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Research Article

Sharp Generalized Seiffert Mean Bounds for Toader Mean

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For $p \in [0,1]$, the generalized Seiffert mean of two positive numbers a and b is defined by $S_p(a,b) = p(a-b)/\arctan[2p(a-b)/(a+b)],\ 0 < p \leq 1,\ a \neq b;\ (a+b)/2,\ p = 0,\ a \neq b;\ a,\ a = b$. In this paper, we find the greatest value α and least value β such that the double inequality $S_\alpha(a,b) < T(a,b) < S_\beta(a,b)$ holds for all $a, b > 0$ with $a \neq b$, and give new bounds for the complete elliptic integrals of the second kind. Here, $T(a,b) = (2/\pi)\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$ denotes the Toader mean of two positive numbers a and b.

1. Introduction

For $p \in [0,1]$, the generalized Seiffert mean of two positive numbers a and b is defined by

$$S_p(a,b) = \begin{cases} \dfrac{p(a-b)}{\arctan\left[2p(a-b)/(a+b)\right]}, & 0 < p \leq 1,\ a \neq b, \\[2ex] \dfrac{a+b}{2}, & p = 0,\ a \neq b, \\[2ex] a, & a = b. \end{cases} \tag{1.1}$$

It is well known that $S_p(a,b)$ is continuous and strictly increasing with respect to $p \in [0,1]$ for fixed $a, b > 0$ with $a \neq b$. In particular, if $p = 1/2$, then the generalized Seiffert mean reduces to the Seiffert mean

$$S(a,b) = \begin{cases} \dfrac{a-b}{2\arctan((a-b)/(a+b))}, & a \neq b, \\ a, & a = b. \end{cases} \tag{1.2}$$

Recently, the Seiffert mean and its generalization have been the subject of intensive research, many remarkable inequalities for these means can be found in the literature [1–5].

In [6], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$\begin{aligned} T(a,b) &= \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta, \\ &= \begin{cases} \dfrac{2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)}{\pi}, & a > b, \\ \dfrac{2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)}{\pi}, & a < b, \\ a, & a = b, \end{cases} \end{aligned} \tag{1.3}$$

where $\mathcal{E}(r) = \int_0^{\pi/2}(1 - r^2\sin^2 t)^{1/2}dt$, $r \in [0,1)$ is the complete elliptic integral of the second kind.

Vuorinen [7] conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$, where

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases} \tag{1.5}$$

is the power mean of order p of two positive numbers a and b. This conjecture was proved by Barnard et al. [8].

In [9], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b) \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to find the greatest value α and least value β such that the double inequality $S_\alpha(a,b) < T(a,b) < S_\beta(a,b)$ holds for all $a, b > 0$ with $a \neq b$ and give new bounds for the complete elliptic integrals of the second kind.

2. Lemmas

In order to establish our main result, we need several formulas and lemmas, which we present in this section.

The following formulas were presented in [10, Appendix E, pages 474-475]: Let $r \in [0,1)$, then

$$\mathcal{K}(r)=\int_0^{\pi/2}\left(1-r^2\sin^2 t\right)^{-1/2}dt,\qquad \mathcal{K}(0)=\frac{\pi}{2},\qquad \mathcal{K}(1^-)=+\infty,$$
$$\mathcal{E}(r)=\int_0^{\pi/2}\left(1-r^2\sin^2 t\right)^{1/2}dt,\qquad \mathcal{E}(0)=\pi/2,\ \mathcal{E}(1^-)=1,$$
$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)},\qquad \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r},\tag{2.1}$$
$$\frac{d\left[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{dr}=r\mathcal{K}(r),\qquad \frac{d[\mathcal{K}(r)-\mathcal{E}(r)]}{dr}=\frac{r\mathcal{E}(r)}{1-r^2},$$
$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{1+r}.$$

Lemma 2.1 (see [10, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f(x), g(x) : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and be differentiable on (a,b), let $g'(x) \neq 0$ for all $x \in (a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)},\qquad \frac{f(x)-f(b)}{g(x)-g(b)}.\tag{2.2}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. (1) $[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,1)$;

(2) $\{[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2-\pi/4\}/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/32, 1-\pi/4)$;

(3) $[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,+\infty)$;

(4) $2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/2,2)$;

(5) $F(r)=[(2-r^2)\mathcal{K}(r)-2\mathcal{E}(r)]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/16,+\infty)$;

(6) $G(r)=[4\pi-\pi r^2-8\mathcal{E}(r)]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(3\pi/16, 3\pi-8)$.

Proof. Parts (1)–(4) can be found in [10, Theorem 3.21(1), Theorem 3.31(6), and Exercise 3.43(11) and (13)].

For part (5), clearly $F(1^-)=+\infty$. Let $F_1(r)=(2-r^2)\mathcal{K}(r)-2\mathcal{E}(r)$ and $F_2(r)=r^4$, then $F(r)=F_1(r)/F_2(r)$, $F_1(0)=F_2(0)=0$ and

$$\frac{F_1'(r)}{F_2'(r)}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}}{4r^2(1-r^2)}.\tag{2.3}$$

It follows from (2.3) and part (1) together with Lemma 2.1 that $F(r)$ is strictly increasing in $(0,1)$ and $F(0^+)=\pi/16$.

For part (6), clearly $G(1^-) = 3\pi - 8$. Let $G_1(r) = 4\pi - \pi r^2 - 8\mathcal{E}(r)$ and $G_2(r) = r^4$, then $G(r) = G_1(r)/G_2(r)$, $G_1(0) = G_2(0) = 0$, and

$$\frac{G_1'(r)}{2G_2'(r)} = \frac{(2-r^2)\mathcal{K}(r) - 2\mathcal{E}(r)}{r^4} + \frac{\left[\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]/r^2 - \pi/4}{r^2}. \tag{2.4}$$

From (2.4), parts (2) and (5) together with Lemma 2.1, we know that $G(r)$ is strictly increasing in $(0,1)$, and $f(0^+) = 3\pi/16$. □

Lemma 2.3. (1) $g(r) = \arctan(\sqrt{3}r/2) - \sqrt{3}\pi r/\{4[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]\}$ *is strictly increasing from* $(0,1)$ *onto* $(0, \arctan(\sqrt{3}/2) - \sqrt{3}\pi/8)$.

(2) $f(r) = \arctan r - \pi r/\{2[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]\} < 0$ *for* $r \in (0,1)$.

Proof. For part (1), clearly $g(0^+) = 0$ and $g(1^-) = \arctan(\sqrt{3}/2) - \sqrt{3}\pi/8 = 0.0335\cdots > 0$. Simple computation leads to

$$\begin{aligned} g'(r) &= \frac{2\sqrt{3}}{4+3r^2} - \frac{\sqrt{3}\pi\mathcal{E}(r)}{4[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]^2} \\ &= \frac{\sqrt{3}r^4\mathcal{E}(r)}{4(4+3r^2)[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]^2} g_1(r), \end{aligned} \tag{2.5}$$

where $g_1(r) = \{8[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]^2 - \pi(4+3r^2)\mathcal{E}(r)\}/[r^4\mathcal{E}(r)]$.

Making use of Lemma 2.2 (1), (2), and (6), we get

$$\begin{aligned} g_1(r) &= \frac{8}{\mathcal{E}(r)} \cdot \left[\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2}\right]^2 + \frac{16\{\left[\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]/r^2 \ - \pi/4\}}{r^2} \\ &\quad - \frac{4\pi - \pi r^2 - 8\mathcal{E}(r)}{r^4} \\ &> \frac{16}{\pi} \cdot \left(\frac{\pi}{4}\right)^2 + 16 \cdot \frac{\pi}{32} - (3\pi - 8) = 8 - \frac{3\pi}{2} > 0. \end{aligned} \tag{2.6}$$

Therefore, part (1) follows from (2.5) and (2.6) together with the limiting values of $g(r)$ at $r = 0$ and $r = 1$.

For part (2), simple computations yield that

$$\lim_{r\to 0^+} f(r) = \lim_{r\to 1^-} f(r) = 0, \tag{2.7}$$

$$f'(r) = \frac{f_1(r)}{2(1+r^2)[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]^2}, \tag{2.8}$$

where $f_1(r) = 2[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]^2 - \pi(1+r^2)\mathcal{E}(r)$. Note that

$$\lim_{r\to 0^+} f_1(r) = 0, \tag{2.9}$$

$$\lim_{r\to 1^-} f_1(r) = 8 - 2\pi > 0, \tag{2.10}$$

$$\begin{aligned} f_1'(r) &= \frac{4\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]\left[\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{r} - 2\pi r\mathcal{E}(r) \\ &\quad -\pi(1+r^2)\frac{\mathcal{E}(r) - \mathcal{K}(r)}{r} \\ &= r f_2(r), \end{aligned} \tag{2.11}$$

where $f_2(r) = 4[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)][\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2-2\pi\mathcal{E}(r)+\pi(1+r^2)[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$.

From Lemma 2.2(1), (3), and (4) together with the monotonicity of $\mathcal{E}(r)$ we know that $f_2(r)$ is strictly increasing in $(0,1)$. Moreover,

$$\lim_{r\to 0^+} f_2(r) = -\frac{\pi^2}{4}, \tag{2.12}$$

$$\lim_{r\to 1^-} f_2(r) = +\infty. \tag{2.13}$$

Equations (2.11)–(2.13) and the monotonicity of $f_2(r)$ lead to the conclusion that there exists $r_0 \in (0,1)$ such that $f_1(r)$ is strictly decreasing in $(0, r_0)$ and strictly increasing in $(r_0, 1)$.

It follows from (2.8)–(2.10) and the piecewise monotonicity of $f_1(r)$ that there exists $r_1 \in (0,1)$ such that $f(r)$ is strictly decreasing in $(0, r_1)$ and strictly increasing in $(r_1, 1)$.

Therefore, part (2) follows from (2.7) and the piecewise monotonicity of $f(r)$. □

3. Main Result

Theorem 3.1. *Inequality $S_{\sqrt{3}/4}(a,b) < T(a,b) < S_{1/2}(a,b)$ holds for all $a,b > 0$ with $a \neq b$, and $S_{\sqrt{3}/4}(a,b)$ and $S_{1/2}(a,b)$ are the best possible lower and upper generalized Seiffert mean bounds for the Toader mean $T(a,b)$, respectively.*

Proof. Firstly, we prove that

$$S_{\sqrt{3}/4}(a,b) < T(a,b) < S_{1/2}(a,b) \tag{3.1}$$

for all $a,b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = b/a < 1$, $r = (1-t)/(1+t)$. Then (1.1) and (1.3) lead to

$$\begin{aligned}
T(a,b) - S_{\sqrt{3}/4}(a,b) &= \frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-t^2}\right) - \frac{\sqrt{3}a(1-t)}{4\arctan\left[\sqrt{3}(1-t)/2(1+t)\right]} \\
&= \frac{2a}{\pi}\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) - \frac{\sqrt{3}ar}{2(1+r)\arctan\left(\left(\sqrt{3}/2\right)r\right)} \\
&= \frac{2a}{\pi}\frac{\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{1+r} - \frac{\sqrt{3}ar}{2(1+r)\arctan\left(\left(\sqrt{3}/2\right)r\right)} \\
&= \frac{2a\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{\pi(1+r)\arctan\left(\left(\sqrt{3}/2\right)r\right)} g(r),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
T(a,b) - S_{1/2}(a,b) &= \frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-t^2}\right) - \frac{a(1-t)}{2\arctan((1-t)/(1+t))} \\
&= \frac{2a}{\pi}\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) - \frac{ar}{(1+r)\arctan r} \\
&= \frac{2a}{\pi}\frac{\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{1+r} - \frac{ar}{(1+r)\arctan r} \\
&= \frac{2a\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{\pi(1+r)\arctan r} f(r),
\end{aligned} \tag{3.3}$$

where $g(r)$ and $f(r)$ are defined as in Lemma 2.3.

Therefore, inequality (3.1) follows from (3.2) and (3.3) together with Lemma 2.3.

Next, we prove that $S_{\sqrt{3}/4}(a,b)$ and $S_{1/2}(a,b)$ are the best possible lower and upper generalized Seiffert mean bounds for the Toader mean $T(a,b)$, respectively.

For any $\varepsilon > 0$ and $0 < x < 1$, from (1.1) and (1.3) one has

$$\lim_{x\to 0}\left[S_{1/2-\varepsilon}(1,x) - T(1,x)\right] = \frac{1-2\varepsilon}{2\arctan(1-2\varepsilon)} - \frac{2}{\pi} < \frac{1}{2\arctan 1} - \frac{2}{\pi} = 0, \tag{3.4}$$

$$S_{\sqrt{3}/4+\varepsilon}(1,1-x) - T(1,1-x) = \frac{J(x)}{\arctan\left[\left(\sqrt{3}+4\varepsilon\right)x/2(2-x)\right]}, \tag{3.5}$$

where $J(x) = (\sqrt{3}/4+\varepsilon)x - 2\mathcal{E}(\sqrt{2x-x^2})\arctan\{[(\sqrt{3}+4\varepsilon)x]/[2(2-x)]\}/\pi$.

Letting $x \to 0$ and making use of Taylor expansion, we get

$$J(x) = \left(\frac{\sqrt{3}}{4}+\varepsilon\right)x - \left(\frac{\sqrt{3}}{4}+\varepsilon\right)x\left[1-\frac{1}{2}x+\frac{1}{16}x^2+o\left(x^2\right)\right] \times \left\{1+\frac{1}{2}x+\left[\frac{1}{4}-\frac{1}{3}\left(\frac{\sqrt{3}}{4}+\varepsilon\right)^2\right]x^2+o\left(x^2\right)\right\} = \frac{\varepsilon}{3}\left(\frac{\sqrt{3}}{2}+\varepsilon\right)\left(\frac{\sqrt{3}}{4}+\varepsilon\right)x^3+o\left(x^3\right). \tag{3.6}$$

Inequality (3.4) and equations (3.5) and (3.6) imply that for any $\varepsilon > 0$ there exist $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$, such that $S_{\sqrt{3}/4+\varepsilon}(1, 1-x) > T(1, 1-x)$ for $x \in (0, \delta_1)$ and $S_{1/2-\varepsilon}(1, x) < T(1, x)$ for $x \in (0, \delta_2)$. □

From Theorem 3.1, we get new bounds for the complete elliptic integrals of the second kind as follows.

Corollary 3.2. *The inequality*

$$\frac{\sqrt{3}\pi\left(1-\sqrt{1-r^2}\right)}{8\arctan\left\{\sqrt{3}\left(1-\sqrt{1-r^2}\right)/\left[2\left(1+\sqrt{1-r^2}\right)\right]\right\}} < E(r) < \frac{\pi\left(1-\sqrt{1-r^2}\right)}{4\arctan\left[\left(1-\sqrt{1-r^2}\right)/\left(1+\sqrt{1-r^2}\right)\right]} \tag{3.7}$$

holds for all $r \in (0, 1)$.

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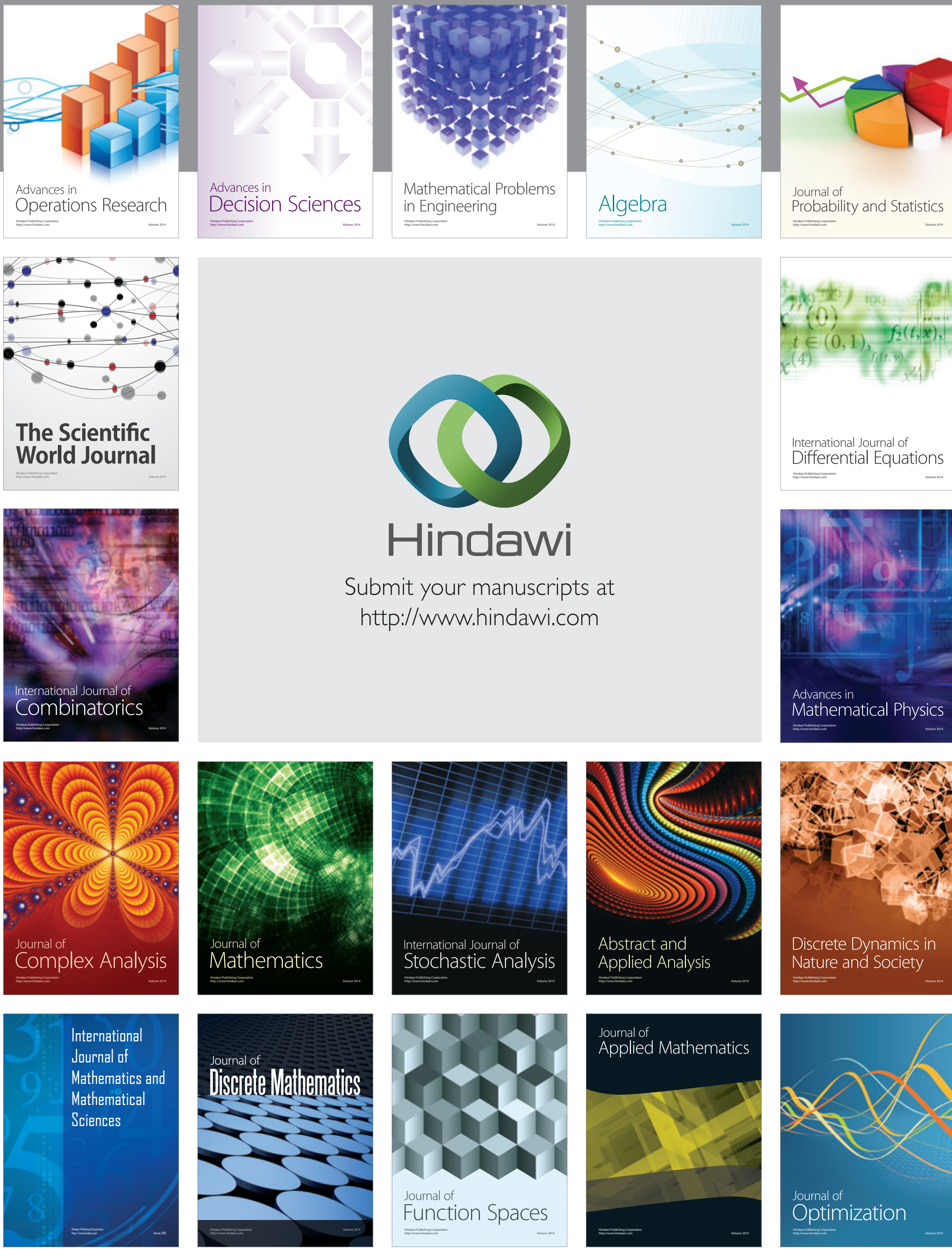
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