

Research Article

A Fixed Point Approach to Superstability of Generalized Derivations on Non-Archimedean Banach Algebras

M. Eshaghi Gordji,^{1,2} M. B. Ghaemi,³ and Badrkhan Alizadeh⁴

¹ Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

² Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran

³ Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran

⁴ The Holy Prophet Higher Education Complex, Tabriz College of Technology, P.O. Box 51745-135, Tabriz, Iran

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

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We investigate the superstability of generalized derivations in non-Archimedean algebras by using a version of fixed point theorem via Cauchy functional equation.

1. Introduction

A functional equation (ξ) is *superstable* if every approximately solution of (ξ) is an exact solution of it.

The stability of functional equations was first introduced by Ulam [1] during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940.

Given a metric group $G(\cdot, \rho)$, a number $\varepsilon > 0$, and a mapping $f : G \rightarrow G$ which satisfies the inequality $\rho(f(x \cdot y), f(x) \cdot f(y)) \leq \varepsilon$ for all x, y in G , does there exist an automorphism a of G and a constant $k > 0$, depending only on G such that $\rho(a(x), f(x)) \leq k\varepsilon$ for all $x \in G$?

If the answer is affirmative, we would call the equation $a(x \cdot y) = a(x) \cdot a(y)$ of automorphism is stable. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. In 1978, Rassias [3] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, ($\varepsilon > 0$, $p \in [0, 1)$). In 1991, Gajda [4] answered the question for the case

$p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias or generalized Hyers-Ulam stability of functional equations [5, 6].

In 1992, Găvruta [7] generalized the Th. M. Rassias Theorem as follows.

Suppose that $(G, +)$ is an abelian group, X is a Banach space $\varphi : G \times G \rightarrow [0, \infty)$ which satisfies

$$\tilde{\varphi}(x, y) = \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty, \quad (1.1)$$

for all $x, y \in G$. If $f : G \rightarrow X$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y), \quad (1.2)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \rightarrow X$ such that $T(x+y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)$ for all $x, y \in G$.

In 1949, Bourgin [8] proved the following result, which is sometimes called the superstability of ring homomorphisms: suppose that A and B are Banach algebras with unit. If $f : A \rightarrow B$ is a surjective mapping such that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \epsilon, \\ \|f(xy) - f(x)f(y)\| &\leq \delta, \end{aligned} \quad (1.3)$$

for some $\epsilon \geq 0$, $\delta \geq 0$ and for all $x, y \in A$, then f is a ring homomorphism.

Badora [9] and Miura et al. [10] proved the Ulam-Hyers stability and the Isac and Rassias-type stability of derivations [11] (see also [12, 13]); Savadkouhi et al. [14] have contributed works regarding the stability of ternary Jordan derivations. Jung and Chang [15] investigated the stability and superstability of higher derivations on rings. Recently, Ansari-Piri and Anjidani [16] discussed the superstability of generalized derivations on Banach algebras. In this paper, we investigate the superstability of generalized derivations on non-Archimedean Banach algebras by using the fixed point methods.

2. Preliminaries

In 1897, Hensel [17] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [18, 19].

A non-Archimedean field is a field \mathbb{K} equipped with a function (valuation) $|\cdot|$ from \mathbb{K} into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$ (see [20, 21]).

Definition 2.1. Let X be a vector space over a scalar field \mathbb{K} with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (NA₁) $\|x\| = 0$ if and only if $x = 0$,
- (NA₂) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$,
- (NA₃) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ (the strong triangle inequality).

A sequence $\{x_m\}$ in a non-Archimedean space is Cauchy if and only if $\{x_{m+1} - x_m\}$ converges to zero. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent. A non-Archimedean normed algebra is a non-Archimedean normed space A with a linear associative multiplication, satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. A non-Archimedean complete normed algebra is called a non-Archimedean Banach algebra (see [22]).

Example 2.2. Let p be a prime number. For any nonzero rational number $x = (a/b)p^{n_x}$ such that a and b are integers not divisible by p , define the p -adic absolute value $|x|_p := p^{-n_x}$. Then, $|\cdot|$ is a non-Archimedean norm on \mathbb{Q} . The completion of \mathbb{Q} with respect to $|\cdot|$ is denoted by \mathbb{Q}_p which is called the p -adic number field.

Definition 2.3. Let X be a nonempty set and $d : X \times X \rightarrow [0, \infty]$ satisfy the following properties:

- (D₁) $d(x, y) = 0$ if and only if $x = y$,
- (D₂) $d(x, y) = d(y, x)$ (symmetry),
- (D₃) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (strong triangle in equality),

for all $x, y, z \in X$. Then, (X, d) is called a non-Archimedean generalized metric space. (X, d) is called complete if every d -Cauchy sequence in X is d -convergent.

Definition 2.4. Let A be a non-Archimedean algebra. An additive mapping $D : A \rightarrow A$ is said to be a ring derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$. An additive mapping $H : A \rightarrow A$ is said to be a generalized ring derivation if there exists a ring derivation $D : A \rightarrow A$ such that

$$H(xy) = xH(y) + D(x)y, \quad (2.1)$$

for all $x, y \in A$.

We need the following fixed point theorem (see [23, 24]).

Theorem 2.5 (non-Archimedean alternative Contraction Principle). *Suppose that (X, d) is a non-Archimedean generalized complete metric space and $\Lambda : X \rightarrow X$ is a strictly contractive mapping; that is,*

$$d(\Lambda x, \Lambda y) \leq Ld(x, y), \quad (x, y \in X), \quad (2.2)$$

for some $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the followings are true:

- (a) the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ ,
- (b) x^* is a unique fixed point of Λ in

$$X^* = \{y \in X \mid d(\Lambda^k x, y) < \infty\}, \quad (2.3)$$

- (c) if $y \in X^*$, then

$$d(y, x^*) \leq d(\Lambda y, y). \quad (2.4)$$

3. Non-Archimedean Superstability of Generalized Derivations

Hereafter, we will assume that A is a non-Archimedean Banach algebra with unit over a non-Archimedean field \mathbb{K} .

Theorem 3.1. *Let $\varphi : A \times A \rightarrow [0, \infty)$ be a function. Suppose that $f, g : A \rightarrow A$ are mappings such that g is additive and*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y), \quad (3.1)$$

$$\|f(xy) - xf(y) - g(x)y\| \leq \varphi(x, y), \quad (3.2)$$

for all $x, y \in A$. If there exists a natural number $k \in \mathbb{K}$ and $0 < L < 1$,

$$|k|^{-1}\varphi(kx, ky), |k|^{-1}\varphi(kx, y), |k|^{-1}\varphi(x, ky) \leq L\varphi(x, y), \quad (3.3)$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

Proof. By induction on i , we prove that

$$\|f(ix) - if(x)\| \leq \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \dots, \varphi((i-1)x, x)\}, \quad (3.4)$$

for all $x \in A$ and $i \geq 2$. Let $x = y$ in (3.1). Then,

$$\|f(2x) - 2f(x)\| \leq \max\{\varphi(0, 0), \varphi(x, x)\}, \quad n \in \mathbb{N}_0, \quad x \in A. \quad (3.5)$$

This proves (3.4) for $i = 2$. Let (3.4) holds for $i = 1, 2, \dots, j$. Replacing x by jx and y by x in (3.1) for each $n \in \mathbb{N}_0$, and for all $x \in A$, we get

$$\|f((j+1)x) - f(jx) - f(x)\| \leq \max\{\varphi(0, 0), \varphi(jx, x)\}. \quad (3.6)$$

Since

$$\begin{aligned} & f((j+1)x) - f(jx) - f(x) \\ &= f((j+1)x) - (j+1)f(x) + (j+1)f(x) - f(jx) - f(x) \\ &= f((j+1)x) - (j+1)f(x) + jf(x) - f(jx), \end{aligned} \quad (3.7)$$

for all $x \in A$, it follows from induction hypothesis and (3.6) that

$$\begin{aligned} & \|f((j+1)x) - (j+1)f(x)\| \\ & \leq \max\{\|f((j+1)x) - f(jx) - f(x)\|, \|jf(x) - f(jx)\|\} \\ & \leq \max\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \dots, \varphi((j)x, x)\}, \end{aligned} \quad (3.8)$$

for all $x \in A$. This proves (3.4) for all $i \geq 2$. In particular,

$$\|f(kx) - kf(x)\| \leq \varphi(x), \quad (3.9)$$

for all $x \in A$ where

$$\varphi(x) = \max\{\varphi(0,0), \varphi(x,x), \varphi(2x,x), \dots, \varphi((k-1)x,x)\} \quad (x \in A). \quad (3.10)$$

Let X be the set of all functions $r : A \rightarrow A$. We define $d : X \times X \rightarrow [0, \infty]$ as follows:

$$d(r, s) = \inf\{\alpha > 0 : \|r(x) - s(x)\| \leq \alpha\varphi(x) \quad \forall x \in A\}. \quad (3.11)$$

It is easy to see that d defines a generalized complete metric on X . Define $J : X \rightarrow X$ by $J(r)(x) = k^{-1}r(kx)$. Then, J is strictly contractive on X , in fact, if

$$\|r(x) - s(x)\| \leq \alpha\varphi(x), \quad (x \in A), \quad (3.12)$$

then by (3.3),

$$\|J(r)(x) - J(s)(x)\| = |k|^{-1}\|r(kx) - s(kx)\| \leq \alpha|k|^{-1}\varphi(kx) \leq L\alpha\varphi(x), \quad (x \in A). \quad (3.13)$$

It follows that

$$d(J(r), J(s)) \leq Ld(r, s) \quad (r, s \in X). \quad (3.14)$$

Hence, J is a strictly contractive mapping with Lipschitz constant L . By (3.9),

$$\begin{aligned} \|(Jf)(x) - f(x)\| &= \|k^{-1}f(kx) - f(x)\|, \\ |k|^{-1}\|f(kx) - kf(x)\| &\leq |k|^{-1}\varphi(x) \quad (x \in A). \end{aligned} \quad (3.15)$$

This means that $d(J(f), f) \leq 1/|k|$. By Theorem 2.5, J has a unique fixed point $h : A \rightarrow A$ in the set

$$U = \{r \in X : d(r, J(f)) < \infty\}, \quad (3.16)$$

and for each $x \in A$,

$$h(x) = \lim_{m \rightarrow \infty} J^m(f(x)) = \lim_{m \rightarrow \infty} k^{-m}f(k^m x). \quad (3.17)$$

Therefore,

$$\begin{aligned}
& \|h(x+y) - h(x) - h(y)\| \\
&= \lim_{m \rightarrow \infty} |k|^{-m} \|f(k^m(x+y)) - f(k^m x) - f(k^m y)\| \\
&\leq \lim_{m \rightarrow \infty} |k|^{-m} \max\{\varphi(0,0), \varphi(k^m x, k^m y)\} \\
&\leq \lim_{m \rightarrow \infty} L^m \varphi(x, y) = 0,
\end{aligned} \tag{3.18}$$

for all $x, y \in A$. This shows that h is additive.

Replacing x by $k^n x$ in (3.2) to get

$$\|f(k^n xy) - k^n x f(y) - g(k^n x) y\| \leq \varphi(k^n x, y), \tag{3.19}$$

and so

$$\left\| \frac{f(k^n xy)}{k^n} - x f(y) - \frac{g(k^n x)}{k^n} y \right\| \leq \frac{1}{|k|^n} \varphi(k^n x, y) \leq L^n \varphi(x, y), \tag{3.20}$$

for all $x, y \in A$ and all $n \in \mathbb{N}$. By taking $n \rightarrow \infty$, we have

$$h(xy) = x f(y) + \lim_{n \rightarrow \infty} \frac{g(k^n x)}{k^n} y, \tag{3.21}$$

for all $x, y \in A$.

Fix $m \in \mathbb{N}$. By (3.21), we have

$$\begin{aligned}
x f(k^m y) &= h(k^m xy) - \lim_{n \rightarrow \infty} \left(\frac{g(k^n x)}{k^n} (k^m y) \right) \\
&= k^m x f(y) + \lim_{n \rightarrow \infty} \left(\frac{g(k^n k^m x)}{k^n} y \right) - k^m \lim_{n \rightarrow \infty} \left(\frac{g(k^n x)}{k^n} y \right) \\
&= k^m x f(y) + k^m \lim_{n \rightarrow \infty} \left(\frac{g(k^{n+m} x)}{k^{n+m}} y \right) - k^m \lim_{n \rightarrow \infty} \left(\frac{g(k^n x)}{k^n} y \right) \\
&= k^m x f(y),
\end{aligned} \tag{3.22}$$

for all $x, y \in A$. Then, $x f(y) = x(f(k^m y)/k^m)$ for all $x, y \in A$ and each $m \in \mathbb{N}$, and so by taking $m \rightarrow \infty$, we have $x f(y) = x h(y)$. Now, we obtain $h = f$, since A is with unit. Replacing y by $k^n y$ in (3.2), we obtain

$$\|f(k^n(xy)) - x f(k^n y) - k^n g(x) y\| \leq \varphi(x, k^n y), \tag{3.23}$$

and hence,

$$\left\| \frac{f(k^n xy)}{k^n} - x \frac{f(k^n y)}{k^n} - g(x)y \right\| \leq \frac{1}{|k|^n} \varphi(x, k^n y) \leq L^n \varphi(x, y), \quad (3.24)$$

for all $x, y \in A$ and each $n \in \mathbb{N}$. Letting n tends to infinite, we have

$$f(xy) = xf(y) + g(x)y. \quad (3.25)$$

Now, we show that g is a ring derivation. By (3.25), we get

$$\begin{aligned} g(xy)z &= f(xyz) - xyf(z) \\ &= xf(yz) + g(x)yz - xyf(z) \\ &= (xg(y) + g(x)y)z, \end{aligned} \quad (3.26)$$

for all $x, y, z \in A$. Therefore, we have $g(xy) = xg(y) + g(x)y$. □

The proof of following theorem is similar to that in Theorem 3.1, hence it is omitted.

Theorem 3.2. *Let $\varphi : A \times A \rightarrow [0, \infty)$ be a function. Suppose that $f, g : A \rightarrow A$ are mappings such that g is additive and*

$$\begin{aligned} \|f(x + y) - f(x) - f(y)\| &\leq \varphi(x, y), \\ \|f(xy) - xf(y) - g(x)y\| &\leq \varphi(x, y), \end{aligned} \quad (3.27)$$

for all $x, y \in A$. If there exists a natural number $k \in \mathbb{K}$ and $0 < L < 1$,

$$|k|\varphi(k^{-1}x, k^{-1}y), |k|\varphi(k^{-1}x, y), |k|\varphi(x, k^{-1}y) \leq L\varphi(x, y), \quad (3.28)$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

The following results are immediate corollaries of Theorems 3.1 and 3.2 and Example 2.3.

Corollary 3.3. *Let A be a non-Archimedean Banach algebra over \mathbb{Q}_p , $\varepsilon > 0$, and $p_1, p_2 \in (1, \infty)$. Suppose that $f, g : A \rightarrow A$ are mappings such that g is additive and*

$$\begin{aligned} \|f(x + y) - f(x) - f(y)\| &\leq \varepsilon(\|x\|^{p_1} \|y\|^{p_2}), \\ \|f(xy) - xf(y) - g(x)y\| &\leq \varepsilon(\|x\|^{p_1} \|y\|^{p_2}), \end{aligned} \quad (3.29)$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

Corollary 3.4. Let A be a non-Archimedean Banach algebra over \mathbb{Q}_p , $\varepsilon > 0$ and $p_1, p_2, p_1 + p_2 \in (-\infty, 1)$. Suppose that $f, g : A \rightarrow A$ are mappings such that g is additive and

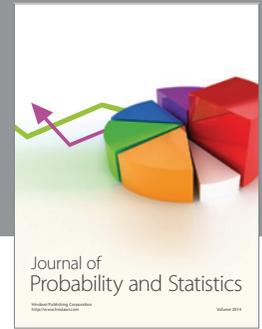
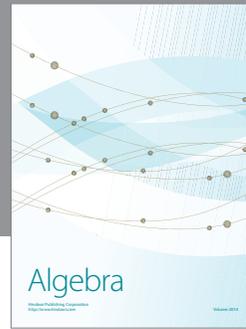
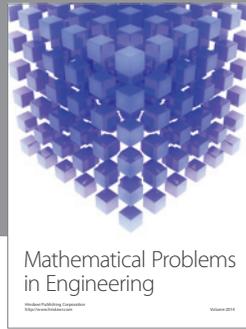
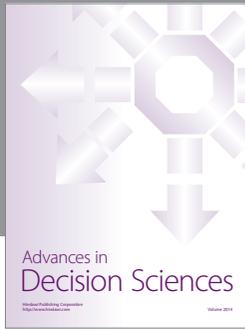
$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \varepsilon(\|x\|^{p_1} \|y\|^{p_2}), \\ \|f(xy) - xf(y) - g(x)y\| &\leq \varepsilon(\|x\|^{p_1} \|y\|^{p_2}), \end{aligned} \quad (3.30)$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

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