_n\|,\ \|y_n - z\|\}$. By (3.15), Lemma 2.3, and conditions (i)–(iii), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.16}$$

Step 3. We shall show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$.

Let $v \in \mathfrak{F}$. Since $u_n = T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}\|v - T_{r_n}x_n\|^2 &= \|T_{r_n}v - T_{r_n}x_n\|^2\\ &\le \langle T_{r_n}v - T_{r_n}x_n, v - x_n\rangle\\ &= \frac{1}{2}\left(\|T_{r_n}x_n - v\|^2 + \|x_n - v\|^2 - \|T_{r_n}x_n - x_n\|^2\right).\end{aligned} \tag{3.17}$$

Hence,

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|u_n - x_n\|^2. \tag{3.18}$$

By (3.18), we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \left\|\alpha_n(u - v) + \beta_n(P_C(I-\lambda A)x_n - v) + \gamma_n(S_nP_C(I-\lambda A)u_n - v)\right\|^2\\ &\le \alpha_n\|u - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|u_n - v\|^2\\ &\le \alpha_n\|u - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\left(\|x_n - v\|^2 - \|u_n - x_n\|^2\right)\\ &\le \alpha_n\|u - v\|^2 + \|x_n - v\|^2 - \gamma_n\|u_n - x_n\|^2\end{aligned} \tag{3.19}$$

it implies that

$$\begin{aligned}\gamma_n\|u_n - x_n\|^2 &\le \alpha_n\|u - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2\\ &\le \alpha_n\|u - v\|^2 + (\|x_n - v\| - \|x_{n+1} - v\|)(\|x_n - v\| + \|x_{n+1} - v\|)\\ &\le \alpha_n\|u - v\|^2 + \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|).\end{aligned} \tag{3.20}$$

By (3.16) and condition (i), we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.21}$$

Let $z \in \mathfrak{F}$ and by nonexpansiveness of $I - \lambda A$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \alpha_n \|u - z\|^2 + \beta_n \|P_C(I - \lambda A)x_n - z\|^2 + \gamma_n \|S_n P_C(I - \lambda A)u_n - z\|^2 \\
&\quad - \beta_n \gamma_n \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)u_n\| \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 \\
&\quad - \beta_n \gamma_n \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)u_n\| \\
&\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 \\
&\quad - \beta_n \gamma_n \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)u_n\|.
\end{aligned} \tag{3.22}$$

It implies that

$$\begin{aligned}
&\beta_n \gamma_n \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)u_n\| \\
&\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&= \alpha_n \|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)(\|x_n - z\| + \|x_{n+1} - z\|) \\
&= \alpha_n \|u - z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|).
\end{aligned} \tag{3.23}$$

By (3.16) and condition (i), we have

$$\lim_{n \to \infty} \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)u_n\| = 0. \tag{3.24}$$

Since

$$\begin{aligned}
\|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)x_n\| &\leq \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)u_n\| \\
&\quad + \|S_n P_C(I - \lambda A)u_n - S_n P_C(I - \lambda A)x_n\| \\
&\leq \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)u_n\| + \|u_n - x_n\|,
\end{aligned} \tag{3.25}$$

by (3.24) and (3.21), we have

$$\lim_{n \to \infty} \|P_C(I - \lambda A)x_n - S_n P_C(I - \lambda A)x_n\| = 0. \tag{3.26}$$

Since

$$\begin{aligned}
\|x_n - P_C(I - \lambda A)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - P_C(I - \lambda A)x_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|u - P_C(I - \lambda A)x_n\| \\
&\quad + \gamma_n \|S_n P_C(I - \lambda A)u_n - P_C(I - \lambda A)x_n\|,
\end{aligned} \tag{3.27}$$

by (3.24), (3.16), and condition (i), we have

$$\lim_{n\to\infty}\|x_n - P_C(I-\lambda A)x_n\| = 0. \tag{3.28}$$

Since

$$\begin{aligned}\|x_n - S_nP_C(I-\lambda A)u_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_nP_C(I-\lambda A)u_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - S_nP_C(I-\lambda A)u_n\| \\ &\quad + \beta_n\|P_C(I-\lambda A)x_n - S_nP_C(I-\lambda A)u_n\|,\end{aligned} \tag{3.29}$$

again by (3.24), (3.16), and condition (i), we have

$$\lim_{n\to\infty}\|x_n - S_nP_C(I-\lambda A)u_n\| = 0. \tag{3.30}$$

Since

$$\begin{aligned}\|x_n - S_nP_C(I-\lambda A)x_n\| &\le \|x_n - S_nP_C(I-\lambda A)u_n\| + \|S_nP_C(I-\lambda A)u_n - S_nP_C(I-\lambda A)x_n\| \\ &\le \|x_n - S_nP_C(I-\lambda A)u_n\| + \|u_n - x_n\|,\end{aligned} \tag{3.31}$$

by (3.21) and (3.30), we have

$$\lim_{n\to\infty}\|x_n - S_nP_C(I-\lambda A)x_n\| = 0. \tag{3.32}$$

Step 4. Putting $z_0 = P_{\mathfrak{F}}u$, we will show that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0. \tag{3.33}$$

To show this inequality, take a subsequence $\{x_{n_m}\}$ of $\{x_n\}$, such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \limsup_{m\to\infty}\langle u - z_0, x_{n_m} - z_0\rangle. \tag{3.34}$$

Without loss of generality, we may assume that $x_{n_m} \rightharpoonup \omega$ as $m \to \infty$ where $\omega \in C$. By nonexpansiveness of $P_C(I-\lambda A)$, (3.28), and Lemma 2.2, we have $\omega \in F(P_C(I-\lambda A))$. By Lemma 2.6, we obtain that $\omega \in \mathrm{VI}(C, A)$. Since $\|u_{n_m} - x_{n_m}\| \to 0$ as $m \to \infty$, we have $u_{n_m} \rightharpoonup \omega$ as $m \to \infty$. Since

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C. \tag{3.35}$$

By (A2), we have

$$\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq F(y,u_n), \quad \forall y \in C. \tag{3.36}$$

In particular,

$$\left\langle y-u_{n_m}, \frac{1}{r_{n_m}}(u_{n_m}-x_{n_m})\right\rangle \geq F(y,u_{n_m}). \tag{3.37}$$

By condition (A4), $F(y,\cdot)$ is lower semicontinuous and convex, and thus weakly semicontinuous. By (3.21) imply that $(1/r_{n_m})(u_{n_m}-x_{n_m}) \to 0$ in norm. Therefore, letting $m \to \infty$ in (3.37), we have

$$F(y,\omega) \leq \lim_{m\to\infty} F(y,u_{n_m}) \leq 0, \quad \forall y \in C. \tag{3.38}$$

Replacing y with $y_t := ty+(1-t)\omega$, $t \in (0,1]$, we have $y_t \in C$, and using (A1), (A4), and (3.38), we obtain

$$0 = F(y_t,y_t) \leq tF(y_t,y)+(1-t)F(y_t,\omega) \leq tF(y_t,y). \tag{3.39}$$

Hence, $F(ty+(1-t)\omega,y) \geq 0$, for all $t \in (0,1]$ and for all $y \in C$. Letting $t \to 0^+$ and using assumption (A3), we can conclude that

$$F(\omega,y) \geq 0, y \in C. \tag{3.40}$$

Therefore, $\omega \in \mathrm{EP}(F)$.

We will show that $\omega \in \bigcap_{i=1}^{\infty} F(T_i)$. By Lemma 2.10, we have $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\omega \neq S\omega$. Using Opial s' property, (3.32), $\omega \in F(P_C(I-\lambda A))$, and Remark 2.9, we have

$$\begin{aligned}
\liminf_{m\to\infty}\|x_{n_m}-\omega\| &< \liminf_{m\to\infty}\|x_{n_m}-S\omega\| \\
&\leq \liminf_{m\to\infty}(\|x_{n_m}-S_{n_m}P_C(I-\lambda A)x_{n_m}\| \\
&\qquad + \|S_{n_m}P_C(I-\lambda A)x_{n_m}-S_{n_m}P_C(I-\lambda A)\omega\| \\
&\qquad +\|S_{n_m}P_C(I-\lambda A)\omega-S\omega\|) \\
&= \liminf_{m\to\infty}(\|x_{n_m}-S_{n_m}P_C(I-\lambda A)x_{n_m}\| \\
&\qquad + \|S_{n_m}P_C(I-\lambda A)x_{n_m}-S_{n_m}P_C(I-\lambda A)\omega\| \\
&\qquad +\|S_{n_m}\omega-S\omega\|) \\
&\leq \liminf_{m\to\infty}\|x_{n_m}-\omega\|.
\end{aligned} \tag{3.41}$$

This is a contradiction, then $\omega \in \bigcap_{i=1}^{\infty} F(T_i)$. Hence, $\omega \in \mathfrak{F}$.

Since $x_{n_m} \rightharpoonup \omega$ and $\omega \in \mathfrak{F}$, we have

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \limsup_{m\to\infty}\langle u - z_0, x_{n_m} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \tag{3.42}$$

Step 5. Finally, we show that $\{x_n\}$ and $\{u_n\}$ converse strongly to $z_0 = P_{\mathfrak{F}}u$. Putting $z_0 = P_{\mathfrak{F}}u$, by nonexpansiveness of $P_C(I - \lambda A)$, S_n, and T_{r_n}, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \left\|\alpha_n(u - z_0) + \beta_n(P_C(I - \lambda A)x_n - z_0) + \gamma_n(S_nP_C(I - \lambda A)u_n - z_0)\right\|^2 \\
&\leq \left\|\beta_n(P_C(I - \lambda A)x_n - z_0) + \gamma_n(S_nP_C(I - \lambda A)u_n - z_0)\right\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&\leq \beta_n\|P_C(I - \lambda A)x_n - z_0\|^2 + \gamma_n\|S_nP_C(I - \lambda A)u_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&\leq \beta_n\|x_n - z_0\|^2 + \gamma_n\|T_{r_n}x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&\leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.
\end{aligned} \tag{3.43}$$

□

From Step 4 and Lemma 2.3, we obtain that $\{x_n\}$ converse strongly to $z_0 = P_{\mathfrak{F}}u$. By using (3.21), we have $\{u_n\}$ converse strongly to $z_0 = P_{\mathfrak{F}}u$.

4. Application

Using our main theorem (Theorem 3.1), we obtain the following strong convergence theorems involving infinite family of κ-strict pseudocontractions.

To prove strong convergence theorem in this section, we need definition and lemma as follows.

Definition 4.1. A mapping $T : C \to C$ is said to be a κ-strongly pseudocontraction mapping, if there exists $\kappa \in [0, 1)$, such that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + \kappa\left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C. \tag{4.1}$$

Lemma 4.2 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a κ-strict pseudocontraction. Define $S : C \to C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for each $x \in C$. Then, as $\alpha \in [\kappa, 1)$ S is nonexpansive, such that $F(S) = F(T)$.*

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $A : C \to H$ be a α-inverse-strongly monotone mapping. Let $\{T_i\}_{i=0}^{\infty}$ be infinite family of κ_i-pseudocontractions mappings with $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \mathrm{EP}(F) \bigcap \mathrm{VI}(C, A)$. Define a mapping T_{κ_i} by $T_{\kappa_i} = \kappa_i x + (1 - \kappa_i)T_i x$, for all $x \in C$, $i \in \mathbb{N}$, and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S be S-mappings generated by $T_{\kappa_n}, \ldots, T_{\kappa_1}$ and $\rho_n, \rho_{n-1}, \ldots, \rho_1$ and $T_{\kappa_n}, T_{\kappa_{n-1}}, \ldots$, and $\rho_n, \rho_{n-1}, \ldots$, respectively. Let $\{x_n\}, \{u_n\}$ be* *sequences generated by* $x_1, u \in C$ *and*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda A)x_n + \gamma_n S_n P_C(I - \lambda A)u_n, \quad \forall n \geq 1, \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$*, such that* $\alpha_n + \beta_n + \gamma_n = 1$*,* $\beta_n \in [c,d] \subset (0,1) r_n \in [a,b] \subset (0, 2\alpha)$*,* $\lambda \subset (0, 2\alpha)$*. Assume that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty}\alpha_1^n < \infty$,

(iii) $\sum_{n=1}^{\infty}|r_{n+1} - r_n|$, $\sum_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n|$, $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

then the sequence $\{x_n\}, \{u_n\}$ *converges strongly to* $z = P_{\mathfrak{F}}u$*.*

Proof. For every $i \in \mathbb{N}$, by Lemma 4.2, we have that T_{κ_i} is nonexpansive mappings. From Theorem 3.1, we could have the desired conclusion. □

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