Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 593436, 17 pages
doi:10.1155/2011/593436

*Research Article*

# Analysis of Compactly Supported Nonstationary Biorthogonal Wavelet Systems Based on Exponential B-Splines

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Received 1 June 2011; Accepted 30 September 2011

Academic Editor: Agacik Zafer



This paper is concerned with analyzing the mathematical properties, such as the regularity and stability of nonstationary biorthogonal wavelet systems based on exponential B-splines. We first discuss the biorthogonality condition of the nonstationary refinable functions, and then we show that the refinable functions based on exponential B-splines have the same regularities as the ones based on the polynomial B-splines of the corresponding orders. In the context of nonstationary wavelets, the stability of wavelet bases is not implied by the stability of a refinable function. For this reason, we prove that the suggested nonstationary wavelets form Riesz bases for the space that they generate.

## 1. Introduction

For the last two decades, the wavelet transforms have become very useful tools in a variety of applications such as signal and image processing and numerical computation. The construction of classical wavelets is now well understood thanks to such pioneering works as [1–3]. Many properties, such as symmetry (or antisymmetry), vanishing moments, regularity, and short support, are required in a practical use for application areas. In particular, polynomial splines have been a common source for wavelet construction [1, 3–6]. A new class of compactly supported biorthogonal wavelet systems that are constructed from pseudosplines was introduced in [7].

Exponential B-splines and polynomials have been found to be quite useful in a number of applications such as computer-aided geometric design, shape-preserving curve fitting, and signal interpolation [8–10]. Exponential B-splines were used as a key ingredient for the construction of wavelets [11, 12] and particularly used in wavelet construction on $\mathbb{S}^2$ and $\mathbb{S}^3$ [13]. In particular, in the approximation and sparse representation of acoustic signals, polynomial-based (stationary) wavelet systems have an important limitation because they do not consider the spectral features (e.g., band limited) of a given signal. However, (non-stationary) wavelet systems based on the exponential B-spline can be tuned to the specific trait of the given signal, yielding better approximations and sparser representations than classical wavelets at strictly the same computational costs. Details on exponential splines can be found in the selected references [10, 11, 14–16]. Related studies on non-stationary wavelets can be found in [11, 12, 15, 17–22].

One natural and convenient way to introduce wavelets is to follow the notion of multiresolution analysis. However, because the refinement masks we are interested in are non-stationary (i.e., scale dependent), we use the structure of non-stationary multiresolution analysis as introduced in [17]. Given an integer $j_0 \in \mathbb{Z}$ and compactly supported refinable functions $\phi_j$, $j \geq j_0$, in $L_2(\mathbb{R})$, we say that a sequence of subspaces

$$V_j = \overline{\operatorname{span}}\left\{\phi_{j,k} := 2^{j/2}\phi_j\left(2^j \cdot -k\right) : k \in \mathbb{Z}\right\}, \quad j \geq j_0 \tag{1.1}$$

forms a non-stationary multiresolution analysis (MRA) of $L_2(\mathbb{R})$ if the following conditions are satisfied:

(1) $V_j \subset V_{j+1}$ for all $j \geq j_0$,

(2) $\bigcup_{j\geq j_0} V_j$ is dense in $L_2(\mathbb{R})$;

(3) the set $\{\phi_{j,k} : k \in \mathbb{Z}\}$ is a Riesz (or stable) basis for $V_j$ for each $j \geq j_0$.

The nested embedding of the spaces $V_j$ implies the existence of a sequence $\mathbf{a}^{[j]} := (a_n^{[j]})_{n\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ that satisfies the *non-stationary refinement equation*

$$\phi_j(x) = \sum_{n\in\mathbb{Z}} a_n^{[j]}\phi_{j+1}(2x-n), \quad j \geq j_0, \tag{1.2}$$

where the sequence $\mathbf{a}^{[j]}$ is usually called *the refinement mask* for $\phi_j$. One should notice that the function $\phi_j(2^j\cdot) \in V_j$ is no longer a dilated version of $\phi_{j_0}$ in $V_{j_0}$. A refinable function $\tilde{\phi}_j \in L_2(\mathbb{R})$ with the mask $\tilde{\mathbf{a}}^{[j]} := (\tilde{a}_n^{[j]})_{n\in\mathbb{Z}}$ is called the dual refinable function of $\phi_j \in L_2(\mathbb{R})$ (or just the *dual* of $\phi_j$ for simplicity) if it satisfies

$$\left\langle \phi_j, \tilde{\phi}_j(\cdot - \ell) \right\rangle = \delta_{0,\ell}, \quad \ell \in \mathbb{Z}. \tag{1.3}$$

Let $(V_j)_{j\geq j_0}$ and $(\tilde{V}_j)_{j\geq j_0}$ be a pair of MRAs generated by a pair of dual refinable functions $\phi_j$ and $\tilde{\phi}_j$, $j \geq j_0$, respectively. The concept of biorthogonal wavelets is to find complement spaces

$W_j$ and $\widetilde{W}_j$ of $V_j$ and $\widetilde{V}_j$, respectively, satisfying $V_{j+1} = V_j \dot{+} W_j$, $W_j \perp \widetilde{V}_j$ and $\widetilde{V}_{j+1} = \widetilde{V}_j \dot{+} \widetilde{W}_j$, $\widetilde{W}_j \perp V_j$. The corresponding biorthogonal wavelets are given by

$$\begin{aligned} \psi_j &= \sum_{n\in\mathbb{Z}} (-1)^n \widetilde{a}^{[j]}_{1-n} \phi_{j+1}(2\cdot -n), \\ \widetilde{\psi}_j &= \sum_{n\in\mathbb{Z}} (-1)^n a^{[j]}_{1-n} \widetilde{\phi}_{j+1}(2\cdot -n). \end{aligned} \tag{1.4}$$

A generalization of the biorthogonal wavelets of Cohen-Daubechies-Feuveau [1] was introduced that was based on exponential B-splines [12]. By generalizing the Strang-Fix conditions, the authors discussed the relationship between the reproduction of exponential polynomials (by $\phi_j$ or $\widetilde{\phi}_j$) and the zeros of the corresponding Laurent polynomials. They also proved that for each $j \geq j_0$, the proposed non-stationary refinable function $\phi_j$ generates a Riesz basis for $V_j$ and that the corresponding Riesz (upper and lower) bounds are independent of $j \geq j_0$. However, the authors did not explicitly address the biorthogonality condition of the corresponding non-stationary refinable functions. Moreover, some fundamental questions concerning the global stability, and regularity, were left unanswered. Therefore, the primary goal of this paper is to address these issues. First, we provide a sufficient condition for the biorthogonality (1.3) of non-stationary refinable functions, and then we prove that the refinable functions based on exponential B-splines have the same regularities as the ones based on the polynomial B-splines of the corresponding orders. In the context of non-stationary wavelets, the stability of the wavelet bases $\{\psi_{j,k} : k \in \mathbb{Z},\ j \geq j_0\}$ is not implied by the stability of a refinable function. Therefore, we prove that the set $\{\psi_{j,k} : k \in \mathbb{Z},\ j \geq j_0\}$ forms a Riesz basis for the space $\dot{+}_{j\geq j_0} W_j$. Furthermore, we show that the set $\{\phi_{j_0,k} : k \in \mathbb{Z}\} \cup \{\psi_{j,k} : k \in \mathbb{Z},\ j \geq j_0\}$ becomes a Riesz basis for the space $L_2(\mathbb{R})$.

This paper is organized as follows. In Section 2, we provide basic notions of exponential B-splines. Section 3 discusses the biorthogonality condition of non-stationary refinable functions and then studies their regularities. In Section 4, we prove the (global) stability of the proposed non-stationary wavelet bases.

## 2. Preliminaries: Exponential B-Splines

Given a set of complex numbers $G = \{\gamma_j \in \mathbb{C} : j = 1, \ldots, N\}$, the corresponding $N$th-order exponential B-splines can be defined as successive convolutions of the first-order B-spline

$$\phi_j := \phi_j^{[G]} := \tau_j \left( e_{\gamma_1 2^{-j}} B_1 * \cdots * e_{\gamma_N 2^{-j}} B_1 \right), \tag{2.1}$$

with a normalization factor $\tau_j$ defined so that $\|\phi_j\|_{L_1(\mathbb{R})} = 1$ (see [10]), where $B_1$ indicates the first-order $B$-spline, that is, $B_1 := \chi_{[0,1]}$, and $e_\gamma : x \mapsto e^{\gamma x}$, $\gamma \in \mathbb{C}$, is the exponential function. For simplicity, we will omit $G$ in $\phi_j^{[G]}$. Obviously, the function $\phi_j$ is a compactly supported piecewise exponential polynomial. The global regularity of $\phi_j$ is $C^{N-2}$ (see [10, 15]). A convenient way to represent an ($N$th-order) exponential B-spline is with the Laurent polynomial

$$a^{[j]}(z) := \sum_{n\in\mathbb{Z}} a_n^{[j]} z^n := 2c_j \prod_{n=1}^{N} \frac{1}{2}\left(1 + e^{\gamma_n 2^{-j-1}} z\right), \quad j \geq j_0, \tag{2.2}$$

where $c_j$ is the normalization factor defined by

$$c_j := c_{j,G} := \prod_{n=1}^{N} \frac{2}{\left(1 + e^{\gamma_n 2^{-j-1}}\right)}. \tag{2.3}$$

We call $a^{[j]}$ the *symbol* of $\phi_j$. Throughout this paper, $\gamma_n$ is assumed to be a real or a pure imaginary number, that is, $\gamma_n \in \mathbb{R}$ or $\gamma_n \in i\mathbb{R}$. Since we want the mask $(a_n^{[j]})_{n\in\mathbb{Z}}$ to be symmetric with respect to its center, it is reasonable to assume that if $\gamma_n \in G$, then $-\gamma_n \in G$. The relationship between the reproduction of exponential polynomials by $\phi_j$ and the zeros of the corresponding Laurent polynomial $a^{[j]}$ is discussed in [12].

It is well known that the integer translates $\phi_0(\cdot - k)$, $k \in \mathbb{Z}$, are linearly independent if and only if $\gamma_\ell - \gamma_n \notin 2\pi i\mathbb{Z}$ for $\gamma_\ell \neq \gamma_n$ [10, 16]. From (2.1) and (2.2), we can easily deduce that for each $j \geq j_0$, the integer translates $\phi_j(\cdot - k)$, $k \in \mathbb{Z}$, are linearly independent if and only if

$$2^{-j}(\gamma_\ell - \gamma_n) \notin 2\pi i\mathbb{Z}, \qquad \gamma_\ell \neq \gamma_n. \tag{2.4}$$

A concept related to (but weaker than) the linear independence is the notion of the stability of $\phi_j$. It is known (see, e.g., [23]) that the set $\{\phi_j(\cdot - k) : k \in \mathbb{Z}\}$ forms a Riesz basis if and only if there exist constants $A_j, B_j > 0$ such that

$$A_j \leq \left[\widehat{\phi}_j, \widehat{\phi}_j\right] \leq B_j, \tag{2.5}$$

where the *bracket product* $[f, g]$ for $f, g \in L_2(\mathbb{R})$ is defined by

$$[f, g](\xi) := \sum_{n\in\mathbb{Z}} f(\xi + 2\pi n)\overline{g(\xi + 2\pi n)}. \tag{2.6}$$

We say that the function $\phi_j \in L_2(\mathbb{R})$ is *stable* if (2.5) is satisfied. If the integer translates of $\phi_j$ are linearly independent, the stability of $\phi_j$ is immediate [24, Theorem 1.2], that is, $\{\phi_j(2^j \cdot -k) : k \in \mathbb{Z}\}$ is a Riesz basis for $V_j$. Finally, the basic requirement on the set $G = \{\gamma_1, \ldots, \gamma_N\}$ can be summarized as follows: (i) each $\gamma_n$ is a real or a pure imaginary number; (ii) both $\gamma_n$ and $-\gamma_n$ belong to $G$, and (iii) for any $j \geq j_0$, $2^{-j}(\gamma_n - \gamma_\ell) \notin 2\pi i\mathbb{Z}$ with $\gamma_n \neq \gamma_\ell$.

## 3. Dual Refinable Functions

### *3.1. Construction of Dual Refinable Functions*

Given refinable functions $\phi_j$ with $j \geq j_0$ the first step in the construction of a biorthogonal wavelet system is to find their dual refinable functions $\widetilde{\phi}_j \in L_2(\mathbb{R})$, $j \geq j_0$ (whose symbol is denoted by $\widetilde{a}^{[j]}$). A necessary condition for $\phi_j$ and $\widetilde{\phi}_j$ to satisfy (1.3) is

$$a^{[j]}(z)\overline{\widetilde{a}^{[j]}(z)} + a^{[j]}(-z)\overline{\widetilde{a}^{[j]}(-z)} = 4, \tag{3.1}$$

where $\overline{z}$ is the conjugate of the complex number $z$. Thus, the construction of $\widetilde{\phi}_j$ starts with the construction of a dual symbol $\widetilde{a}^{[j]}(z) := \sum_{n\in\mathbb{Z}} \widetilde{a}_n^{[j]} z^n$ such that (3.1) is satisfied. The algorithm to find $\widetilde{a}^{[j]}$ is analogous to the classical method [1], except for the main difference that $\widetilde{a}^{[j]}$, $j \geq j_0$, are scale dependent. However, in order to ensure the existence of $\widetilde{a}^{[j]}$ satisfying (3.1), we need to prove that $a^{[j]}$ has no roots of opposite signs. For this purpose, denoting

$$N = 2L + \tau, \qquad \tau \in \{0, 1\}, \tag{3.2}$$

and setting $z = e^{-i\xi}$ and $y = \sin^2\xi/2$ (i.e., $y = -(1-z)^2/4z$), we represent $a^{[j]}$ as follows:

$$a^{[j]}\left(e^{-i\xi}\right) = 2e^{-iL\xi}\left(\frac{1+e^{-i\xi}}{2}\right)^{\tau}\prod_{n=1}^{L}\left(1 - \nu_{j,n} y\right), \tag{3.3}$$

where

$$\nu_{j,n} = \frac{4}{e^{\gamma_n 2^{-j-1}} + e^{-\gamma_n 2^{-j-1}} + 2}. \tag{3.4}$$

Indeed, we are looking for a Laurent polynomial $\widetilde{a}^{[j]}$ of the form $\widetilde{a}^{[j]}(e^{-i\xi}) = a^{[j]}(e^{-i\xi})Q^{[j]}(y)$ such that (3.1) holds. Therefore, if we define $P^{[j]}(y) := a^{[j]}(e^{-i\xi})\overline{a^{[j]}(e^{-i\xi})}/4$, that is,

$$P^{[j]}(y) := (1-y)^{\tau}\prod_{n=1}^{L}\left(1 - \nu_{j,n} y\right)^2, \tag{3.5}$$

then the problem of finding $\widetilde{a}^{[j]}(z)$ is reduced to constructing $Q^{[j]}(y)$ which satisfies the equation

$$P^{[j]}(y)Q^{[j]}(y) + P^{[j]}(1-y)Q^{[j]}(1-y) = 1, \qquad \forall y \in [0,1]. \tag{3.6}$$

Let us now prove that there is no common zero of $P^{[j]}$ and $P^{[j]}(1-\cdot)$ on $[0,1]$.

**Proposition 3.1.** *Let the polynomials $P^{[j]}$, $j \geq j_0$, be given as in* (3.5). *Assume that for any $j \geq j_0$, $2^{-j}(\gamma_n - \gamma_\ell) \notin 2\pi i\mathbb{Z}$ for any $\gamma_n, \gamma_\ell \in G$ with $\gamma_n \neq \gamma_\ell$, then $P^{[j]}$ and $P^{[j]}(1-\cdot)$ have no common roots on* $[0,1]$.

*Proof.* Assume that $P^{[j]}$ and $P^{[j]}(1-\cdot)$ have a common root on $[0,1]$. This is equivalent to the existence of a number $z_0 \in \mathbb{C}\setminus\{0\}$ such that

$$a^{[j]}(z_0) = a^{[j]}(-z_0) = 0. \tag{3.7}$$

From (2.2), we can deduce that for some $\gamma_\ell \neq \gamma_n$, $1 + e^{-2^{-j-1}\gamma_\ell}z_0 = 0$ and $1 + e^{-2^{-j-1}\gamma_n}(-z_0) = 0$. It follows that $2^{-j}(\gamma_n - \gamma_\ell) \in 2\pi i\mathbb{Z}$, which contradicts the initial assumption. $\square$

By virtue of Proposition 3.1, the Bezout theorem guarantees the existence of a unique polynomial $Q^{[j]}$ of degree $N-1$ that satisfies (3.6). One may look for a polynomial $Q^{[j]}$ of degree higher than $N-1$. However, if the corresponding dual refinable functions are to have the shortest possible support, the degree of $Q^{[j]}$ must be constrained to be $N-1$. On the other hand, it is of interest to see that by regrouping the simple fractions of $Q^{[j]}$ into two groups, that is,

$$Q^{[j]}(y) = b^{[j]}(z)\tilde{b}^{[j]}(z), \qquad y = -\frac{(1-z)^2}{4z}, \tag{3.8}$$

then we can define the adjusted Laurent polynomials $a^{[j]}$ and $\tilde{a}^{[j]}$ by

$$\begin{aligned} a^{[j]}(z) &:= 2c_j b^{[j]}(z)\prod_{n=1}^{N}\frac{1}{2}\left(1+e^{\gamma_n 2^{-j-1}}z\right), \\ \tilde{a}^{[j]}(z) &:= 2c_j \tilde{b}^{[j]}(z)\prod_{n=1}^{N}\frac{1}{2}\left(1+e^{\gamma_n 2^{-j-1}}z\right), \end{aligned} \tag{3.9}$$

so that the lengths of $a^{[j]}$ and $\tilde{a}^{[j]}$ are very close. This allows us to construct generalized non-stationary refinable functions $\phi_j$ and $\tilde{\phi}_j$. If $b^{[j]} = 1$, the resulting function $\phi_j$ becomes an exponential B-spline of order $N$. On the other hand, the classical counterparts of $a^{[j]}$ and $\tilde{a}^{[j]}$ (which are obtained by setting $\gamma_n = 0$ for all $\gamma_n \in G$) can be written as

$$a(z) = 2^{-N+1}(1+z)^N b(z), \qquad \tilde{a}(z) = 2^{-N+1}(1+z)^N \tilde{b}(z). \tag{3.10}$$

For notational simplicity, we will write

$$b^{[j]}(z) = \sum_{n\in\mathbb{Z}} b_n^{[j]} z^n, \qquad b(z) = \sum_{n\in\mathbb{Z}} b_n z^n. \tag{3.11}$$

**Lemma 3.2.** *Let* $\mathbf{b}^{[j]} := (b_n^{[j]} : n \in \mathbb{Z})$ *and* $\mathbf{b} := (b_n : n \in \mathbb{Z})$ *with* $j \ge j_0$, *then, as* $j$ *tends to* $\infty$, *one has* $\|\mathbf{b}^{[j]} - \mathbf{b}\|_1 = O(2^{-j})$.

*Proof.* This is a direct consequence of [12, Lemma 2]. □

For the given Laurent polynomials $\tilde{a}^{[j]}$ with $j \ge j_0$, there corresponds a potential candidate for the refinable function $\tilde{\phi}_j$ which is defined in terms of the Fourier transform as

$$\widehat{\tilde{\phi}}_j(\xi) = \prod_{n=0}^{\infty} \frac{1}{2}\tilde{A}^{[j+n]}\left(\frac{\xi}{2^{n+1}}\right), \tag{3.12}$$

where

$$\tilde{A}^{[j]}(\xi) = \tilde{a}^{[j]}\left(e^{-i\xi}\right), \tag{3.13}$$

provided that $\widehat{\tilde{\phi}}_j(\xi/2^n) \to 1$ as $n \to \infty$. In fact, $\tilde{\phi}_j$ in (3.12) is the only candidate for the refinable function associated with $\tilde{a}^{[j]}$ such that $\tilde{\phi}_j$ is a dual of $\phi_j$. Although the infinite product in (3.12) converges pointwise and $\tilde{\phi}_j$ is in $L_2(\mathbb{R})$ whenever $\tilde{\phi}$ is in $L_2(\mathbb{R})$ [12], it still needs to be ensured that the function $\tilde{\phi}_j$ in (3.12) is indeed a dual of $\phi_j$. The following results address this issue. For simplicity, using $b^{[j]}$ and $\tilde{b}^{[j]}$ in (3.9), we introduce the notation

$$B_{k,j} := \sup_{n\geq j} \max_{\xi} \left| \prod_{\ell=1}^{k} b^{[n+\ell-1]}\left(e^{-i\xi/2^{\ell}}\right) \right|^{1/k},$$
$$\tilde{B}_{\tilde{k},j} := \sup_{n\geq j} \max_{\xi} \left| \prod_{\ell=1}^{\tilde{k}} \tilde{b}^{[n+\ell-1]}\left(e^{-i\xi/2^{\ell}}\right) \right|^{1/\tilde{k}}. \tag{3.14}$$

**Lemma 3.3.** *Let $\tilde{\phi}_j$, $j \geq j_0$, be given as in* (3.12) *with the symbol $\tilde{a}^{[j]}$ in* (3.9). *Suppose that for some integer $\tilde{k} > 0$, $\tilde{B}_{\tilde{k},j} < 2^{N-1/2}$, then*

$$\left|\hat{\tilde{\phi}}_j\right| \leq c\,\left(1+|\cdot|\right)^{-N+\log_2 \tilde{B}_{\tilde{k},j}}, \tag{3.15}$$

*which implies that $\tilde{\phi}_j \in L_2(\mathbb{R})$. Moreover, the function $\hat{\phi}_j$ can be defined as in* (3.12), *then this lemma also holds for $\phi_j$.*

*Proof.* Let $m \in \mathbb{N}$ and define $\tilde{u}_{j,m}$ by

$$\tilde{u}_{j,m}(\xi) := \left[\prod_{\ell=1}^{m} \frac{1}{2}\tilde{A}^{[j+\ell-1]}\left(\xi 2^{-\ell}\right)\right]\chi_{[-\pi,\pi]}\left(\xi 2^{-m}\right). \tag{3.16}$$

For the given $\gamma_n \in \mathbb{C}$ with $n = 1, \ldots, N$, set $\xi_{\gamma_n,j} := \xi + i\gamma_n 2^{-j-1}$. Then, we get the identity

$$\prod_{\ell=1}^{m} \left| \frac{1+e^{\gamma 2^{-j-\ell-1}} e^{-i\xi 2^{-\ell}}}{2} \right| = \prod_{\ell=1}^{m} \left| \frac{\sin\left(\xi_{n,j}2^{-\ell}\right)}{2\sin\left(\xi_{n,j}2^{-\ell-1}\right)} \right| = \left| \frac{\sin\left(\xi_{n,j}2^{-1}\right)}{2^m \sin\left(\xi_{n,j}2^{-m-1}\right)} \right|. \tag{3.17}$$

Here, we can see that there exist $\eta > 0$ and $M \in \mathbb{N}$ such that if $m \geq M$ and $|\xi 2^{-m-1}| \leq \eta$, $|\sin(\xi_{n,j}2^{-m-1})| \geq 2^{-1}|\xi_{n,j}2^{-m-1}|$. Also, it is obvious that $|\sin(\xi_{n,j}2^{-1})| \leq c_1|\xi_{n,j}2^{-1}|$ for some constant $c_1 > 0$ independent of $\xi$ but dependent on $\gamma_n$. Therefore, we have

$$\left| \frac{\sin\left(\xi_{n,j}2^{-1}\right)}{2^m \sin\left(\xi_{n,j}2^{-m-1}\right)} \right| \leq c_1 \left| \frac{\xi_{n,j}2^{-m-1}}{\sin\left(\xi_{n,j}2^{-m-1}\right)} \right| \leq c_2, \tag{3.18}$$

with a constant $c_2 > 0$ depending on $\gamma_n$. Next, consider the case of $\eta < |\xi 2^{-m-1}| \le \pi/2$ and let $\epsilon > 0$ be a sufficiently small number so that $\epsilon < \eta/2$. Noting that $|\sin\xi| \ge (2/\pi)|\xi|$ for all $\eta < |\xi| \le \pi/2$, it is not difficult to see that for a sufficiently large $j \ge J$,

$$\begin{aligned} 2^m\left|\sin\left(\xi_{n,j}2^{-m-1}\right)\right| &\ge 2^m\left(2^{-1}\left|\sin\left(\xi 2^{-m-1}\right)\right| - \left|\sin\left(\gamma_n 2^{-j-m-2}\right)\right|\right) \\ &\ge c_3(|\xi| - \epsilon) \ge c_4|\xi|, \end{aligned} \tag{3.19}$$

then we obtain that

$$\left|\frac{\sin\left(\xi_{n,j}2^{-1}\right)}{2^m \sin\left(\xi_{n,j}2^{-m-1}\right)}\right| \le \frac{c}{1+|\xi|}, \tag{3.20}$$

for a constant $c > 0$ depending on $\gamma_n$. Consequently, invoking the fact that $\widetilde{A}^{[j]}(\xi) = \widetilde{a}^{[j]}(e^{-i\xi})$ with $\widetilde{a}^{[j]}$ in (3.9), we obtain

$$\left|\widetilde{u}_{j,m}(\xi)\right| \le \frac{c}{(1+|\xi|)^N}\prod_{\ell=1}^{m}\left|\widetilde{b}^{[j+\ell-1]}\left(e^{-i\xi 2^{-\ell}}\right)\right|\chi_{[-\pi,\pi]}\left(2^{-m}\xi\right), \tag{3.21}$$

for any $j \ge J$. Here, using the same argument in [1, Proposition 4.8] (see point 3), we can get

$$\prod_{\ell=1}^{m}\widetilde{b}^{[j+\ell-1]}\left(e^{-i\xi 2^{-\ell}}\right) \le c(1+|\xi|)^{\log_2 \widetilde{B}_{\widetilde{k},j}}. \tag{3.22}$$

This together with (3.21) implies that $|\widetilde{u}_{j,m}(\xi)| \le c(1+|\xi|)^{-N+\log_2\widetilde{B}_{\widetilde{k},j}}$ for any $j \ge J$, where $c > 0$ is independent of $m$. Since $\widetilde{u}_{j,m}(\xi) \to \widehat{\widetilde{\phi}}_j(\xi)$ pointwise as $m \to \infty$, we get the relation in (3.15) with $j \ge J$. For $j < J$, applying an inductive argument based on the refinement equation $\widehat{\widetilde{\phi}}_j = \widetilde{A}^{[j]}(\cdot/2)\,\widehat{\widetilde{\phi}}_{j+1}(\cdot/2)$, we obtain the required result. The case of $\widehat{\phi}_j$ can be done similarly. □

**Lemma 3.4.** *Let $a^{[j]}$ and $\widetilde{a}^{[j]}$, $j \ge j_0$, be given as in* (3.9)*. Suppose that for some integers $k, \widetilde{k} > 0$,*

$$B_{k,j}, \widetilde{B}_{\widetilde{k},j} < 2^{N-1/2}, \tag{3.23}$$

*with $B_{k,j}$ and $\widetilde{B}_{k,j}$ in* (3.14)*, then, for any $j \ge j_0$, one has $\langle \phi_{j,\ell}, \widetilde{\phi}_{j,n}\rangle = \delta_{\ell,n}$ for all $\ell, n \in \mathbb{Z}$.*

*Proof.* Recalling the definition of $\widetilde{u}_{j,m}$ in (3.16), define $u_{j,m}$ by

$$u_{j,m}(\xi) := \left[\prod_{\ell=1}^{m}\frac{1}{2}A^{[j+\ell-1]}\left(\xi 2^{-\ell}\right)\right]\chi_{[-\pi,\pi]}\left(\xi 2^{-m}\right), \tag{3.24}$$

Then $u_{j,m}$ and $\widetilde{u}_{j,m}$ converge pointwise to $\widehat{\phi}_j$ and $\widehat{\widetilde{\phi}}_j$, respectively. Moreover, using (3.1), we can derive the relation

$$\begin{aligned}\widehat{u_{j,m}\overline{\widetilde{u}}_{j,m}}(\ell) &= \int_0^{2^{m+1}\pi}\left(\prod_{\ell=1}^{m}\frac{1}{4}A^{[j+\ell-1]}\left(\xi 2^{-\ell}\right)\overline{\widetilde{A}}^{[j+\ell-1]}\left(\xi 2^{-\ell}\right)\right)e^{i\ell\xi}d\xi \\ &= \int_0^{2^{m}\pi}\left(\prod_{\ell=1}^{m-1}\frac{1}{4}A^{[j+\ell-1]}\left(\xi 2^{-\ell}\right)\overline{\widetilde{A}}^{[j+\ell-1]}\left(\xi 2^{-\ell}\right)\right)e^{i\ell\xi}d\xi \\ &= \widehat{u_{j,m-1}\overline{\widetilde{u}}_{j,m-1}}(\ell).\end{aligned} \tag{3.25}$$

Repeating this process yields the identity $\widehat{u_{j,m}\overline{\widetilde{u}}_{j,m}}(\ell) = \widehat{u_{j,1}\overline{\widetilde{u}}_{j,1}}(\ell) = 2\pi\delta_{0,\ell}$. By Lemma 3.3, $\phi_j, \widetilde{\phi}_j \in L_2(\mathbb{R})$, then it is immediate from the Lebesgue-dominated convergence theorem that $u_{j,m}$ and $\widetilde{u}_{j,m}$ converge to $\widehat{\phi}_j$ and $\widehat{\widetilde{\phi}}_j$, respectively, in $L_2$. This in turn implies that $u_{j,m}\overline{\widetilde{u}}_{j,m} \to \widehat{\phi}_j\overline{\widehat{\widetilde{\phi}}}_j$ in $L_1$, as $m \to \infty$. Applying Plancherel's theorem, we arrive at the conclusion that $\langle \phi_j, \widetilde{\phi}_j(\cdot - \ell)\rangle = \delta_{0,\ell}$ for any $\ell \in \mathbb{Z}$. □

This result proves that $\langle \phi_{j,0}, \widetilde{\phi}_{j,n}\rangle = \delta_{0,n}$, $n \in \mathbb{Z}$, for some $j$ (in fact, for a sufficiently large $j$) which guarantees the condition (3.23). But the following proposition indeed proves that this duality condition holds for any $j \geq j_0$, under some suitable condition on the symbols $a$ and $\widetilde{a}$ in (3.10). In the following analysis, it is useful to use the notation

$$B_k := \max_{\xi}\left|\prod_{\ell=1}^{k} b\left(e^{-i\xi/2^{\ell}}\right)\right|^{1/k}, \qquad \widetilde{B}_{\widetilde{k}} := \max_{\xi}\left|\prod_{\ell=1}^{\widetilde{k}} \widetilde{b}\left(e^{-i\xi/2^{\ell}}\right)\right|^{1/\widetilde{k}}, \tag{3.26}$$

with $b$ and $\widetilde{b}$ in (3.10).

**Proposition 3.5.** *Let $a^{[j]}$ and $\widetilde{a}^{[j]}$, $j \geq j_0$, be given as in* (3.9). *Assume that for some integers $k, \widetilde{k} > 0$,*

$$B_k, \widetilde{B}_{\widetilde{k}} < 2^{N-1/2}, \tag{3.27}$$

*then, for any $j \geq j_0$, one has $\langle \phi_{j,0}, \widetilde{\phi}_{j,n}\rangle = \delta_{0,n}$ for all $n \in \mathbb{Z}$.*

*Proof.* Due to Lemma 3.2, we find that as $j \to \infty$, $b^{[j]}$ and $\widetilde{b}^{[j]}$ converge uniformly on $|z| = 1$ to $b$ and $\widetilde{b}$, respectively. Thus, we can deduce that there exists a large $J \in \mathbb{N}$ such that

$$B_{k,j}, \widetilde{B}_{\widetilde{k},j} < 2^{N-1/2}, \quad \forall j \geq J. \tag{3.28}$$

It follows from Lemma 3.4 that for any $j \geq J$, $\langle \phi_{j,0}, \widetilde{\phi}_{j,n} \rangle = \delta_{0,n}$ for all $n \in \mathbb{Z}$. For the case $j < J$, this property can be derived by using an inductive argument based on the non-stationary refinement equation. Specifically, applying (1.2), we get

$$\begin{aligned} \left\langle \phi_{j,0}, \widetilde{\phi}_{j,\ell} \right\rangle &= \frac{1}{2} \sum_{n \in \mathbb{Z}} a_n^{[j]} \sum_{k \in \mathbb{Z}} \widetilde{a}_{k-2\ell}^{[j]} \left\langle \phi_{j+1,n}, \widetilde{\phi}_{j+1,k} \right\rangle \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}} a_n^{[j]} \sum_{k \in \mathbb{Z}} \widetilde{a}_{k-2\ell}^{[j]} \delta_{n,k} \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}} a_n^{[j]} \widetilde{a}_{n-2\ell}^{[j]} = \delta_{0,\ell}, \end{aligned} \tag{3.29}$$

for any $\ell \in \mathbb{Z}$. This completes the proof. □

### 3.2. Smoothness of Refinable Functions

For a given $\kappa = n + s$ with $n \in \mathbb{N}$ and $s \in [0,1]$, the Hölder space $H^{\kappa}$ (e.g., see [3]) is defined to be the space of $n$-times continuously differentiable functions $f$ whose $n$th derivative $f^{(n)}$ satisfies the Lipschitz condition

$$\sup_{x,h \in \mathbb{R}} \frac{\left| f^{(n)}(x+h) - f^{(n)}(x) \right|}{|h|^s} \leq C. \tag{3.30}$$

In particular, the regularity of $f$ also can be analyzed by estimating the decay of $\widehat{f}$ around $\infty$. If $|\widehat{f}(\xi)| \leq c(1+|\xi|)^{-1-\kappa-\epsilon}$, $\xi \in \mathbb{R}$, for an arbitrary small $\epsilon > 0$, then $f$ belongs to the space $H^{\kappa}$. In the following theorem, under some more stringent condition on $B_k$ (than (3.27)), we derive the smoothness of non-stationary refinable functions $\widetilde{\phi}_j$, $j \geq j_0$, in (3.12). The smoothness of $\phi_j$ can be shown in a similar way.

**Theorem 3.6.** *Assume that the Laurent polynomial $\widetilde{a}$ in* (3.10) *can be rewritten as*

$$\widetilde{a}(z) = 2^{-\widetilde{K}} (1+z)^{\widetilde{K}+1} \widetilde{b}(z), \quad \widetilde{K} < N, \tag{3.31}$$

*such that $\widetilde{B}_{\widetilde{k}} < 1$ for some $\widetilde{k} \in \mathbb{N}$. Let $\widetilde{\phi}_j$, $j \geq j_0$, be given as in* (3.12) *with the symbol $\widetilde{a}^{[j]}$ in* (3.9)*, then $\widetilde{\phi}_j \in H^{\widetilde{K}+\widetilde{\nu}}$ for some $\widetilde{\nu} \in (0,1)$.*

*Proof.* The Laurent polynomial $\widetilde{a}^{[j]}$ in (3.9) can be written as

$$\widetilde{a}^{[j]}(z) = 2c_j \widetilde{b}^{[j]}(z) \prod_{n=1}^{\widetilde{K}+1} \frac{1}{2} \left( 1 + e^{\gamma_n 2^{-j-1}} z \right), \quad \widetilde{K} < N. \tag{3.32}$$

From Lemma 3.2, we can deduce that $\widetilde{b}^{[j]}(z) \to \widetilde{b}(z)$ uniformly on $|z| = 1$ as $j \to \infty$. This implies that $\widetilde{B}_{\widetilde{k},j}$ in (3.14) converges to $\widetilde{B}_{\widetilde{k}}$ ($< 1$) as $j \to \infty$. Thus, there exists a sufficiently large $J \in \mathbb{N}$ such that $\widetilde{B}_{\widetilde{k},j} < 1$ for any $j \geq J$, which means that $\log_2 \widetilde{B}_{\widetilde{k},j} < 0$. Hence, recalling from (3.15) that

$$\left|\widehat{\widetilde{\phi}}_j\right| \leq c(1 + |\cdot|)^{-\widetilde{K}-1+\log_2 \widetilde{B}_{\widetilde{k},j}}, \tag{3.33}$$

we see that for any $j \geq J$, $\widetilde{\phi}_j \in H^{\widetilde{K}+\widetilde{\nu}}$ with $\widetilde{\nu} \in (0,1)$. Next, consider the case of $j < J$. By applying an inductive argument based on the refinement equation

$$\widetilde{\phi}_j = \sum_{n\in\mathbb{Z}} a_n^{[j]} \widetilde{\phi}_{j+1}(\cdot - n), \tag{3.34}$$

we show that $\widetilde{\phi}_j \in H^{\widetilde{K}+\widetilde{\nu}}$ for any $j < J$. □

*Remark 3.7.* It is known (e.g., see [1]) that the Fourier transform of $\widetilde{\phi}$ with the symbol $\widetilde{a}$ in (3.31) has the decay rate

$$\left|\widehat{\widetilde{\phi}}\right| \leq c(1 + |\cdot|)^{-\widetilde{K}-1+\log_2 \widetilde{B}_{\widetilde{k}}}. \tag{3.35}$$

Since $\widetilde{B}_{\widetilde{k},j}$ converges to $\widetilde{B}_{\widetilde{k}}$ as $j$ tends to $\infty$, we can deduce that the functions $\widetilde{\phi}_j$, $j \geq j_0$, have the same regularity as $\widetilde{\phi}$. For the details about the (optimal) regularity of $\widetilde{\phi}$, the reader is referred to [1]. One may investigate the regularity of $\widetilde{\phi}_j$ by using the concept of asymptotical equivalent subdivision schemes (see [15, 19]). However, the methods in [15, 19] are mainly concerned with the integer smoothness of the refinable functions.

## 4. Stability of Nonstationary Biorthogonal Wavelets

For each $j \geq j_0$, if the condition (2.4) holds, the integer translates $\phi_j(\cdot - k)$, $k \in \mathbb{Z}$, are linearly independent, and in particular, the set $\{\phi_j(\cdot - k) : k \in \mathbb{Z}\}$ forms a Riesz basis for $V_j$, that is, there exist constants $A_j, B_j > 0$ such that

$$A_j \leq \left[\widehat{\phi}_j, \widehat{\phi}_j\right] \leq B_j. \tag{4.1}$$

In [12], it was proved that there exists some constants $0 < A, B < \infty$ independent of $j \geq j_0$ (but dependent on $j_0$) such that

$$A \leq \left[\widehat{\phi}_j, \widehat{\phi}_j\right] \leq B, \quad \forall j \geq j_0. \tag{4.2}$$

However, in the context of non-stationary wavelets, the stability of $\phi_j$ does not imply the global stability of the wavelet bases $\{\psi_{j,k} : (j,k) \in \mathcal{J}\}$, where

$$\mathcal{J} := \left\{(j,k) \in \mathbb{Z}^2 : j \geq j_0,\ k \in \mathbb{Z}\right\}. \tag{4.3}$$

The results in this section fill the gap. Specifically, we prove that the set $\{\psi_{j,k} : (j,k) \in \mathcal{I}\}$ is a Riesz basis for $+_{j\geq j_0} W_j$. Further, we show that the set

$$\mathcal{B} := \{\phi_{j_0,k} : k \in \mathbb{Z}\} \cup \{\psi_{j,k} : (j,k) \in \mathcal{I}\} \tag{4.4}$$

becomes a Riesz basis for the space $L_2(\mathbb{R})$.

In the sequel, we will use the notation $\phi^a$, $\tilde{\phi}^{\tilde{a}}$ when referring to the pair of dual refinable functions based on the $N$th-order polynomial B-spline, and $\psi^a$, $\tilde{\psi}^{\tilde{a}}$ for their corresponding biorthogonal wavelets (i.e., $\gamma_n = 0$ for all $n = 1, \ldots, N$).

**Lemma 4.1.** *Let $\psi_j$ and $\tilde{\psi}_j$ be a pair of biorthogonal wavelet functions associated with $\phi_j$ and $\tilde{\phi}_j$, $j \geq j_0$, then $\psi_j$ (resp., $\tilde{\psi}_j$) converges to $\psi^a$ (resp., $\tilde{\psi}^{\tilde{a}}$) in $L_2(\mathbb{R})$, as $j \to \infty$, with the convergence rate $O(2^{-j})$.*

*Proof.* With the refinement masks $\mathbf{a}^{[j]}$ and $\mathbf{a}$ of $\phi_j$ and $\phi^a$, respectively, it is apparent from (1.4) that

$$\|\psi_j - \psi^a\|_{L_2(\mathbb{R})} \leq \left\|\mathbf{a}^{[j]} - \mathbf{a}\right\|_1 \|\phi_j\|_{L_2(\mathbb{R})} + \|\mathbf{a}\|_1 \|\phi_j - \phi^a\|_{L_2(\mathbb{R})}. \tag{4.5}$$

It has been proved in the proof of Theorem 3 in [12] that $\|\phi_j - \phi^a\|_{L_2(\mathbb{R})} = O(2^{-j})$ as $j \to \infty$. Thus, by Lemma 3.2, it is obvious that $\|\psi_j - \psi^a\|_{L_2(\mathbb{R})} = O(2^{-j})$. Similarly, the convergence of $\tilde{\psi}_j$ to $\tilde{\psi}^{\tilde{a}}$ (as $j \to \infty$) can be proved. □

**Proposition 4.2.** *Assume that $\psi_j$ and $\tilde{\psi}_j$ are a pair of biorthogonal wavelet functions associated with $\phi_j$ and $\tilde{\phi}_j$, $j \geq j_0$. Let $\psi_{j,k} := 2^{j/2}\psi_j(2^j \cdot -k)$ and $\tilde{\psi}_{j,k} := 2^{j/2}\tilde{\psi}_j(2^j \cdot -k)$, then, for any $j \geq j_0$, the sets $\{\psi_{j,k} : k \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{j,k} : k \in \mathbb{Z}\}$ form Riesz bases for $W_j$ and $\widetilde{W}_j$, respectively. Furthermore, there exist constants $C, D, \tilde{C}, \tilde{D} > 0$ independent of $j \geq j_0$ (but dependent on $j_0$) such that*

$$C \leq [\hat{\psi}_j, \hat{\psi}_j] \leq D, \quad \tilde{C} \leq \left[\hat{\tilde{\psi}}_j, \hat{\tilde{\psi}}_j\right] \leq \tilde{D}, \quad \forall j \geq j_0. \tag{4.6}$$

*Proof.* Since $\psi_j$ and $\tilde{\psi}_j$ are compactly supported functions, there exist constants $D_j, \tilde{D}_j > 0$ such that

$$[\hat{\psi}_j, \hat{\psi}_j] \leq D_j, \qquad \left[\hat{\tilde{\psi}}_j, \hat{\tilde{\psi}}_j\right] \leq \tilde{D}_j. \tag{4.7}$$

Notice that the duality condition $\langle \psi_j(\cdot - k), \tilde{\psi}_j \rangle = \delta_{0,k}$ is equivalent to $[\hat{\psi}_j, \hat{\tilde{\psi}}_j] = 1$, then, by using the Cauchy-Schwartz inequality, we get

$$[\hat{\psi}_j, \hat{\psi}_j]\left[\hat{\tilde{\psi}}_j, \hat{\tilde{\psi}}_j\right] \geq 1. \tag{4.8}$$

Putting $C_j = \widetilde{D}_j^{-1}$, we obtain from (4.7) that $C_j \le [\widehat{\varphi}_j, \widehat{\varphi}_j] \le D_j$. Next, we prove that the Riesz bounds $C_j$ and $D_j$, $j \ge j_0$, are in some interval $[C, D]$ with $0 < C, D < \infty$. For this, we see that the $2\pi$ periodic function $[\widehat{\varphi}_j, \widehat{\varphi}_j]$ is of the form

$$[\widehat{\varphi}_j, \widehat{\varphi}_j](\xi) =: \sum_{n\in\mathbb{Z}} c_n^{[j]} e^{in\xi}, \qquad \text{where } c_n^{[j]} = \langle \psi_j, \psi_j(\cdot - n)\rangle. \tag{4.9}$$

Here, by Lemma 4.1, $\psi_j$ converges to $\psi^a$ in $L_2(\mathbb{R})$ as $j \to \infty$. Thus, it follows that for any $n \in \mathbb{Z}$, $c_n^{[j]}$ converge to $c_n = \langle \psi^a, \psi^a(\cdot - n)\rangle$ as $j \to \infty$. Moreover, since $\psi_j$ is compactly supported, only a finite number of $c_n^{[j]}$ is nonzero. It yields that

$$[\widehat{\varphi}_j, \widehat{\varphi}_j](\xi) \longrightarrow \sum_{n\in\mathbb{Z}} c_n e^{in\xi} = [\psi^a, \psi^a](\xi), \quad \text{as } j \longrightarrow \infty. \tag{4.10}$$

Therefore, we can deduce that the Riesz bounds $C_j$ and $D_j$ converge to $C_{\psi^a}$ and $D_{\psi^a}$ (resp.), as $j \to \infty$, which are the Riesz (upper and lower) bounds of $\psi^a$. It concludes that

$$C \le [\widehat{\varphi}_j, \widehat{\varphi}_j] \le D, \quad \forall j \ge j_0, \tag{4.11}$$

with some constants $C, D > 0$ independent of $j \ge j_0$. In a similar fashion, it can be proved that $\widetilde{C} \le [\widehat{\widetilde{\varphi}}_j, \widehat{\widetilde{\varphi}}_j] \le \widetilde{D}$ with some constants $\widetilde{C}, \widetilde{D} > 0$ independent of $j \ge j_0$. □

The above proposition discusses the stability of wavelet functions at each fixed level. The real problem is the global stability of the set $\{\psi_{j,k} : (j,k) \in \mathcal{J}\}$ (resp., $\{\widetilde{\psi}_{j,k} : (j,k) \in \mathcal{J}\}$) with $\mathcal{J}$ in (4.3). The following results treat this problem.

**Lemma 4.3.** *Let $\psi_j$ and $\widetilde{\psi}_j$ be a pair of biorthogonal wavelet functions associated with $\phi_j$ and $\widetilde{\phi}_j$, $j \ge j_0$, then there exist two constants $D, \widetilde{D} > 0$ depending on $j_0$ such that for all $f \in L_2(\mathbb{R})$,*

$$\sum_{(j,k)\in\mathcal{J}} \left|\langle f, \psi_{j,k}\rangle\right|^2 \le D\|f\|^2_{L_2(\mathbb{R})}, \qquad \sum_{(j,k)\in\mathcal{J}} \left|\langle f, \widetilde{\psi}_{j,k}\rangle\right|^2 \le \widetilde{D}\|f\|^2_{L_2(\mathbb{R})}, \tag{4.12}$$

*where $\mathcal{J}$ is given in* (4.3).

*Proof.* Let $\psi^a_{j,k} := 2^{j/2}\psi^a(2^j - k)$, and invoke that $\psi^a$ has the same support as $\psi_j$ for any $j \ge j_0$. Put $S_k := \operatorname{supp}\ \psi^a(\cdot - k)$ with $k \in \mathbb{Z}$, then, we observe that

$$\begin{aligned}\left|\left\langle f, \psi_{j,k} - \psi^a_{j,k}\right\rangle\right| &= \left|\int_{\mathbb{R}} f(x)\left(\psi_{j,k}(x) - \psi^a_{j,k}(x)\right)dx\right| \\ &\le 2^{-j/2}\int_{S_k}\left|f\left(2^{-j}x\right)\left[\psi_j(x-k) - \psi^a_j(x-k)\right]\right|dx\end{aligned}$$

$$\le 2^{-j/2}\|\psi_j - \psi^a\|_{L_2(\mathbb{R})}\left(\int_{S_k}\left|f\left(2^{-j}x\right)\right|^2 dx\right)^{1/2}$$
$$\le c2^{-j}\left(\int_{2^{-j}S_k}|f(x)|^2 dx\right)^{1/2}, \tag{4.13}$$

where the last bound is obtained by Lemma 4.1. Then, we get

$$\begin{aligned}
\sum_{(j,k)\in\mathcal{J}}\left|\left\langle f,\psi_{j,k}-\psi^a_{j,k}\right\rangle\right|^2 &\le c\sum_{(j,k)\in\mathcal{J}}2^{-2j}\int_{2^{-j}S_k}|f(x)|^2dx \\
&= c\sum_{j\ge j_0}2^{-2j}\sum_{k\in\mathbb{Z}}\int_{2^{-j}S_k}|f(x)|^2dx \\
&\le c\sum_{j\ge j_0}2^{-2j}\|f\|^2_{L_2(\mathbb{R})} \\
&\le c\|f\|^2_{L_2(\mathbb{R})}.
\end{aligned} \tag{4.14}$$

Putting all together, it is immediate that

$$\begin{aligned}
\left(\sum_{(j,k)\in\mathcal{J}}|\langle f,\psi_{j,k}\rangle|^2\right)^{1/2} &\le \left(\sum_{(j,k)\in\mathcal{J}}\left|\left\langle f,\psi^a_{j,k}\right\rangle\right|^2\right)^{1/2} + \left(\sum_{(j,k)\in\mathcal{J}}\left|\left\langle f,\psi_{j,k}-\psi^a_{j,k}\right\rangle\right|^2\right)^{1/2} \\
&\le D_{\psi^a}\|f\|_{L_2(\mathbb{R})} + c\|f\|_{L_2(\mathbb{R})} = D\|f\|_{L_2(\mathbb{R})},
\end{aligned} \tag{4.15}$$

for some constant $D > 0$, where $D_{\psi^a}$ is the Riesz upper bound of $\psi^a$. Similarly, we can prove the second relation in (4.12). □

We now arrive at the central results of this section.

**Theorem 4.4.** *Let $\psi_j$ and $\tilde{\psi}_j$ be a pair of biorthogonal wavelet functions associated with $\phi_j$ and $\tilde{\phi}_j$, $j \ge j_0$, then the sets $\{\psi_{j,k} : (j,k) \in \mathcal{J}\}$ and $\{\tilde{\psi}_{j,k} : (j,k) \in \mathcal{J}\}$ form Riesz bases for the spaces $+_{j\ge j_0}W_j$ and $+_{j\ge j_0}\widetilde{W}_j$, respectively, where $\mathcal{J}$ is given in* (4.3).

*Proof.* Let $f$ be a function in the space $\dot{+}_{j\geq j_0} W_j$. By the Cauchy-Schwartz inequality and Lemma 4.3, we obtain

$$\begin{aligned}
\|f\|^2_{L_2(\mathbb{R})} &= \lim_{J\to\infty} \sum_{j=j_0}^{J} \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{j,k}\rangle \overline{\langle f, \varphi_{j,k}\rangle} \\
&\leq \left( \sum_{(j,k)\in\mathcal{J}} \left|\langle f, \widetilde{\varphi}_{j,k}\rangle\right|^2 \right)^{1/2} \left( \sum_{(j,k)\in\mathcal{J}} \left|\langle f, \varphi_{j,k}\rangle\right|^2 \right)^{1/2} \\
&\leq \widetilde{D}^{1/2} \|f\|_{L_2(\mathbb{R})} \left( \sum_{(j,k)\in\mathcal{J}} \left|\langle f, \varphi_{j,k}\rangle\right|^2 \right)^{1/2},
\end{aligned} \tag{4.16}$$

for some constant $\widetilde{D} > 0$. Putting $C = \widetilde{D}^{-1}$, we have

$$C\|f\|^2_{L_2(\mathbb{R})} \leq \sum_{(j,k)\in\mathcal{J}} \left|\langle f, \varphi_{j,k}\rangle\right|^2. \tag{4.17}$$

Since the upper bound is proved in Lemma 4.3, it concludes that

$$C\|f\|^2_{L_2(\mathbb{R})} \leq \sum_{(j,k)\in\mathcal{J}} \left|\langle f, \varphi_{j,k}\rangle\right|^2 \leq D\|f\|^2_{L_2(\mathbb{R})}, \tag{4.18}$$

for some constant $D > 0$. Further, it is obvious that $\varphi_{j,k}$, $(j,k) \in \mathcal{J}$, are linearly independent. It indeed proves that the set $\{\varphi_{j,k} : (j,k) \in \mathcal{J}\}$ constitutes a Riesz basis for the space $\dot{+}_{j\geq j_0} W_j$ (see [4, Theorem 3.20]). Similarly, we can show that $\{\widetilde{\varphi}_{j,k} : (j,k) \in \mathcal{J}\}$ forms a Riesz basis for the space $\dot{+}_{j\geq j_0} \widetilde{W}_j$. □

Since $\phi_j \to \phi_a$ and $\widetilde{\phi}_j \to \widetilde{\phi}_a$ in $L_2(\mathbb{R})$ as $j \to \infty$, it is immediate that for any $f \in L_2(\mathbb{R})$, the biorthogonal projection

$$P_j f = \sum_{k\in\mathbb{Z}} \left\langle f, \widetilde{\phi}_{j,k} \right\rangle \phi_{j,k} \in V_j \tag{4.19}$$

converges to $f$ in the $L_2$-norm as $j \to \infty$, then based on this observation, we get the following result.

**Theorem 4.5.** *Let $\varphi_j$ and $\widetilde{\varphi}_j$ be a pair of biorthogonal wavelet functions associated with $\phi_j$ and $\widetilde{\phi}_j$, $j \geq j_0$, then the set $\mathcal{B} = \{\phi_{j_0,k} : k \in \mathbb{Z}\} \cup \{\varphi_{j,k} : (j,k) \in \mathcal{J}\}$ forms a Riesz basis for $L_2(\mathbb{R})$. This result also applies to the set of dual functions $\widetilde{\mathcal{B}} := \{\widetilde{\phi}_{j_0,k} : k \in \mathbb{Z}\} \cup \{\widetilde{\varphi}_{j,k} : (j,k) \in \mathcal{J}\}$.*

*Proof.* For a given function $f \in L_2(\mathbb{R})$, for notational simplicity, we define the following sequences:

$$\mathbf{c} := \left(\langle f, \phi_{j_0,k}\rangle\right)_{k\in\mathbb{Z}}, \quad \mathbf{d}_j := \left(\langle f, \varphi_{j,k}\rangle\right)_{k\in\mathbb{Z}}, \quad j \geq j_0. \tag{4.20}$$

Similarly, let $\widetilde{\mathbf{c}}$ and $\widetilde{\mathbf{d}}_j$, $j \geq j_0$, be the sequences consisting of the terms $\langle f, \widetilde{\phi}_{j_0,k} \rangle$ and $\langle f, \widetilde{\varphi}_{j,k} \rangle$, $k \in \mathbb{Z}$, respectively. It has been proven in [12] that there exist constants $\widetilde{A}, \widetilde{B} > 0$ such that

$$\widetilde{A}\|f\|^2_{L_2(\mathbb{R})} \leq \|\widetilde{\mathbf{c}}\|_2^2 \leq \widetilde{B}\|f\|^2_{L_2(\mathbb{R})}. \tag{4.21}$$

Also, by Theorem 4.4,

$$\sum_{j=j_0}^{\infty} \left\|\widetilde{\mathbf{d}}_j\right\|_2^2 \leq \widetilde{D}\|f\|^2_{L_2(\mathbb{R})}, \tag{4.22}$$

for some constant $\widetilde{D} > 0$, then, by the Cauchy-Schwartz inequality and Theorem 4.4, we get

$$\begin{aligned} \|f\|^2_{L_2(\mathbb{R})} &= \lim_{J \to \infty} \left( \sum_{k \in \mathbb{Z}} \left\langle f, \widetilde{\phi}_{j_0,k} \right\rangle \langle f, \phi_{j_0,k} \rangle + \sum_{j=j_0}^{J} \sum_{k \in \mathbb{Z}} \langle f, \widetilde{\varphi}_{j,k} \rangle \langle f, \varphi_{j,k} \rangle \right) \\ &\leq \left( \|\mathbf{c}\|_2^2 + \sum_{j=j_0}^{\infty} \|\mathbf{d}_j\|_2^2 \right)^{1/2} \left( \|\widetilde{\mathbf{c}}\|_2^2 + \sum_{j=j_0}^{\infty} \left\|\widetilde{\mathbf{d}}_j\right\|_2^2 \right)^{1/2} \\ &\leq \max\left(\widetilde{B}, \widetilde{D}\right)^{1/2} \|f\|_{L_2(\mathbb{R})} \left( \|\mathbf{c}\|_2^2 + \sum_{j=j_0}^{\infty} \|\mathbf{d}_j\|_2^2 \right)^{1/2}. \end{aligned} \tag{4.23}$$

Putting $E := \max(\widetilde{B}, \widetilde{D})^{-1}$, we have

$$E\|f\|^2_{L_2(\mathbb{R})} \leq \|\mathbf{c}\|_2^2 + \sum_{j=j_0}^{\infty} \|\mathbf{d}_j\|_2^2. \tag{4.24}$$

Since the upper bound is clear from Lemma 4.3, it concludes that

$$E\|f\|^2_{L_2(\mathbb{R})} \leq \|\mathbf{c}\|_2^2 + \sum_{j=j_0}^{\infty} \|\mathbf{d}_j\|_2^2 \leq F\|f\|^2_{L_2(\mathbb{R})}, \tag{4.25}$$

for some constant $F > 0$. It proves that the set $\mathcal{B}$ constitutes a Riesz basis for $L_2(\mathbb{R})$ (see [25, Theorem 6.1.1]). Similarly, we can show that $\widetilde{\mathcal{B}}$ forms a Riesz basis for $L_2(\mathbb{R})$. □

## Acknowledgments

The authors are grateful to the anonymous referees for their valuable suggestions on this paper. The work of Y. J. Lee was supported by Basic Science Research Program (2009-0068156) and J. Yoon was supported by Mid-Career Researcher Program (2009-0084583) and Basic Science Research Program (2010-0016257), through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology.

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