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Research Article

Bounds of Solutions of Integrodifferential Equations

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Some new integral inequalities are given, and bounds of solutions of the following integro-differential equation are determined: $x'(t)-\digamma(t,x(t),\int_0^t k(t,s,x(t),x(s))ds) = h(t)$, $x(0) = x_0$, where $h : R_+ \to R$, $k : R_+^2 \times R^2 \to R$, $\digamma : R_+ \times R^2 \to R$ are continuous functions, $R_+ = [0,\infty)$.

1. Introduction

Ou Yang [1] established and applied the following useful nonlinear integral inequality.

Theorem 1.1. *Let u and h be nonnegative and continuous functions defined on R_+ and let $c \geq 0$ be a constant. Then, the nonlinear integral inequality*

$$u^2(t) \leq c^2 + 2\int_0^t h(s)u(s)ds, \quad t \in R_+ \tag{1.1}$$

implies

$$u(t) \leq c + \int_0^t h(s)ds, \quad t \in R_+. \tag{1.2}$$

This result has been frequently used by authors to obtain global existence, uniqueness, boundedness, and stability of solutions of various nonlinear integral, differential, and integrodifferential equations. On the other hand, Theorem 1.1 has also been extended and generalized by many authors; see, for example, [2–19]. Like Gronwall-type inequalities, Theorem 1.1 is also used to obtain *a priori* bounds to unknown functions. Therefore, integral inequalities of this type are usually known as Gronwall-Ou Yang type inequalities.

In the last few years there have been a number of papers written on the discrete inequalities of Gronwall inequality and its nonlinear version to the Bihari type, see [13, 16, 20]. Some applications discrete versions of integral inequalities are given in papers [21–23].

Pachpatte [11, 12, 14–16] and Salem [24] have given some new integral inequalities of the Gronwall-Ou Yang type involving functions and their derivatives. Lipovan [7] used the modified Gronwall-Ou Yang inequality with logarithmic factor in the integrand to the study of wave equation with logarithmic nonlinearity. Engler [5] used a slight variant of the Haraux's inequality for determination of global regular solutions of the dynamic antiplane shear problem in nonlinear viscoelasticity. Dragomir [3] applied his inequality to the stability, boundedness, and asymptotic behaviour of solutions of nonlinear Volterra integral equations.

In this paper, we present new integral inequalities which come out from above-mentioned inequalities and extend Pachpatte's results (see [11, 16]) especially. Obtained results are applied to certain classes of integrodifferential equations.

2. Integral Inequalities

Lemma 2.1. *Let u, f, and g be nonnegative continuous functions defined on R_+. If the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s)+u(\tau))d\tau\right)ds \tag{2.1}$$

holds where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \leq u_0\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right] \tag{2.2}$$

for $t \in R_+$.

Proof. Define a function $v(t)$ by the right-hand side of (2.1)

$$v(t) = u_0 + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s)+u(\tau))d\tau\right)ds. \tag{2.3}$$

Then, $v(0) = u_0$, $u(t) \leq v(t)$ and

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_0^t g(s)(u(t)+u(s))ds \\ &\leq f(t)v(t) + f(t)\int_0^t g(s)(v(t)+v(s))ds. \end{aligned} \tag{2.4}$$

Define a function $m(t)$ by

$$m(t) = v(t) + \int_0^t g(s)v(s)ds + v(t)\int_0^t g(s)ds, \tag{2.5}$$

then $m(0) = v(0) = u_0$, $v(\mathrm{t}) \leq m(t)$,

$$v'(t) \leq f(t)m(t), \tag{2.6}$$

$$\begin{aligned} m'(t) &= 2g(t)v(t) + v'(t)\left(1 + \int_0^t g(s)ds\right) \\ &\leq m(t)\left[2g(t) + f(t)\left(1 + \int_0^t g(s)ds\right)\right]. \end{aligned} \tag{2.7}$$

Integrating (2.7) from 0 to t, we have

$$m(t) \leq u_0 \exp\left(\int_0^t \left(2g(s) + f(s)\left(1 + \int_0^s g(\sigma)d\sigma\right)\right)ds\right). \tag{2.8}$$

Using (2.8) in (2.6), we obtain

$$v'(t) \leq u_0 f(t) \exp\left(\int_0^t \left(2g(s) + f(s)\left(1 + \int_0^s g(\sigma)d\sigma\right)\right)ds\right). \tag{2.9}$$

Integrating from 0 to t and using $u(t) \leq v(t)$, we get inequality (2.2). The proof is complete. □

Lemma 2.2. *Let u, f, and g be nonnegative continuous functions defined on R_+, $w(t)$ be a positive nondecreasing continuous function defined on R_+. If the inequality*

$$u(t) \leq w(t) + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau\right)ds, \tag{2.10}$$

holds, where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \leq w(t)\left[1 + \int_0^t f(s) \exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \tag{2.11}$$

where $t \in R_+$.

Proof. Since the function $w(t)$ is positive and nondecreasing, we obtain from (2.10)

$$\frac{u(t)}{w(t)} \le 1 + \int_0^t f(s)\left(\frac{u(s)}{w(s)} + \int_0^s g(\tau)\left(\frac{u(s)}{w(s)} + \frac{u(\tau)}{w(\tau)}\right)d\tau\right)ds. \tag{2.12}$$

Applying Lemma 2.1 to inequality (2.12), we obtain desired inequality (2.11). □

Lemma 2.3. *Let u, f, g, and h be nonnegative continuous functions defined on R_+, and let c be a nonnegative constant.*

If the inequality

$$u^2(t) \le c^2 + 2\left[\int_0^t f(s)u(s)\left(u(s) + \int_0^s g(\tau)(u(\tau) + u(s))d\tau\right) + h(s)u(s)\right]ds \tag{2.13}$$

holds for $t \in R_+$, then

$$u(t) \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \tag{2.14}$$

where

$$p(t) = c + \int_0^t h(s)ds. \tag{2.15}$$

Proof. Define a function $z(t)$ by the right-hand side of (2.13)

$$z(t) = c^2 + 2\left[\int_0^t f(s)u(s)\left(u(s) + \int_0^s g(\tau)(u(\tau) + u(s))d\tau\right) + h(s)u(s)\right]ds. \tag{2.16}$$

Then $z(0) = c^2$, $u(t) \le \sqrt{z(t)}$ and

$$\begin{aligned} z'(t) &= 2\left[f(t)u(t)\left(u(t) + \int_0^t g(s)(u(t) + u(s))ds\right) + h(t)u(t)\right] \\ &\le 2\sqrt{z(t)}\left[f(t)\left(\sqrt{z(t)} + \int_0^t g(s)\left(\sqrt{z(t)} + \sqrt{z(s)}\right)ds\right) + h(t)\right]. \end{aligned} \tag{2.17}$$

Differentiating $\sqrt{z(t)}$ and using (2.17), we get

$$\begin{aligned} \frac{d}{dt}\left(\sqrt{z(t)}\right) &= \frac{z'(t)}{2\sqrt{z(t)}} \\ &\le f(t)\left(\sqrt{z(t)} + \int_0^t g(s)\left(\sqrt{z(t)} + \sqrt{z(s)}\right)ds\right) + h(t). \end{aligned} \tag{2.18}$$

Integrating inequality (2.18) from 0 to t, we have

$$\sqrt{z(t)} \le p(t) + \int_0^t f(s)\left(\sqrt{z(s)} + \int_0^s g(\tau)\left(\sqrt{z(s)} + \sqrt{z(\tau)}\right)d\tau\right)ds, \tag{2.19}$$

where $p(t)$ is defined by (2.15), $p(t)$ is positive and nondecreasing for $t \in R_+$. Now, applying Lemma 2.2 to inequality (2.19), we get

$$\sqrt{z(t)} \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right]. \tag{2.20}$$

Using (2.20) and the fact that $u(t) \le \sqrt{z(t)}$, we obtain desired inequality (2.14). □

3. Application of Integral Inequalities

Consider the following initial value problem

$$x'(t) - \digamma\left(t, x(t), \int_0^t k(t, s, x(t), x(s))ds\right) = h(t), \quad x(0) = x_0, \tag{3.1}$$

where $h : R_+ \to R$, $k : R_+^2 \times R^2 \to R$, $\digamma : R_+ \times R^2 \to R$ are continuous functions. We assume that a solution $x(t)$ of (3.1) exists on R_+.

Theorem 3.1. *Suppose that*

$$\begin{gathered}|k(t, s, u_1, u_2)| \le f(t)g(s)(|u_1| + |u_2|) \quad \text{for } (t, s, u_1, u_2) \in R_+^2 \times R^2, \\ |\digamma(t, u_1, v_1)| \le f(t)|u_1| + |v_1| \quad \text{for } (t, u_1, v_1) \in R_+ \times R^2,\end{gathered} \tag{3.2}$$

where f, g are nonnegative continuous functions defined on R_+. Then, for the solution $x(t)$ of (3.1) the inequality

$$\begin{gathered}|x(t)| \le r(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \\ r(t) = |x_0| + \int_0^t |h(t)|dt\end{gathered} \tag{3.3}$$

holds on R_+.

Proof. Multiplying both sides of (3.1) by $x(t)$ and integrating from 0 to t we obtain

$$x^2(t) = x_0^2 + 2\int_0^t \left[x(s)\digamma\left(s, x(s), \int_0^s k(s, \tau, x(s), x(\tau))d\tau\right) + x(s)h(s)\right]ds. \tag{3.4}$$

From (3.2) and (3.4), we get

$$|x(t)|^2 \le |x_0|^2 + 2\int_0^t \left[f(s)|x(s)| \times \left(|x(s)| + \int_0^s g(\tau)(|x(s)| + |x(\tau)|)d\tau \right) + |h(s)||x(s)| \right] ds. \tag{3.5}$$

Using inequality (2.14) in Lemma 2.3, we have

$$|x(t)| \le r(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma \right) \right) d\tau \right) ds \right], \tag{3.6}$$

where

$$r(t) = |x_0| + \int_0^t |h(t)|dt, \tag{3.7}$$

which is the desired inequality (3.3). □

Remark 3.2. It is obvious that inequality (3.3) gives the bound of the solution $x(t)$ of (3.1) in terms of the known functions.

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