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Research Article

A Study on the Fermionic p-Adic q-Integral Representation on $\mathbb{Z}_p$ Associated with Weighted q-Bernstein and q-Genocchi Polynomials

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We consider weighted q-Genocchi numbers and polynomials. We investigated some interesting properties of the weighted q-Genocchi numbers related to weighted q-Bernstein polynomials by using fermionic p-adic integrals on $\mathbb{Z}_p$.

1. Introduction, Definitions, and Notations

The main motivation of this paper is [1] by Kim, in which he introduced and studied properties of q-Bernoulli numbers and polynomials with weight α. Recently, many mathematicians have studied weighted special polynomials (see [1–5]).

This numbers and polynomials are used in not only number theory, complex analysis, and the other branch of mathematics, but also in other parts of the p-adic analysis and mathematical physics. Kurt Hensel (1861–1941) invented the so-called p-adic numbers around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within scientific community [6] although they have penetrated several mathematical fields such as number theory, algebraic geometry, algebraic topology, analysis, and mathematical physics (see, for details, [6–8]).

The p-adic q-integral (or q-Volkenborn integral) are originally constructed by Kim [9]. The q-Volkenborn integral is used in mathematical physics, for example, the functional equation of the q-zeta function, the q-Stirling numbers, and q-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L function.

Let p be a fixed odd prime number. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The p-adic absolute value is defined by $|p|_p = 1/p$. In this paper, we assume $|q-1|_p < 1$ as an indeterminate. In [10–12], let UD ($\mathbb{Z}_p$) be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in$ UD ($\mathbb{Z}_p$), the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} q^x f(x)(-1)^x. \tag{1.1}$$

For $\alpha, k, n \in \mathbb{N}^*$ and $x \in [0,1]$, Kim et al. defined weighted q-Bernstein polynomials as follows:

$$B_{k,n}^{(\alpha)}(x,q) = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}, \tag{1.2}$$

(see [13, 14]). When we put $q \to 1$ and $\alpha = 1$ in (1.2), $[x]_{q^\alpha}^k \to x^k$, $[1-x]_{q^{-\alpha}}^{n-k} \to (1-x)^{n-k}$, and we obtain the classical Bernstein polynomials (see [13, 14]), where $[x]_q$ is a q-extension of x which is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.3}$$

(see [1–4, 7, 9–12, 14–26]). Note that $\lim_{q\to 1}[x]_q = x$.

In [3], For $n \in \mathbb{N}^*$, S. Araci et al. defined weighted q-Genocchi polynomials as follows:

$$\begin{aligned} \frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1} &= \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_{-q}(y) = \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha \ell x} \frac{1}{1+q^{\alpha\ell+1}} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [m+x]_{q^\alpha}^n. \end{aligned} \tag{1.4}$$

In the special case, $x = 0$, $\widetilde{G}_{n,q}^{(\alpha)}(0) = \widetilde{G}_{n,q}^{(\alpha)}$ are called the q-Genocchi numbers with weight α.

In [3], For $\alpha \in \mathbb{N}^*$ and $n \in \mathbb{N}$, S. Araci et al. defined q-Genocchi numbers with weight α as follows:

$$\widetilde{G}_{0,q}^{(\alpha)} = 0, \qquad q\widetilde{G}_{n,q}^{(\alpha)}(1) + \widetilde{G}_{n,q}^{(\alpha)} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases} \tag{1.5}$$

In this paper we obtained some relations between the weighted q-Bernstein polynomials and the q-Genocchi numbers. From these relations, we derive some interesting identities on the q-Genocchi numbers and polynomials with weight α.

2. On the Weighted q-Genocchi Numbers and Polynomials

By the definition of q-Genocchi polynomials with weight α, we easily get

$$\begin{aligned}\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1} &= \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_{-q}(y) = \int_{\mathbb{Z}_p} \left([x]_{q^\alpha} + q^{\alpha x}[y]_{q^\alpha}\right)^n d\mu_{-q}(y) \\ &= \sum_{k=0}^{n}\binom{n}{k}[x]_{q^\alpha}^{n-k} q^{\alpha k x}\int_{\mathbb{Z}_p}[y]_{q^\alpha}^k d\mu_{-q}(y) = \sum_{k=0}^{n}\binom{n}{k}[x]_{q^\alpha}^{n-k} q^{\alpha k x}\frac{\widetilde{G}_{k+1,q}^{(\alpha)}}{k+1}.\end{aligned} \tag{2.1}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n, \alpha \in \mathbb{N}^*$, one has*

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = q^{-\alpha x}\sum_{k=0}^{n}\binom{n}{k} q^{\alpha k x}\widetilde{G}_{k,q}^{(\alpha)}[x]_{q^\alpha}^{n-k}, \tag{2.2}$$

with usual convention about replacing $(\widetilde{G}_q^{(\alpha)})^n$ by $\widetilde{G}_{n,q}^{(\alpha)}$.

By Theorem 2.1, we have

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = q^{-\alpha x}\left(q^{\alpha x}\widetilde{G}_q^{(\alpha)} + [x]_{q^\alpha}\right)^n. \tag{2.3}$$

By (1.4), we get

$$\begin{aligned}\frac{\widetilde{G}_{n+1,q^{-1}}^{(\alpha)}(1-x)}{n+1} &= \int_{\mathbb{Z}_p}[1-x+y]_{q^{-\alpha}}^n d\mu_{-q}(y) \\ &= \frac{[2]_{q^{-1}}}{(1-q^{-\alpha})^n}\sum_{l=0}^{n}\binom{n}{l} q^{-\alpha l(1-x)}(-1)^l\frac{1}{1+q^{-\alpha l-1}} \\ &= (-1)^n q^{\alpha n}\frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{1}{1+q^{\alpha l+1}} \\ &= (-1)^n q^{\alpha n}\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1}.\end{aligned} \tag{2.4}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n, \alpha \in \mathbb{N}^*$, one has*

$$\widetilde{G}_{n,q^{-1}}^{(\alpha)}(1-x) = (-1)^{n-1} q^{\alpha(n-1)}\widetilde{G}_{n,q}^{(\alpha)}(x). \tag{2.5}$$

From (1.5) and Theorem 2.1, we have the following theorem.

Theorem 2.3. *For $n, \alpha \in \mathbb{N}^*$, one has*

$$\widetilde{G}_{0,q}^{(\alpha)} = 0, \qquad q^{1-\alpha}\left(q^{\alpha}\widetilde{G}_{q}^{(\alpha)} + 1\right)^{n} + \widetilde{G}_{n,q}^{(\alpha)} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases} \tag{2.6}$$

with usual convention about replacing $(G_q^{(\alpha)})^n$ by $G_{n,q}^{(\alpha)}$.

For $n, \alpha \in \mathbb{N}$, by Theorem 2.3, we note that

$$\begin{aligned} q^{2\alpha}\widetilde{G}_{n,q}^{(\alpha)}(2) &= \left(q^{\alpha}\left(q^{\alpha}\widetilde{G}_{q}^{(\alpha)} + 1\right) + 1\right)^{n} \\ &= \sum_{k=0}^{n}\binom{n}{k} q^{k\alpha}\left(q^{\alpha}\widetilde{G}_{q} + 1\right)^{k} \\ &= nq^{2\alpha-1}\left([2]_q - \widetilde{G}_{1,q}^{(\alpha)}\right) - q^{\alpha-1}\sum_{k=2}^{n}\binom{n}{k} q^{k\alpha}\widetilde{G}_{k,q}^{(\alpha)} \\ &= q^{\alpha-1}[2]_q - q^{\alpha-1}\sum_{k=1}^{n}\binom{n}{k} q^{\alpha k}\widetilde{G}_{k,q}^{(\alpha)} \\ &= q^{\alpha-1}[2]_q + q^{2\alpha-2}\widetilde{G}_{n,q}^{(\alpha)} \quad \text{if } n > 1. \end{aligned} \tag{2.7}$$

Therefore, we have the following theorem.

Theorem 2.4. *For $n \in \mathbb{N}$, one has*

$$\widetilde{G}_{n,q}^{(\alpha)}(2) = \frac{[2]_q}{q^{\alpha+1}} + \frac{1}{q^2}\widetilde{G}_{n,q}^{(\alpha)}. \tag{2.8}$$

From Theorem 2.2 and (2.5), we see that

$$\begin{aligned} &(n+1)\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n} d\mu_{-q}(x) = (n+1)(-1)^n q^{n\alpha}\int_{\mathbb{Z}_p} [x-1]_{q^{\alpha}}^{n} d\mu_{-q}(x) \\ &\quad = (-1)^n q^{n\alpha}\widetilde{G}_{n+1,q}^{(\alpha)}(-1) = \widetilde{G}_{n+1,q^{-1}}^{(\alpha)}(2). \end{aligned} \tag{2.9}$$

Therefore, we get the following theorem.

Theorem 2.5. *For $n, \alpha \in \mathbb{N}^*$, one has*

$$(n+1)\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n} d\mu_{-q}(x) = \widetilde{G}_{n+1,q^{-1}}^{(\alpha)}(2). \tag{2.10}$$

Let $n, \alpha \in \mathbb{N}$. By Theorems 2.4 and 2.5, we get

$$(n+1)\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n} d\mu_{-q}(x) = q^{\alpha}[2]_q + q^2 \widetilde{G}_{n+1,q^{-1}}^{(\alpha)}. \tag{2.11}$$

From (2.11), we get the following corollary.

Corollary 2.6. *For $n, \alpha \in \mathbb{N}^*$, one has*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n} d\mu_{-q}(x) = \frac{q^{\alpha}}{n+1}[2]_q + q^2 \frac{\widetilde{G}_{n+1,q^{-1}}^{(\alpha)}}{n+1}. \tag{2.12}$$

3. Novel Identities on the Weighted q-Genocchi Numbers

In this section, we derive concerning the some interesting properties of q-Genocchi numbers via the p-adic q-integral on $\mathbb{Z}_p$, in the sense of fermionic and weighted q-Bernstein polynomials.

$$B_{k,n}^{(\alpha)}(x,q) = \binom{n}{k}[x]_{q^{\alpha}}^{k}[1-x]_{q^{-\alpha}}^{n-k}, \quad \text{where } n,k,\alpha \in \mathbb{N}^*. \tag{3.1}$$

By (3.1), Kim et al. get the symmetry of q-Bernstein polynomials weighted α as follows:

$$B_{k,n}^{(\alpha)}(x,q) = B_{n-k,n}^{(\alpha)}\left(1-x,q^{-1}\right), \tag{3.2}$$

(see [4]). Thus, from Corollary 2.6, (3.1), and (3.2), we see that

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}^{(\alpha)}\left(1-x,q^{-1}\right)d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n-l}d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(\frac{q^{\alpha}[2]_q}{n-l+1} + q^2\frac{\widetilde{G}_{n-l+1,q^{-1}}^{(\alpha)}}{n-l+1}\right).
\end{aligned} \tag{3.3}$$

For $n, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n > k$, we obtain

$$\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q}(x) = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(\frac{q^\alpha [2]_q}{n-l+1} + q^2 \frac{\widetilde{G}^{(\alpha)}_{n-l+1,q^{-1}}}{n-l+1} \right) = \begin{cases} \dfrac{q^\alpha [2]_q}{n+1} + q^2 \dfrac{\widetilde{G}^{(\alpha)}_{n+1,q^{-1}}}{n+1}, & \text{if } k = 0, \\ \displaystyle\sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(\frac{q^\alpha [2]_q}{n-l+1} + q^2 \frac{\widetilde{G}^{(\alpha)}_{n-l+1,q^{-1}}}{n-l+1} \right), & \text{if } k \neq 0. \end{cases} \quad (3.4)$$

Let us take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the weighted q-Bernstein polynomials of degree n as follows:

$$\int_{\mathbb{Z}_p} B^{(\alpha)}_{k,n}(x,q) d\mu_{-q}(x) = \binom{n}{k} \int_{\mathbb{Z}_p} [x]^k_{q^\alpha} [1-x]^{n-k}_{q^{-\alpha}} d\mu_{-q}(x) = \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{\widetilde{G}^{(\alpha)}_{l+k+1,q}}{l+k+1}. \quad (3.5)$$

Therefore, by (3.4) and (3.5), we obtain the following theorem.

Theorem 3.1. *For $n, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n > k$, one has*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{\widetilde{G}^{(\alpha)}_{l+k+1,q}}{l+k+1} = \begin{cases} \dfrac{q^2 [2]_q}{n+1} + q^2 \dfrac{\widetilde{G}^{(\alpha)}_{n+1,q^{-1}}}{n+1}, & \text{if } k = 0, \\ \displaystyle\sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(\frac{q^\alpha [2]_q}{n-l+1} + q^2 \frac{\widetilde{G}^{(\alpha)}_{n-l+1,q^{-1}}}{n-l+1} \right), & \text{if } k \neq 0. \end{cases} \quad (3.6)$$

Let $n_1, n_2, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n_1 + n_2 > 2k$. Then, we get

$$\int_{\mathbb{Z}_p} B^{(\alpha)}_{k,n_1}(x,q) B^{(\alpha)}_{k,n_2}(x,q) d\mu_{-q}(x) = \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}_p} [1-x]^{n_1+n_2-l}_{q^{-\alpha}} d\mu_{-q}(x)$$

$$= \left(\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k+l}\left(\frac{q^{\alpha}[2]_q}{n_1+n_2-l+1} + q^2\frac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1} \right) \right)$$

$$= \begin{cases} \dfrac{q^{\alpha}[2]_q}{n_1+n_2+1} + q^2\dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2+1,q^{-1}}}{n_1+n_2+1}, & \text{if } k=0, \\ \dbinom{n_1}{k}\dbinom{n_2}{k}\displaystyle\sum_{l=0}^{2k}\dbinom{2k}{l}(-1)^{2k+l}\left(\dfrac{q^{\alpha}[2]_q}{n_1+n_2-l+1} + q^2\dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1} \right), & \text{if } k\neq 0. \end{cases} \tag{3.7}$$

Therefore, we obtain the following theorem.

Theorem 3.2. *For $n_1, n_2, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n_1 + n_2 > 2k$, one has*

$$\int_{\mathbb{Z}_p} B^{(\alpha)}_{k,n_1}(x,q)B^{(\alpha)}_{k,n_2}(x,q)\,d\mu_{-q}(x)$$
$$= \begin{cases} \dfrac{q^{\alpha}[2]_q}{n_1+n_2+1} + q^2\dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2+1,q^{-1}}}{n_1+n_2+1}, & \text{if } k=0, \\ \dbinom{n_1}{k}\dbinom{n_2}{k}\displaystyle\sum_{l=0}^{2k}\dbinom{2k}{l}(-1)^{2k+l}\left(\dfrac{q^{\alpha}[2]_q}{n_1+n_2-l+1} + q^2\dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1} \right), & \text{if } k\neq 0. \end{cases} \tag{3.8}$$

From the binomial theorem, we can derive

$$\begin{aligned} &\int_{\mathbb{Z}_p} B^{(\alpha)}_{k,n_1}(x,q)B^{(\alpha)}_{k,n_2}(x,q)\,d\mu_{-q}(x) \\ &\quad= \prod_{i=1}^{2}\binom{n_i}{k}\sum_{l=0}^{n_1+n_2-2k}\binom{n_1+n_2-2k}{l}(-1)^l\int_{\mathbb{Z}_p}[x]^{2k+l}_{q^{\alpha}}\,d\mu_{-q}(x) \\ &\quad= \prod_{i=1}^{2}\binom{n_i}{k}\sum_{l=0}^{n_1+n_2-2k}\binom{n_1+n_2-2k}{l}(-1)^l\frac{\widetilde{G}^{(\alpha)}_{l+2k+1,q}}{l+2k+1}. \end{aligned} \tag{3.9}$$

Thus, for Theorem 3.4 and (3.13), we can obtain the following corollary.

Corollary 3.3. *For $n_1, n_2, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n_1 + n_2 > 2k$, one has*

$$\sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \frac{\widetilde{G}^{(\alpha)}_{l+2k+1,q}}{l+2k+1} = \begin{cases} \dfrac{q^\alpha [2]_q}{n_1+n_2+1} + q^2 \dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2+1,q^{-1}}}{n_1+n_2+1}, & \text{if } k = 0, \\ \displaystyle\sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left(\frac{q^\alpha [2]_q}{n_1+n_2-l+1} + q^2 \frac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1} \right), & \text{if } k \neq 0. \end{cases} \tag{3.10}$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $\sum_{l=1}^{s} n_l > sk$. Then, we take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the weighted q-Bernstein polynomials of degree n as follows:

$$\begin{aligned} &\int_{\mathbb{Z}_p} \underbrace{B^{(\alpha)}_{k,n_1}(x,q) B^{(\alpha)}_{k,n_2}(x,q) \cdots B^{(\alpha)}_{k,n_s}(x,q)}_{s\text{-times}} d\mu_{-q}(x) \\ &= \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} [x]^{sk}_{q^\alpha} [1-x]^{n_1+n_2+\cdots+n_s-sk}_{q^{-\alpha}} d\mu_{-q}(x) \\ &= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} [1-x]^{n_1+n_2+\cdots+n_s-sk}_{q^{-\alpha}} d\mu_{-q}(x) \\ &= \begin{cases} \dfrac{q^\alpha [2]_q}{n_1+n_2+\cdots+n_s+1} + q^2 \dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2+\cdots+n_s+1,q^{-1}}}{n_1+n_2+\cdots+n_s+1}, & \text{if } k = 0, \\ \displaystyle\prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} & \\ \quad \times \left(\dfrac{q^\alpha [2]_q}{n_1+n_2+\cdots+n_s-l+1} + q^2 \dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2+\cdots+n_s-l+1,q^{-1}}}{n_1+n_2+\cdots+n_s-l+1} \right), & \text{if } k \neq 0. \end{cases} \end{aligned} \tag{3.11}$$

Therefore, we obtain the following theorem.

Theorem 3.4. *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s$, $k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $\sum_{l=1}^{s} n_l > sk$. Then, one has*

$$\int_{\mathbb{Z}_p} \prod_{i=1}^{s} B^{(\alpha)}_{k,n_i}(x) d\mu_{-q}(x) = \begin{cases} \dfrac{q^\alpha [2]_q}{n_1+n_2+\cdots+n_s+1} q^2 \dfrac{\widetilde{G}^{(\alpha)}_{n_1+n_2+\cdots+n_s+1,q^{-1}}}{n_1+n_2+\cdots+n_s-l+1}, & \text{if } k = 0, \\ \displaystyle\prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left(\frac{q^\alpha [2]_q}{n_1+n_2-l+1} + q^2 \frac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1} \right), & \text{if } k \neq 0. \end{cases} \tag{3.12}$$

From the definition of weighted q-Bernstein polynomials and the binomial theorem, we easily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}^{(\alpha)}(x,q) B_{k,n_2}^{(\alpha)}(x,q) \cdots B_{k,n_s}^{(\alpha)}(x,q)}_{s\text{-times}} d\mu_{-q}(x) \\
&\quad = \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{\sum_{d=1}^{s}(n_d-k)}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_{q^\alpha}^{sk+l} d\mu_{-q}(x) \\
&\quad = \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=1}^{n_1+\cdots+n_s-sk} \binom{\sum_{d=1}^{s}(n_d-k)}{l} (-1)^l \frac{\widetilde{G}_{l+sk+1,q}^{(\alpha)}}{l+sk+1}.
\end{aligned} \tag{3.13}$$

Therefore, from (3.13) and Theorem 3.4, we have the following corollary.

Corollary 3.5. *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $\sum_{l=1}^{s} n_l > sk$. One has*

$$\begin{aligned}
&\sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{\sum_{d=1}^{s}(n_d-k)}{l} (-1)^l \frac{\widetilde{G}_{l+sk+1,q}^{(\alpha)}}{l+sk+1} \\
&= \begin{cases} \dfrac{q^\alpha [2]_q}{n_1+n_2+\cdots+n_s+1} + q^2 \dfrac{\widetilde{G}_{n_1+n_2+\cdots+n_s+1,q^{-1}}^{(\alpha)}}{n_1+n_2+\cdots+n_s+1}, & \text{if } k=0, \\ \displaystyle\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left(\frac{q^\alpha [2]_q}{n_1+n_2+\cdots+n_s-l+1} + q^2 \frac{\widetilde{G}_{n_1+n_2+\cdots+n_s-l+1,q^{-1}}^{(\alpha)}}{n_1+n_2+\cdots+n_s-l+1} \right), & \text{if } k \neq 0. \end{cases}
\end{aligned} \tag{3.14}$$

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