Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 630583, 11 pages
doi:10.1155/2011/630583

Research Article

On Complete Convergence for Weighted Sums of Arrays of Dependent Random Variables

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Received 24 September 2011; Accepted 17 November 2011

Academic Editor: Zhenya Yan

A rate of complete convergence for weighted sums of arrays of rowwise independent random variables was obtained by Sung and Volodin (2011). In this paper, we extend this result to negatively associated and negatively dependent random variables. Similar results for sequences of φ-mixing and ρ^*-mixing random variables are also obtained. Our results improve and generalize the results of Baek et al. (2008), Kuczmaszewska (2009), and Wang et al. (2010).

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [1]. A sequence $\{X_n,\ n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \epsilon) < \infty \quad \forall \epsilon > 0. \tag{1.1}$$

In view of the Borel-Cantelli lemma, this implies that $X_n \rightarrow \theta$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdös [2] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors.

Ahmed et al. [3] obtained complete convergence for weighted sums of arrays of rowwise independent Banach-space-valued random elements.

We recall that the array $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if

$$P(|X_{ni}| > x) \leq CP(|X| > x) \quad \forall x > 0, \quad \forall i \geq 1,\ n \geq 1, \tag{1.2}$$

where C is a positive constant.

Theorem 1.1 (Ahmed et al. [3]). *Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of rowwise independent random elements which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of constants satisfying*

$$\sup_{i\geq 1}|a_{ni}| = O(n^{-\gamma}) \quad \text{for some } \gamma > 0, \tag{1.3}$$

$$\sum_{i=1}^{\infty}|a_{ni}| = O(n^{\alpha}) \quad \text{for some } \alpha < \gamma. \tag{1.4}$$

Suppose that there exists $\delta > 1$ such that $1 + \alpha/\gamma < \delta \leq 2$. Let $\beta \neq -1 - \alpha$ and $\nu = \max\{1 + (1 + \alpha + \beta)/\gamma, \delta\}$. If $E|X|^{\nu} < \infty$ and $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left\|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right\| > \epsilon\right) < \infty \quad \forall \epsilon > 0. \tag{1.5}$$

Note that there was a typographical error in Ahmed et al. [3] (the relation $\delta > 0$ should be $\delta > 1$). If $\beta < -1$, then the conclusion of Theorem 1.1 is immediate. Hence, Theorem 1.1 is of interest only for $\beta \geq -1$.

Baek et al. [4] extended Theorem 1.1 to negatively associated random variables.

Theorem 1.2 (Baek et al. [4]). *Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of rowwise negatively associated random variables which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of constants satisfying* (1.3) *and* (1.4). *Suppose that there exists $\delta > 0$ such that $1+\alpha/\gamma < \delta \leq 2$. Let $\beta \geq -1$ and $\nu = \max\{1 + (1 + \alpha + \beta)/\gamma, \delta\}$. If $EX_{ni} = 0$, for all $i \geq 1$ and $n \geq 1$, and*

$$\begin{gathered} E|X|\log|X| < \infty, \quad \text{for } 1 + \alpha + \beta = 0, \\ E|X|^{\nu} < \infty, \quad \text{for } 1 + \alpha + \beta > 0, \end{gathered} \tag{1.6}$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right| > \epsilon\right) < \infty \quad \forall \epsilon > 0. \tag{1.7}$$

Sung and Volodin [5] improved Theorem 1.1 as follows.

Theorem 1.3 (Sung and Volodin [5]). *Suppose that $\beta \geq -1$. Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of rowwise independent random elements which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of constants satisfying* (1.3) *and* (1.4). *Assume that $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$*

in probability. If

$$E|X|\log|X| < \infty, \quad \text{for } 1+\alpha+\beta = 0,$$
$$E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty, \quad \text{for } 1+\alpha+\beta > 0, \tag{1.8}$$

then (1.5) *holds.*

In this paper, we extend Theorem 1.3 to negatively associated and negatively dependent random variables. We also obtain similar results for sequences of φ-mixing and ρ^*-mixing random variables. Our results improve and generalize the results of Baek et al. [4], Kuczmaszewska [6], and Wang et al. [7].

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance. It proves convenient to define $\log x = \max\{1, \ln x\}$, where $\ln x$ denotes the natural logarithm.

2. Preliminaries

In this section, we present some background materials which will be useful in the proofs of our main results.

The following lemma is well known, and its proof is standard.

Lemma 2.1. *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of random variables which are stochastically dominated by a random variable* X*. For any* $\alpha > 0$ *and* $b > 0$*, the following statements hold:*

(i) $E|X_n|^\alpha I(|X_n| \leq b) \leq C\{E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\}$,

(ii) $E|X_n|^\alpha I(|X_n| > b) \leq CE|X|^\alpha I(|X| > b)$.

Lemma 2.2 (Sung [8]). *Let* X *be a random variable with* $E|X|^r < \infty$ *for some* $r > 0$*. For any* $t > 0$*, the following statements hold:*

(i) $\sum_{n=1}^{\infty} n^{-1-t\delta} E|X|^{r+\delta} I(|X| \leq n^t) \leq CE|X|^r$ *for any* $\delta > 0$,

(ii) $\sum_{n=1}^{\infty} n^{-1+t\delta} E|X|^{r-\delta} I(|X| > n^t) \leq CE|X|^r$ *for any* $\delta > 0$ *such that* $r - \delta > 0$,

(iii) $\sum_{n=1}^{\infty} n^{-1+tr} P(|X| > n^t) \leq CE|X|^r$.

The Rosenthal-type inequality plays an important role in establishing complete convergence. The Rosenthal-type inequalities for sequences of dependent random variables have been established by many authors.

The concept of negatively associated random variables was introduced by Alam and Saxena [9] and carefully studied by Joag-Dev and Proschan [10]. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\big(f_1(X_i,\ i \in A),\ f_2(X_j,\ j \in B)\big) \leq 0, \tag{2.1}$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

The following lemma is a Rosenthal-type inequality for negatively associated random variables.

Lemma 2.3 (Shao [11]). *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of negatively associated random variables with* $EX_n = 0$ *and* $E|X_n|^q < \infty$ *for some* $q \geq 2$ *and all* $n \geq 1$. *Then there exists a constant* $C > 0$ *depending only on* q *such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^q\right) \leq C\left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right\}. \tag{2.2}$$

The concept of negatively dependent random variables was given by Lehmann [12]. A finite family of random variables $\{X_1, \dots, X_n\}$ is said to be negatively dependent (or negatively orthant dependent) if for each $n \geq 2$, the following two inequalities hold:

$$\begin{gathered} P(X_1 \leq x_1, \dots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i), \\ P(X_1 > x_1, \dots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i), \end{gathered} \tag{2.3}$$

for all real numbers $x_1, \dots, x_n$. An infinite family of random variables is negatively dependent if every finite subfamily is negatively dependent.

Obviously, negative association implies negative dependence, but the converse is not true.

The following lemma is a Rosenthal-type inequality for negatively dependent random variables.

Lemma 2.4 (Asadian et al. [13]). *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of negatively dependent random variables with* $EX_n = 0$ *and* $E|X_n|^q < \infty$ *for some* $q \geq 2$ *and all* $n \geq 1$. *Then there exists a constant* $C > 0$ *depending only on* q *such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^q \leq C\left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right\}. \tag{2.4}$$

For a sequence $\{X_n,\ n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, let $\mathcal{F}_n^m$ denote the σ-algebra generated by the random variables $X_n, X_{n+1}, \dots, X_m$. Define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k\geq 1} \sup\left\{|P(B \mid A) - P(B)|,\ A \in \mathcal{F}_1^k,\ P(A) \neq 0,\ B \in \mathcal{F}_{k+n}^{\infty}\right\}. \tag{2.5}$$

The sequence $\{X_n,\ n \geq 1\}$ is called φ-mixing (or ϕ-mixing) if $\varphi(n) \to 0$ as $n \to \infty$.

For any $S \subset \mathbb{N}$, let $\mathcal{F}_S = \sigma(X_i,\ i \in S)$. Define the ρ^*-mixing coefficients by

$$\rho^*(n) = \sup \operatorname{corr}(f, g), \tag{2.6}$$

where the supremum is taken over all $S,T \subset \mathbb{N}$ with $\operatorname{dist}(S,T) \geq n$, and all $f \in L_2(\mathcal{F}_S)$ and $g \in L_2(\mathcal{F}_T)$. The sequence $\{X_n,\ n \geq 1\}$ is called ρ^*-mixing (or $\tilde{\rho}$-mixing) if there exists $k \in \mathbb{N}$ such that $\rho^*(k) < 1$.

Note that if $\{X_n,\ n \geq 1\}$ is a sequence of independent random variables, then $\varphi(n) = 0$ and $\rho^*(n) = 0$ for all $n \geq 1$.

The following lemma is a Rosenthal-type inequality for φ-mixing random variables.

Lemma 2.5 (Wang et al. [7]). *Let $\{X_n, n \geq 1\}$ be a sequence of φ-mixing random variables with $EX_n = 0$ and $E|X_n|^q < \infty$ for some $q \geq 2$ and all $n \geq 1$. Assume that $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$. Then there exists a constant $C > 0$ depending only on q and $\varphi(\cdot)$ such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_i\right|^q\right) \leq C\left\{\sum_{i=1}^{n}E|X_i|^q + \left(\sum_{i=1}^{n}EX_i^2\right)^{q/2}\right\}. \tag{2.7}$$

The following lemma is a Rosenthal-type inequality for ρ^*-mixing random variables.

Lemma 2.6 (Utev and Peligrad [14]). *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables with $EX_n = 0$ and $E|X_n|^q < \infty$ for some $q \geq 2$ and all $n \geq 1$. If $\rho^*(k) < 1$ for some k, then there exists a constant $C > 0$ depending only on q, k, and $\rho^*(k)$ such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_i\right|^q\right) \leq C\left\{\sum_{i=1}^{n}E|X_i|^q + \left(\sum_{i=1}^{n}EX_i^2\right)^{q/2}\right\}. \tag{2.8}$$

3. Main Results

In this section, we extend Theorem 1.3 to negatively associated and negatively dependent random variables. We also obtain similar results for sequences of φ-mixing and ρ^*-mixing random variables.

The following theorem extends Theorem 1.3 to negatively associated random variables.

Theorem 3.1. *Suppose that $\beta \geq -1$. Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of rowwise negatively associated random variables which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of constants satisfying* (1.3) *and* (1.4). *If $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$, and* (1.8) *holds, then*

$$\sum_{n=1}^{\infty}n^{\beta}P\left(\sup_{j\geq 1}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right| > \epsilon\right) < \infty \quad \forall\epsilon > 0. \tag{3.1}$$

Proof. Since $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, we may assume that $a_{ni} \geq 0$. For $i \geq 1$ and $n \geq 1$, define

$$X'_{ni} = X_{ni}I(|X_{ni}| \leq n^{\gamma}) + n^{\gamma}I(X_{ni} > n^{\gamma}) - n^{\gamma}I(X_{ni} < -n^{\gamma}), \qquad X''_{ni} = X_{ni} - X'_{ni}. \tag{3.2}$$

Then $\{X'_{ni},\ i \geq 1,\ n \geq 1\}$ is still an array of rowwise negatively associated random variables. Moreover, $\{a_{ni}X'_{ni},\ i \geq 1,\ n \geq 1\}$ is also an array of rowwise negatively associated random variables. Since $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$, it suffices to show that

$$I_1 =: \sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{j\geq 1} \left| \sum_{i=1}^{j} a_{ni}\left(X'_{ni} - EX'_{ni}\right) \right| > \epsilon \right) < \infty,$$
$$I_2 =: \sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{j\geq 1} \left| \sum_{i=1}^{j} a_{ni}\left(X''_{ni} - EX''_{ni}\right) \right| > \epsilon \right) < \infty. \tag{3.3}$$

We will prove (3.3) with three cases.

Case 1 $(1+(1+\alpha+\beta)/\gamma = 1(\text{i.e., } 1+\alpha+\beta = 0))$. For I_1, we get by Markov's inequality, Lemmas 2.1–2.3, (1.3), and (1.4) that

$$\begin{aligned}
I_1 &\leq \epsilon^{-2} \sum_{n=1}^{\infty} n^{\beta} E \sup_{j\geq 1} \left| \sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni}) \right|^2 \\
&\leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}|^2 E\left|X'_{ni}\right|^2 \quad \text{(by Lemma 2.3)} \\
&\leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}|^2 \left\{ E|X|^2 I(|X| \leq n^{\gamma}) + n^{2\gamma} P(|X| > n^{\gamma}) \right\} \quad \text{(by Lemma 2.1)} \\
&\leq C \sum_{n=1}^{\infty} n^{\beta} n^{-\gamma} n^{\alpha} \left\{ E|X|^2 I(|X| \leq n^{\gamma}) + n^{2\gamma} P(|X| > n^{\gamma}) \right\} \quad \text{(by (1.3) and (1.4))} \\
&\leq CE|X|^{1+(1+\alpha+\beta)/\gamma} < \infty.
\end{aligned} \tag{3.4}$$

The fifth inequality follows from Lemma 2.2.

For I_2, we get by Markov's inequality, stochastic domination, and (1.4) that

$$\begin{aligned}
I_2 &\leq \epsilon^{-1} \sum_{n=1}^{\infty} n^{\beta} E \sup_{j\geq 1} \left| \sum_{i=1}^{j} a_{ni}\left(X''_{ni} - EX''_{ni}\right) \right| \\
&\leq 2\epsilon^{-1} \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}| E\left|X''_{ni}\right| \\
&\leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}| E|X| I(|X| > n^{\gamma}) \\
&\leq C \sum_{n=1}^{\infty} n^{\beta} n^{\alpha} E|X| I(|X| > n^{\gamma}) \\
&= C \sum_{n=1}^{\infty} n^{-1} \sum_{i=n}^{\infty} E|X| I\left(i^{\gamma} < |X| \leq (i+1)^{\gamma}\right)
\end{aligned}$$

$$= C\sum_{i=1}^{\infty} E|X| I\left(i^{\gamma} < |X| \le (i+1)^{\gamma}\right) \sum_{n=1}^{i} n^{-1}$$
$$\le CE|X| \log|X| < \infty. \tag{3.5}$$

Case 2 $(1 < 1 + (1+\alpha+\beta)/\gamma < 2)$. As in Case 1, we have that $I_1 \le CE|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$.

Similar to I_2 in Case 1, we have that

$$\begin{aligned} I_2 &\le C\sum_{n=1}^{\infty} n^{\alpha+\beta} E|X| I(|X| > n^{\gamma}) \\ &= C\sum_{n=1}^{\infty} n^{\alpha+\beta} \sum_{i=n}^{\infty} E|X| I\left(i^{\gamma} < |X| \le (i+1)^{\gamma}\right) \\ &= C\sum_{i=1}^{\infty} E|X| I\left(i^{\gamma} < |X| \le (i+1)^{\gamma}\right) \sum_{n=1}^{i} n^{\alpha+\beta} \\ &\le CE|X|^{1+(1+\alpha+\beta)/\gamma} < \infty. \end{aligned} \tag{3.6}$$

Case 3 $(1 + (1+\alpha+\beta)/\gamma \ge 2)$. For I_1, we take $t > 0$ sufficiently large such that $(\gamma - \alpha)(1 + (1 + \alpha + \beta)/\gamma + t)/2 > 1 + \beta$. Then we obtain by Markov's inequality and Lemma 2.3 that

$$\begin{aligned} I_1 &\le \epsilon^{-1-(1+\alpha+\beta)/\gamma - t} \sum_{n=1}^{\infty} n^{\beta} E \sup_{j \ge 1} \left| \sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni}) \right|^{1+(1+\alpha+\beta)/\gamma+t} \\ &\le C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E\left|a_{ni}X'_{ni}\right|^{1+(1+\alpha+\beta)/\gamma+t} \\ &\quad + C\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{\infty} E\left|a_{ni}X'_{ni}\right|^{2} \right)^{(1+(1+\alpha+\beta)/\gamma+t)/2} \\ &=: I_3 + I_4. \end{aligned} \tag{3.7}$$

Similar to I_1 in Case 1, we obtain that

$$\begin{aligned} I_3 &\le C\sum_{n=1}^{\infty} n^{\beta} n^{-\gamma((1+\alpha+\beta)/\gamma+t)} n^{\alpha} \left\{ E|X|^{1+(1+\alpha+\beta)/\gamma+t} I(|X| \le n^{\gamma}) + n^{\gamma(1+(1+\alpha+\beta)/\gamma+t)} P(|X| > n^{\gamma}) \right\} \\ &= C\sum_{n=1}^{\infty} n^{-1-\gamma t} E|X|^{1+(1+\alpha+\beta)/\gamma+t} I(|X| \le n^{\gamma}) + C\sum_{n=1}^{\infty} n^{\alpha+\beta+\gamma} P(|X| > n^{\gamma}) \\ &\le CE|X|^{1+(1+\alpha+\beta)/\gamma} < \infty. \end{aligned} \tag{3.8}$$

Noting $E|X'_{ni}|^2 \leq CE|X|^2$, we obtain by (1.3) and (1.4) that

$$\begin{aligned} I_4 &\leq C\sum_{n=1}^{\infty} n^{\beta}\left(CE|X|^2\sum_{i=1}^{\infty}|a_{ni}|^2\right)^{(1+(1+\alpha+\beta)/\gamma+t)/2} \\ &\leq C\sum_{n=1}^{\infty} n^{\beta}\left(CE|X|^2 n^{\alpha-\gamma}\right)^{(1+(1+\alpha+\beta)/\gamma+t)/2} < \infty, \end{aligned} \tag{3.9}$$

since $(\gamma-\alpha)(1+(1+\alpha+\beta)/\gamma+t)/2-\beta > 1$. Hence, $I_1 < \infty$. As in Case 2, we obtain $I_2 \leq CE|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$. □

Remark 3.2. The moment condition of Theorem 3.1 is weaker than that of Theorem 1.2. Also, the conclusion of Theorem 3.1 implies the conclusion of Theorem 1.2. Hence, Theorem 3.1 improves Theorem 1.2. Moreover, the method of the proof of Theorem 3.1 is simpler than that of the proof of Theorem 1.2.

Corollary 3.3. *Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of rowwise negatively associated random variables which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be a Toeplitz array satisfying*

$$\sup_{i\geq 1}|a_{ni}| = O\left(n^{1/t-\delta}\right) \quad \textit{for some } t > 0,\ \delta > 0. \tag{3.10}$$

If

$$\begin{aligned} E|X| < \infty, &\quad \textit{for } 0 < t < 1, \\ E|X|\log|X| < \infty, &\quad \textit{for } t = 1, \\ E|X|^{1+(1-1/t)/\delta} < \infty, &\quad \textit{for } t > 1, \end{aligned} \tag{3.11}$$

then

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \epsilon n^{1/t}\right) < \infty \quad \forall \epsilon > 0. \tag{3.12}$$

Proof. For the case $0 < t < 1$, the result can be easily proved by

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \epsilon n^{1/t}\right) &\leq \epsilon^{-1}\sum_{n=1}^{\infty} n^{-1/t} E\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| \\ &\leq \epsilon^{-1}\sum_{n=1}^{\infty} n^{-1/t}\sum_{i=1}^{n}|a_{ni}|E|X_{ni}| \\ &\leq CE|X|\sum_{n=1}^{\infty} n^{-1/t} < \infty. \end{aligned} \tag{3.13}$$

For the case $t \geq 1$, we let $b_{ni} = a_{ni} n^{-1/t}$. Observe that

$$\sup_{i\geq 1} |b_{ni}| = O\left(n^{-\delta}\right), \qquad \sum_{i=1}^{\infty} |b_{ni}| = O\left(n^{-1/t}\right). \tag{3.14}$$

By Theorem 3.1 with $\alpha = -1/t$, $\beta = 0$, $\gamma = \delta$, and a_{ni} replaced by b_{ni}, we get that

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} b_{ni} (X_{ni} - EX_{ni}) \right| > \epsilon \right) < \infty \quad \forall \epsilon > 0. \tag{3.15}$$

To complete the proof, we only prove that

$$J =: \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} b_{ni} EX_{ni} \right| \longrightarrow 0, \tag{3.16}$$

but $J \leq \sum_{i=1}^{n} |b_{ni}| E|X_{ni}| \leq CE|X| n^{-1/t} \to 0$ as $n \to \infty$. □

Remark 3.4. When $0 < t < 1$, Corollary 3.3 holds without negative association. Kuczmaszewska [6, Corollary 2.4], proved Corollary 3.3 under the stronger moment condition $E|X|^{1+1/\delta} < \infty$.

The following theorem extends Theorem 1.3 to negatively dependent random variables.

Theorem 3.5. *Suppose that $\beta \geq -1$. Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of rowwise negatively dependent random variables which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of constants satisfying* (1.3) *and* (1.4). *If $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$, and* (1.8) *holds, then* (1.7) *holds.*

Proof. The proof is the same as that of Theorem 3.1 except that we use Lemma 2.4 instead of Lemma 2.3. □

If the array $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ in Theorem 3.1 is replaced by the sequence $\{X_n,\ n \geq 1\}$, then we can extend Theorem 3.1 to φ-mixing and ρ^*-mixing random variables.

Theorem 3.6. *Suppose that $\beta \geq -1$. Let $\{X_n,\ n \geq 1\}$ be a sequence of φ-mixing random variables which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of constants satisfying* (1.3) *and* (1.4). *Assume that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. If $EX_n = 0$ for all $n \geq 1$, and* (1.8) *holds, then*

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{j\geq 1} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \epsilon \right) < \infty \quad \forall \epsilon > 0. \tag{3.17}$$

Proof. Since $EX_n = 0$ for all $n \geq 1$, it suffices to show that

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{j\geq 1} \left| \sum_{i=1}^{j} a_{ni}(X_i I(|X_i| \leq n^{\gamma}) - EX_i I(|X_i| \leq n^{\gamma})) \right| > \epsilon \right) < \infty,$$
$$\sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{j\geq 1} \left| \sum_{i=1}^{j} a_{ni}(X_i I(|X_i| > n^{\gamma}) - EX_i I(|X_i| > n^{\gamma})) \right| > \epsilon \right) < \infty. \tag{3.18}$$

The rest of the proof is the same as that of Theorem 3.1 except that we use Lemma 2.5 instead of Lemma 2.3 and it is omitted. □

Remark 3.7. Can Theorem 3.6 be extended to the array $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ of rowwise φ-mixing random variables? Let $\{\varphi_n(i),\ i \geq 1\}$ be the sequence of φ-mixing coefficients for the nth row $\{X_{n1}, X_{n2}, \ldots\}$ of the array $\{X_{ni}\}$. When we apply Lemma 2.5 to the nth row, the constant C depends on both q and $\varphi_n(\cdot)$. That is, the constant C depends on n. Hence we cannot extend Theorem 3.6 to the array by using the method of the proof of Theorem 3.1.

Corollary 3.8. *Let $\{X_n,\ n \geq 1\}$ be a sequence of φ-mixing random variables which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be a Toeplitz array satisfying* (3.10)*. Assume that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. If* (3.11) *holds, then*

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \epsilon n^{1/t} \right) < \infty \quad \forall \epsilon > 0. \tag{3.19}$$

Proof. The proof is the same as that of Corollary 3.3 except that we use Theorem 3.6 instead of Theorem 3.1. □

Remark 3.9. When $0 < t < 1$, Corollary 3.8 holds without φ-mixing. Wang et al. [7, Theorem 2.5] proved Corollary 3.8 under the stronger moment condition $E|X|^{\max\{2/\delta, 1+1/\delta\}} < \infty$.

Theorem 3.10. *Suppose that $\beta \geq -1$. Let $\{X_n,\ n \geq 1\}$ be a sequence of ρ^*-mixing random variables which are stochastically dominated by a random variable X. Let $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of constants satisfying* (1.3) *and* (1.4)*. If $EX_n = 0$ for all $n \geq 1$, and* (1.8) *holds, then* (3.17) *holds.*

Proof. The proof is the same as that of Theorem 3.6 except that we use Lemma 2.6 instead of Lemma 2.5. □

Remark 3.11. Likewise in Remark 3.7, we also cannot extend Theorem 3.10 to the array by using the method of the proof of Theorem 3.1.

Acknowledgments

The author would like to thank the Editor Zhenya Yan and an anonymous referee for the helpful comments. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0013131).

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