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Research Article

Monotonicity, Convexity, and Inequalities Involving the Agard Distortion Function

Yu-Ming Chu,[1] Miao-Kun Wang,[1] Yue-Ping Jiang,[2] and Song-Liang Qiu[3]

[1] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*
[2] *College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*
[3] *Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

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We present some monotonicity, convexity, and inequalities for the Agard distortion function $\eta_K(t)$ and improve some well-known results.

1. Introduction

For $r \in [0,1]$, Lengedre's complete elliptic integrals of the first and second kind [1] are defined by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{-1/2} d\theta,$$
$$\mathcal{K}'(r) = \mathcal{K}(r'), \qquad \mathcal{K}(0) = \frac{\pi}{2}, \qquad \mathcal{K}(1) = \infty, \tag{1.1}$$

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{1/2} d\theta,$$
$$\mathcal{E}'(r) = \mathcal{E}(r'), \qquad \mathcal{E}(0) = \frac{\pi}{2}, \qquad \mathcal{E}(1) = 1, \tag{1.2}$$

respectively. Here and in what follows, we set $r' = \sqrt{1 - r^2}$.

Let $\mu(r)$ be the modulus of the plan Grötzsch ring $\mathbf{B}^2 \setminus [0,r]$ for $r \in (0,1)$, where $\mathbf{B}^2$ is the unit disk. Then, it follows from [2] that

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}. \tag{1.3}$$

For $K \in (0,\infty)$, the Hersch-Pfluger distortion function $\varphi_K(r)$ is defined as

$$\varphi_K(r) = \mu^{-1}\left(\frac{\mu(r)}{K}\right) \quad \text{for } r \in (0,1), \qquad \varphi_K(0) = \varphi_K(1) - 1 = 0, \tag{1.4}$$

while the Agard distortion function $\eta_K(t)$ and the linear distortion function $\lambda(K)$ are defined by

$$\eta_K(t) = \left[\frac{\varphi_K(r)}{\varphi_{1/K}(r')}\right], \quad \lambda(K) = \eta_K(1), \quad r = \sqrt{\frac{t}{1+t}} \quad t \in (0,\infty), \tag{1.5}$$

respectively.

It is well known that the functions $\eta_K(t)$ and $\lambda(K)$ play a very important role in quasiconformal theory, quasiregular theory, and some other related fields [3–8]. For example, Martin [8] found that the sharp upper bound in Schottky's theorem can be expressed by $\eta_K(t)$, and in [9–15] the authors established a number of remarkable properties for the Agard distortion function $\eta_K(t)$.

In [14], the authors proved that

$$e^{\pi(K-1)} < \lambda(K) < e^{a(K-1)}, \tag{1.6}$$

$$e^{b(K-1/K)} < \lambda(K) < e^{\pi(K-1/K)} \tag{1.7}$$

for all $K \in (1,\infty)$, where $a = (4/\pi)\mathcal{K}(1/\sqrt{2})^2 = 4.3768\ldots$, $b = a/2$. Recently, Anderson et al. [15] established that

$$\lambda(K) < e^{(\pi+b/K)(K-1)}, \tag{1.8}$$

$$e^{[\log 2+(a-\log 2)/K](K-1)} < \lambda(K) < e^{[\pi+(a-\log 2)/K](K-1)} \tag{1.9}$$

for all $K \in (1,\infty)$, where a and b are defined as in inequalities (1.6) and (1.7), respectively.

The purpose of this paper is to present the new monotonicity, convexity, and inequalities for the Agard distortion function $\eta_K(t)$ and improve inequalities (1.6)–(1.9).

Our main results are Theorems 1.1 and 1.2 as follows.

Theorem 1.1. *Let $K \in (1,\infty)$, $a = (4/\pi)\mathcal{K}(1/\sqrt{2})^2 = 4.3768\ldots$, $b = a/2$, and $c \in \mathbb{R}$. Then, the following statements are true.*

(1) *$f(K) = \lambda(K)/K^c$ is strictly increasing from $(1,\infty)$ onto $(1,\infty)$ for $c \le a$; if $c > a$, then there exists $K_0 \in (1,\infty)$, such that f is strictly decreasing in $(1,K_0)$ and strictly increasing*

in (K_0, ∞)*. In particular, the inequality* $\lambda(K) \geq K^c$ *holds for all* $K \in (1, \infty)$ *with the best possible constant* $c = a$*.*

(2) $g(K) = [\log \eta_K(t) - \log t]/(K-1)$ *is convex in* $(1, \infty)$ *for fixed* $t \in (0, \infty)$*.*

(3) *If* $t \geq 1$ *and* $r = \sqrt{t/(1+t)}$*, then* $h(K) = [\log \eta_K(t) - \log t]/(K - 1/K)$ *is strictly increasing from* $(1, \infty)$ *onto* $(2\mathcal{K}(r)\mathcal{K}'(r)/\pi, \pi\mathcal{K}(r)/\mathcal{K}'(r))$*.*

Theorem 1.2. *Let* $t \in (0, \infty)$*,* $r = \sqrt{t/(1+t)}$*,* $a = (4/\pi)\mathcal{K}(1/\sqrt{2})^2$*,* $b = a/2$*,* $A(r) = \pi^2/(2\mu(r))$*,* $B(r) = 8\mathcal{K}(r)\mathcal{K}'(r)^2[\mathcal{E}(r) - r'^2\mathcal{K}(r)]/\pi^2$*, and* $F_c(K) = K[(\log \eta_K(t) - \log t)/(K-1) - c]$*. Then, the following statements are true.*

(1) $F_c(K)$ *is strictly decreasing from* $(1, \infty)$ *onto* $(-\infty, 4\mathcal{K}(r)\mathcal{K}'(r)/\pi - c)$ *for* $c > A(r)$*. If* $c = A(r)$*, then* $F_c(K)$ *is strictly decreasing from* $(1, \infty)$ *onto* $(A(r) - 4\log 2 - \log t, 4\mathcal{K}(r)\mathcal{K}'(r)/\pi - A(r))$*. Moreover,*

$$te^{(K-1)(A(r)+((A(r)-4\log 2-\log t)/K))} < \eta_K(t) < te^{(K-1)(A(r)+((4\mathcal{K}(r)\mathcal{K}'(r)/\pi-A(r))/K))} \tag{1.10}$$

for all $t \in (0, \infty)$ *and* $K \in (1, \infty)$*. In particular, if* $t = 1$*, then* (1.10) *becomes*

$$e^{(K-1)(\pi+((\pi-4\log 2)/K))} < \lambda(K) < e^{(K-1)(\pi+((a-\pi)/K))}. \tag{1.11}$$

(2) *If* $c \leq B(r)$*, then* $F_c(K)$ *is strictly increasing from* $(1, \infty)$ *onto* $(4\mathcal{K}(r)\mathcal{K}'(r)/\pi - c, \infty)$*. Moreover,*

$$\eta_K(t) > te^{(K-1)(B(r)+((4\mathcal{K}(r)\mathcal{K}'(r)/\pi-B(r))/K))} \tag{1.12}$$

for all $t \in (0, \infty)$ *and* $K \in (1, \infty)$*. In particular, if* $t = 1$*, then* (1.12) *becomes*

$$\lambda(K) > e^{(K-1)(b+(b/K))} = e^{b(K-(1/K))}. \tag{1.13}$$

(3) *If* $B(r) < c < A(r)$*, then there exists* $K_1 \in (1, \infty)$ *such that* $F_c(K)$ *is strictly decreasing on* $(1, K_1)$ *and strictly increasing on* (K_1, ∞)*.*

(4) $F_c(K)$ *is convex in* $(1, \infty)$*.*

2. Lemmas

In order to prove our main results, we need several formulas and lemmas, which we present in this section.

The following formulas were presented in [14, Appendix E, pp. 474-475]. Let $t \in (0,\infty)$, $K \in (0,\infty)$, $r = \sqrt{t/(1+t)} \in (0,1)$, and $s = \varphi_K(r)$. Then,

$$\begin{gathered}
\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2}, \qquad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r}, \\
\mathcal{K}(r)\mathcal{E}'(r) + \mathcal{K}'(r)\mathcal{E}(r) - \mathcal{K}(r)\mathcal{K}'(r) = \frac{\pi}{2}, \\
\frac{d\mu(r)}{dr} = -\frac{\pi^2}{4rr'^2\mathcal{K}(r)^2}, \\
\frac{\partial s}{\partial r} = \frac{ss'^2\mathcal{K}(s)\mathcal{K}'(s)}{rr'^2\mathcal{K}(r)\mathcal{K}'(r)}, \qquad \frac{\partial s}{\partial K} = \frac{2}{\pi K} ss'^2\mathcal{K}(s)\mathcal{K}'(s), \\
\varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1, \\
\eta_K(t) = \left(\frac{s}{s'}\right)^2, \qquad \frac{\partial \eta_K(t)}{\partial K} = \frac{4}{\pi K}\eta_K(t)\mathcal{K}(s)\mathcal{K}'(s) = \frac{2}{\mu(r)}\mathcal{K}'(s)^2\eta_K(t).
\end{gathered} \tag{2.1}$$

Lemma 2.1 (see [14, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \qquad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{2.2}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The following lemma can be found in [14, Theorem 3.21(1) and (7), Lemma 3.32(1) and Theorem 5.13(2)].

Lemma 2.2. (1) $[\mathcal{E}(r) - r'^2\mathcal{K}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4, 1)$;
(2) $r'^c\mathcal{K}(r)$ is strictly decreasing from $(0,1)$ onto $(0,\pi/2)$ if and only if $c \geq 1/2$;
(3) $\mathcal{K}(r)\mathcal{K}'(r)$ is strictly decreasing in $(0,\sqrt{2}/2)$ and strictly increasing in $(\sqrt{2}/2, 1)$;
(4) $\mu(r) + \log r$ is strictly decreasing from $(0,1)$ onto $(0, \log 4)$.

Lemma 2.3. *Let $r \in [1/\sqrt{2}, 1)$, $K \in (1,\infty)$, and $s = \varphi_K(r)$. Then, $G(K) \equiv \{\pi/[2\mathcal{K}(s)]\}^2 + [\mu(r)/\mathcal{K}'(s)]^2$ is strictly decreasing from $(1,\infty)$ onto $(\mathcal{K}'(r)^2/\mathcal{K}(r)^2, \pi^2/[2\mathcal{K}(r)^2])$.*

Proof. Clearly $G(1^+) = \pi^2/(2\mathcal{K}(r)^2)$, $G(+\infty) = \mathcal{K}'(r)^2/\mathcal{K}(r)^2$. Differentiating $G(K)$, one has

$$\begin{aligned}
G'(K) &= \frac{4}{\pi K}\mu(r)^2\mathcal{K}(s)\mathcal{K}'(s)^{-2}\left[\mathcal{E}'(s) - s^2\mathcal{K}'(s)\right] \\
&\quad - \frac{\pi}{K}\mathcal{K}(s)^{-2}\mathcal{K}'(s)\left[\mathcal{E}(s) - s'^2\mathcal{K}(s)\right] \\
&= \frac{4}{\pi K}\mathcal{K}(s)^{-2}\mathcal{K}'(s)^{-2}G_1(K),
\end{aligned} \tag{2.3}$$

where $G_1(K) = [\mathcal{E}'(s) - s^2\mathcal{K}'(s)]\mathcal{K}(s)^3\mu(r)^2 - \pi^2[\mathcal{E}(s) - s'^2\mathcal{K}(s)]\mathcal{K}'(s)^3/4$.

From Lemma 2.2(1) and (2), we clearly see that $G_1(K)$ is strictly decreasing in $(1,\infty)$. Moreover,

$$\lim_{K\to 1^+} G_1(K) = \left[\mathcal{E}'(r) - r^2\mathcal{K}'(r)\right]\mathcal{K}(r)^3\mu(r)^2 - \frac{\pi^2}{4}\left[\mathcal{E}(r) - r'^2\mathcal{K}(r)\right]\mathcal{K}'(r)^3 = \frac{\pi^2}{4}\mathcal{K}'(r)^2 G_2(r), \tag{2.4}$$

where $G_2(r) = \mathcal{K}(r)[\mathcal{E}'(r)-r^2\mathcal{K}'(r)]-\mathcal{K}'(r)[\mathcal{E}(r)-r'^2\mathcal{K}(r)]$ is also strictly decreasing in $(0,1)$. Thus, $G_2(r) \le G_2(\sqrt{2}/2) = 0$ for $r \in [1/\sqrt{2}, 1)$, and $G_1(K) < G_1(1^+) \le 0$ for $K \in (1,\infty)$.

Therefore, the monotonicity of $G(K)$ follows from (2.3) and (2.4) together with the fact that $G_1(K) < 0$ for $K \in (1,\infty)$. □

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. For part (1), clearly $f(1^+) = 1$. Let $r = \mu^{-1}[\pi/(2K)]$ for $K \in (1,\infty)$, then $\lambda(K) = (r/r')^2$, $r \in (1/\sqrt{2}, 1)$,

$$\frac{dr}{dK} = \frac{2}{\pi} r r'^2 \mathcal{K}'(r)^2, \qquad \frac{d\lambda(K)}{dK} = \frac{4}{\pi}\lambda(K)\mathcal{K}'(r)^2, \tag{3.1}$$

$$\lim_{K\to+\infty} f(K) = \lim_{r\to 1}\frac{r^2\mathcal{K}'(r)}{r'^2\mathcal{K}(r)} = +\infty. \tag{3.2}$$

Making use of (3.1), we have

$$\frac{K^{c+1} f'(K)}{\lambda(K)} = f_1(K) \equiv \frac{4}{\pi}\mathcal{K}'(r)\mathcal{K}(r) - c. \tag{3.3}$$

It follows from Lemma 2.2(3) that $f_1(K)$ is strictly increasing from $(1,\infty)$ onto $(a - c, \infty)$. Then, from (3.2) and (3.3), we know that f is strictly increasing from $(1,\infty)$ onto $(1,\infty)$ for $c \le a$. If $c > a$, then there exists $K_0 \in (1,\infty)$ such that f is strictly decreasing in $(1, K_0)$ and strictly increasing in (K_0,∞). Moreover, the inequality $\lambda(K) \ge K^c$ holds for all $K \in (1,\infty)$ with the best possible constant $c = a$.

For part (2), denote $r = \sqrt{t/(1+t)}$. Differentiating $g(K)$, we get

$$g'(K) = \frac{2\mathcal{K}'(s)^2(K-1)/\mu(r) - \left(\log\eta_K(t) - \log t\right)}{(K-1)^2}. \tag{3.4}$$

Let $g_1(K) = 2\mathcal{K}'(s)^2(K-1)/\mu(r) - (\log\eta_K(t) - \log t)$ and $g_2(K) = (K-1)^2$, then $g_1(1) = g_2(1) = 0$, $g'(K) = g_1(K)/g_2(K)$ and

$$\frac{g_1{}'(K)}{g_2{}'(K)} = g_3(K) \equiv -\frac{2}{\mu(r)^2}\left[\mathcal{E}'(s) - s^2\mathcal{K}'(s)\right]\mathcal{K}'(s)^3. \tag{3.5}$$

Clearly, $g_3(K)$ is strictly increasing in $(1,\infty)$. Then, (3.5) and Lemma 2.1 lead to the conclusion that $g'(K)$ is strictly increasing in $(1,\infty)$. Therefore, $g(K)$ is convex in $(1,\infty)$.

For part (3), if $t \geq 1$, then $r \geq \sqrt{2}/2$. Let $h_1(K) = \log \eta_K(t) - \log t$ and $h_2(K) = K - 1/K$, then $h_1(1) = h_2(1) = 0$, $h(K) = h_1(K)/h_2(K)$, and

$$\frac{h_1{}'(K)}{h_2{}'(K)} = \frac{2\mathcal{K}'(s)^2/\mu(r)}{1+K^{-2}} = \frac{2\mu(r)}{G(K)}, \tag{3.6}$$

where $G(K)$ is defined as in Lemma 2.2.

Therefore, $h(K)$ is strictly increasing in $(1,\infty)$ for $t \geq 1$ follows from Lemmas 2.1 and 2.2 together with (3.6). Moreover, making use of l'Hôpital's rule, we have $h(1^+) = 2\mathcal{K}(r)\mathcal{K}'(r)/\pi$, $h(\infty) = \pi\mathcal{K}(r)/\mathcal{K}'(r)$. □

Proof of Theorem 1.2. Differentiating $F_c(K)$ gives

$$\begin{aligned} F_c'(K) &= \frac{\log \eta_K(t) - \log t}{K-1} - c + K\left[\frac{\left(2\mathcal{K}'(s)^2(K-1)\right)/\mu(r) \; - (\log \eta_K(t) - \log t)}{(K-1)^2}\right] \\ &= \frac{2\mathcal{K}'(s)^2K(K-1)/\mu(r) - (\log \eta_K(t) - \log t)}{(K-1)^2} - c. \end{aligned} \tag{3.7}$$

Let

$$H(K) = \frac{\left[2\mathcal{K}'(s)^2K(K-1)/\mu(r)\right] - \left[\log \eta_K(t) - \log t\right]}{(K-1)^2}, \tag{3.8}$$

$H_1(K) = 2\mathcal{K}'(s)^2K(K-1)/\mu(r) - (\log \eta_K(t) - \log t)$, and $H_2(K) = (K-1)^2$, then $H(K) = H_1(K)/H_2(K)$, $H_1(1) = H_2(1) = 0$, and

$$\frac{H_1{}'(K)}{H_2{}'(K)} = H_3(K) \equiv \frac{4}{\pi\mu(r)}\left[\mathcal{E}(s) - s'^2\mathcal{K}(s)\right]\mathcal{K}'(s)^3. \tag{3.9}$$

Clearly, that $H_3(K)$ is strictly increasing in $(1,\infty)$ follows from Lemma 2.2(1) and (2). Then, from (3.8) and (3.9) together with Lemma 2.1, we know that $H(K)$ is strictly increasing in $(1,\infty)$. Moreover, l'Hôpital's rule leads to

$$\lim_{K\to 1} H(K) = B(r), \qquad \lim_{K\to\infty} H(K) = A(r). \tag{3.10}$$

For part (1), if $c > A(r)$, then from (3.7) and (3.8), we know that $F_c'(K) < 0$ for $K \in (1,\infty)$ and $F_c(K)$ is strictly decreasing in $(1,\infty)$. Moreover,

$$\lim_{K\to 1} F_c(K) = \left[4\mathcal{K}(r)\mathcal{K}'(r)/\pi\right] - c, \qquad \lim_{K\to\infty} F_c(K) = -\infty. \tag{3.11}$$

If $c = A(r)$, then $F_c(K)$ is also strictly decreasing in $(1, \infty)$ and $F_c(1^+) = [4\mathcal{K}(r)\mathcal{K}'(r)/\pi] - A(r)$, and from Lemma 2.2(4) we get

$$\lim_{K\to\infty} F_c(K) = \lim_{K\to\infty} \frac{K}{K-1}\left[-2\log(s') - 2\mu(s') + A(r) - \log t\right] \\ = A(r) - 4\log 2 - \log t. \tag{3.12}$$

Therefore, inequalities (1.10) and (1.11) follows from (3.12) and the monotonicity of $F_c(K)$ when $c = A(r)$.

For part (2), if $c \le B(r)$, then that $F_c(K)$ is strictly increasing in $(1, \infty)$ follows from (3.7) and (3.8). Note that

$$\lim_{K\to 1} F_c(K) = \left[4\mathcal{K}(r)\mathcal{K}'(r)/\pi\right] - c, \qquad \lim_{K\to\infty} F_c(K) = +\infty. \tag{3.13}$$

Therefore, inequalities (1.12) and (1.13) follow from (3.13) and the monotonicity of $F_c(K)$ when $c = B(r)$.

For part (3), if $B(r) < c < A(r)$, then from (3.7) and (3.8) together with the monotonicity of $H(K)$ we clearly see that there exists $K_1 \in (1, \infty)$, such that $F_c'(K) < 0$ for $K \in (1, K_1)$ and $F_c'(K) > 0$ for $K \in (K_1, \infty)$. Hence, $F_c(K)$ is strictly decreasing in $(1, K_1)$ and strictly increasing in (K_1, ∞).

Part (4) follows from (3.7) and (3.8) together with the monotonicity of $H(K)$. □

Taking $t = 1$ in Theorem 1.2, we get the following corollary.

Corollary 3.1. *Let a and b be defined as in Theorem 1.2, $c \in \mathbb{R}$, and $f_c(K) = K\{[\log \lambda(K)]/(K-1) - c\}$. Then,*

(1) *if $c > \pi$, then $f_c(K)$ is strictly decreasing from $(1, \infty)$ onto $(-\infty, a - c)$; if $c = \pi$, then $f_c(K)$ is strictly decreasing from $(1, \infty)$ onto $(\pi - 4\log 2, a - \pi)$;*

(2) *if $c \le b$, then $f_c(K)$ is strictly increasing from $(1, \infty)$ onto $(a - c, \infty)$;*

(3) *if $b < c < \pi$, then there exists $K_2 \in (1, \infty)$, such that $f_c(K)$ is strictly decreasing in $(1, K_2)$ and strictly increasing in (K_2, ∞);*

(4) *$f_c(K)$ is convex in $(1, \infty)$.*

Inequalities (1.11) and (1.13) lead to the following corollary, which improve inequalities (1.6)–(1.9).

Corollary 3.2. *Let a and b be defined as in Theorem 1.2, then the following inequality*

$$\max\left\{e^{(K-1)(\pi+((\pi-4\log 2)/K))}, e^{b(K-(1/K))}\right\} < \lambda(K) < e^{(K-1)(\pi+((a-\pi)/K))}. \tag{3.14}$$

holds for all $K \in (1, \infty)$.

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