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Research Article

Asymptotic Behavior of Solutions of Delayed Difference Equations

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This contribution is devoted to the investigation of the asymptotic behavior of delayed difference equations with an integer delay. We prove that under appropriate conditions there exists at least one solution with its graph staying in a prescribed domain. This is achieved by the application of a more general theorem which deals with systems of first-order difference equations. In the proof of this theorem we show that a good way is to connect two techniques—the so-called retract-type technique and Liapunov-type approach. In the end, we study a special class of delayed discrete equations and we show that there exists a positive and vanishing solution of such equations.

1. Introduction

Throughout this paper, we use the following notation: for an integer q, we define

$$\mathbb{Z}_q^{\infty} := \{q, q+1, \dots\}. \tag{1.1}$$

We investigate the asymptotic behavior for $n \to \infty$ of the solutions of the discrete delayed equation of the $(k+1)$-th order

$$\Delta v(n) = f(n, v(n), v(n-1), \dots, v(n-k)), \tag{1.2}$$

where n is the independent variable assuming values from the set $\mathbb{Z}_a^{\infty}$ with a fixed $a \in \mathbb{N}$. The number $k \in \mathbb{N}$, $k \geq 1$ is the fixed delay, $\Delta v(n) = v(n+1) - v(n)$, and $f : \mathbb{Z}_a^{\infty} \times \mathbb{R}^{k+1} \to \mathbb{R}$.

A function $v : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ is a solution of (1.2) if it satisfies (1.2) for every $n \in \mathbb{Z}_{a}^{\infty}$.

We will study (1.2) together with $k+1$ initial conditions

$$v(a+s-k) = v^{a+s-k} \in \mathbb{R}, \quad s = 0, 1, \ldots, k. \tag{1.3}$$

Initial problem (1.2), (1.3) obviously has a unique solution, defined for every $n \in \mathbb{Z}_{a-k}^{\infty}$. If the function f is continuous with respect to its last $k+1$ arguments, then the solution of (1.2) continuously depends on initial conditions (1.3).

Now we give a general description of the problem solved in this paper.

Problem 1. Let $b, c : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ be functions such that $b(n) < c(n)$ for every $n \in \mathbb{Z}_{a-k}^{\infty}$. The problem under consideration is to find sufficient conditions for the right-hand side of (1.2) that will guarantee the existence of a solution $v = v^*(n)$ of initial problem (1.2), (1.3) such that

$$b(n) < v^*(n) < c(n), \quad n \in \mathbb{Z}_{a-k}^{\infty}. \tag{1.4}$$

This problem can be solved with help of a result which is valid for systems of first-order difference equations and which will be presented in the next section. This is possible because the considered equation (1.2) can be rewritten as a system of $k+1$ first-order difference equations, similarly as a differential equation of a higher order can be transformed to a special system of first-order differential equations. Although the process of transforming a $(k+1)$-st order difference equation to a system of first order equations is simple and well-known (it is described in Section 3), the determination of the asymptotic properties of the solutions of the resulting system using either Liapunov approach or retract-type method is not trivial. These analogies of classical approaches, known from the qualitative theory of differential equations, were developed for difference systems in [1] (where an approach based on Liapunov method was formulated) and in [2–5] (where retract-type analysis was modified for discrete equations). It occurs that for the mentioned analysis of asymptotic problems of system (1.2), neither the ideas of Liapunov, nor the retract-type technique can be applied directly. However, in spite of the fact that each of the two mentioned methods fails when used independently, it appears that the combination of both these techniques works for this type of systems. Therefore, in Section 2 we prove the relevant result suitable for the asymptotic analysis of systems arising by transformation of (1.2) to a system of first-order differential equations (Theorem 2.1), where the assumptions put to the right-hand side of the system are of both types: those caused by the application of the Liapunov approach and those which are typical for the retract-type technique. Such an idea was applied in a particular case of investigation of asymptotic properties of solutions of the discrete analogue of the Emden-Fowler equation in [6, 7]. The approach is demonstrated in Section 3 where, moreover, its usefulness is illustrated on the problem of detecting the existence of positive solutions of linear equations with a single delay (in Section 3.4) and asymptotic estimation of solutions (in Section 3.3).

Advantages of our approach can be summarized as follows. We give a general method of analysis which is different from the well-known comparison method (see, e.g., [8, 9]). Comparing our approach with the scheme of investigation in [10, 11] which is based on a result from [12], we can see that the presented method is more general because it unifies the investigation of systems of discrete equations and delayed discrete equations thanks to the Liapunov-retract-type technique.

For related results concerning positive solutions and the asymptotics of solutions of discrete equations, the reader is referred also to [13–25].

2. The Result for Systems of First-Order Equations

Consider the system of m difference equations

$$\Delta u(n) = F(n, u(n)), \tag{2.1}$$

where $n \in \mathbb{Z}_a^\infty$, $u = (u_1, \dots, u_m)$, and $F : \mathbb{Z}_a^\infty \times \mathbb{R}^m \to \mathbb{R}^m$, $F = (F_1, \dots, F_m)$. The solution of system (2.1) is defined as a vector function $u : \mathbb{Z}_a^\infty \to \mathbb{R}^m$ such that for every $n \in \mathbb{Z}_a^\infty$, (2.1) is fulfilled. Again, if we prescribe initial conditions

$$u_i(a) = u_i^a \in \mathbb{R}, \quad i = 1, \dots, m \tag{2.2}$$

the initial problem (2.1), (2.2) has a unique solution. Let us define a set $\Omega \subset \mathbb{Z}_a^\infty \times \mathbb{R}^m$ as

$$\Omega := \bigcup_{n=a}^{\infty} \Omega(n), \tag{2.3}$$

where

$$\Omega(n) := \{(n, u) : n \in \mathbb{Z}_a^\infty,\ u_i \in \mathbb{R},\ b_i(n) < u_i < c_i(n),\ i = 1, \dots, m\} \tag{2.4}$$

with $b_i, c_i : \mathbb{Z}_a^\infty \to \mathbb{R}$, $i = 1, \dots, m$, being auxiliary functions such that $b_i(n) < c_i(n)$ for each $n \in \mathbb{Z}_a^\infty$. Such set Ω is called a *polyfacial set*.

Our aim (in this part) is to solve, in correspondence with formulated Problem 1, the following similar problem for systems of difference equations.

Problem 2. Derive sufficient conditions with respect to the right-hand sides of system (2.1) which guarantee the existence of at least one solution $u(n) = (u_1^*(n), \dots, u_m^*(n))$, $n \in \mathbb{Z}_a^\infty$, satisfying

$$\left(n, u_1^*(n), \dots, u_m^*(n)\right) \in \Omega(n) \tag{2.5}$$

for every $n \in \mathbb{Z}_a^\infty$.

As we mentioned above, in [1] the above described problem is solved via a Liapunov-type technique. Here we will combine this technique with the retract-type technique which was used in [2–5] so as the result can be applied easily to the system arising after transformation of (1.2). This brings a significant increase in the range of systems we are able to investigate. Before we start, we recall some basic notions that will be used.

2.1. Consequent Point

Define the mapping $\mathcal{C} : \mathbb{Z}_a^\infty \times \mathbb{R}^m \to \mathbb{Z}_a^\infty \times \mathbb{R}^m$ as

$$\mathcal{C} : (n, u) \longmapsto (n + 1, u + F(n, u)). \tag{2.6}$$

For any point $M = (n, u) \in \mathbb{Z}_a^\infty \times \mathbb{R}^m$, the point $\mathcal{C}(M)$ is called the *first consequent point* of the point M. The geometrical meaning is that if a point M lies on the graph of some solution of system (2.1), then its first consequent point $\mathcal{C}(M)$ is the next point on this graph.

2.2. Liapunov-Type Polyfacial Set

We say that a polyfacial set Ω is *Liapunov-type* with respect to discrete system (2.1) if

$$b_i(n + 1) < u_i + F_i(n, u) < c_i(n + 1) \tag{2.7}$$

for every $i = 1, \dots, m$ and every $(n, u) \in \Omega$. The geometrical meaning of this property is this: if a point $M = (n, u)$ lies inside the set $\Omega(n)$, then its first consequent point $\mathcal{C}(M)$ stays inside $\Omega(n + 1)$.

In this contribution we will deal with sets that need not be of Liapunov-type, but they will have, in a certain sense, a similar property. We say that a polyfacial set Ω is *Liapunov-type with respect to the jth variable* ($j \in \{1, \dots, m\}$) and to discrete system (2.1) if

$$(n, u) \in \Omega \Longrightarrow b_j(n + 1) < u_j + F_j(n, u) < c_j(n + 1). \tag{2.8}$$

The geometrical meaning is that if $M = (n, u) \in \Omega(n)$, then the u_j-coordinate of its first consequent point stays between $b_j(n + 1)$ and $c_j(n + 1)$, meanwhile the other coordinates of $\mathcal{C}(M)$ may be arbitrary.

2.3. Points of Strict Egress and Their Geometrical Sense

An important role in the application of the retract-type technique is played by the so called strict egress points. Before we define these points, let us describe the boundaries of the sets $\Omega(n)$, $n \in \mathbb{Z}_a^\infty$, in detail. As one can easily see,

$$\bigcup_{n \in \mathbb{Z}_a^\infty} \partial\Omega(n) = \left(\bigcup_{j=1}^{m} \Omega_B^j \right) \cup \left(\bigcup_{j=1}^{m} \Omega_C^j \right) \tag{2.9}$$

with

$$\begin{aligned} \Omega_B^j &:= \{(n, u) : n \in \mathbb{Z}_a^\infty,\ u_j = b_j(n),\ b_i(n) \le u_i \le c_i(n),\ i = 1, \dots, m,\ i \ne j\}, \\ \Omega_C^j &:= \{(n, u) : n \in \mathbb{Z}_a^\infty,\ u_j = c_j(n),\ b_i(n) \le u_i \le c_i(n),\ i = 1, \dots, m,\ i \ne j\}. \end{aligned} \tag{2.10}$$

In accordance with [3, Lemmas 1 and 2], a point $(n, u) \in \partial\Omega(n)$ is a *point of the type of strict egress* for the polyfacial set Ω with respect to discrete system (2.1) if and only if for some $j \in \{1, \dots, m\}$

$$u_j = b_j(n), \quad F_j(n, u) < b_j(n+1) - b_j(n), \tag{2.11}$$

or

$$u_j = c_j(n), \quad F_j(n, u) > c_j(n+1) - c_j(n). \tag{2.12}$$

Geometrically these inequalities mean the following: if a point $M = (n, u) \in \partial\Omega(n)$ is a point of the type of strict egress, then the first consequent point $\mathcal{C}(M) \notin \overline{\Omega(n+1)}$.

2.4. Retract and Retraction

If $A \subset B$ are any two sets in a topological space and $\pi : B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. If there exists a retraction of B onto A, then A is called a *retract* of B.

2.5. The Existence Theorem for the System of First-Order Equations (Solution of Problem 2)

The following result, solving Problem 2, gives sufficient conditions with respect to the right-hand sides of (2.1) which guarantee the existence of at least one solution satisfying (2.5) for every $n \in \mathbb{Z}_a^\infty$.

Theorem 2.1. *Let $b_i(n), c_i(n), b_i(n) < c_i(n)$, $i = 1, \dots, m$, be real functions defined on $\mathbb{Z}_a^\infty$ and let $F_i : \mathbb{Z}_a^\infty \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$, be continuous functions. Suppose that for one fixed $j \in \{1, \dots, m\}$ all the points of the sets Ω_B^j, Ω_C^j are points of strict egress, that is, if $(n, u) \in \Omega_B^j$, then*

$$F_j(n, u) < b_j(n+1) - b_j(n), \tag{2.13}$$

and if $(n, u) \in \Omega_C^j$, then

$$F_j(n, u) > c_j(n+1) - c_j(n). \tag{2.14}$$

Further suppose that the set Ω is of Liapunov-type with respect to the ith variable for every $i \in \{1, \dots, m\}$, $i \neq j$, that is, that for every $(n, u) \in \Omega$

$$b_i(n+1) < u_i + F_i(n, u) < c_i(n+1). \tag{2.15}$$

Then there exists a solution $u = (u_1^*(n), \dots, u_m^*(n))$ of system (2.1) satisfying the inequalities

$$b_i(n) < u_i^*(n) < c_i(n), \quad i = 1, \dots, m, \tag{2.16}$$

for every $n \in \mathbb{Z}_a^\infty$.

Proof. The proof will be by contradiction. We will suppose that there exists no solution satisfying inequalities (2.16) for every $n \in \mathbb{Z}_a^\infty$. Under this supposition we prove that there exists a continuous mapping (a retraction) of a closed interval onto both its endpoints which is, by the intermediate value theorem of calculus, impossible.

Without the loss of generality we may suppose that the index j in Theorem 2.1 is equal to 1, that is, all the points of the sets Ω_B^1 and Ω_C^1 are strict egress points. Each solution of system (2.1) is uniquely determined by the chosen initial condition

$$u(a) = (u_1(a), \dots, u_m(a)) = \left(u_1^a, \dots, u_m^a\right) = u^a. \tag{2.17}$$

For the following considerations, let u_i^a with $u_i^a \in (b_i(a), c_i(a))$, $i = 2, \dots, m$, be chosen arbitrarily but fixed. Now the solution of (2.1) is given just by the choice of u_1^a, we can write

$$u = u\left(n, u_1^a\right) = \left(u_1\left(n, u_1^a\right), \dots, u_m\left(n, u_1^a\right)\right). \tag{2.18}$$

Define the closed interval $I := [b_1(a), c_1(a)]$. Hereafter we show that, under the supposition that there exists no solution satisfying inequalities (2.16), there exists a retraction $\mathcal{R}$ (which will be a composition of two auxiliary mappings $\mathcal{R}_1$ and $\mathcal{R}_2$ defined below) of the set $B := I$ onto the set $A := \partial I = \{b_1(a), c_1(a)\}$. This contradiction will prove our result. To arrive at such a contradiction, we divide the remaining part of the proof into several steps.

Construction of the Leaving Value n^*

Let a point $\tilde{u}_1 \in I$ be fixed. The initial condition $u_1(a) = \tilde{u}_1$ defines a solution $u = u(n, \tilde{u}_1) = (u_1(n, \tilde{u}_1), \dots, u_m(n, \tilde{u}_1))$. According to our supposition, this solution does not satisfy inequalities (2.16) for every $n \in \mathbb{Z}_a^\infty$. We will study the moment the solution leaves the domain Ω for the first time. The first value of n for which inequalities (2.16) are not valid will be denoted as s.

(I) First consider the case $\tilde{u}_1 \in \operatorname{int} I$. Then there exists a value $s > 1$ in $\mathbb{Z}_{a+1}^\infty$ such that

$$(s, u(s, \tilde{u}_1)) \notin \Omega(s) \tag{2.19}$$

while

$$(r, u(r, \tilde{u}_1)) \in \Omega(r) \quad \text{for } a \le r \le s-1. \tag{2.20}$$

As the set Ω is of the Liapunov-type with respect to all variables except the first one and $(s-1, u(s-1, \tilde{u}_1)) \in \Omega(s-1)$, then

$$b_i(s) < u_i(s, \tilde{u}_1) < c_i(s), \quad i = 2, \dots, m. \tag{2.21}$$

Because $j = 1$ was assumed, and Ω is of Liapunov-type for each variable u_i, $i \neq j$, then the validity of inequalities (2.16) has to be violated in the u_1-coordinate. The geometrical meaning was explained in Section 2.2.

Now, two cases are possible: either $(s, u(s, \tilde{u}_1)) \notin \overline{\Omega(s)}$ or $(s, u(s, \tilde{u}_1)) \in \partial\Omega(s)$. In the first case $u_1(s, \tilde{u}_1) < b_1(s)$ or $u_1(s, \tilde{u}_1) > c_1(s)$. In the second case $u_1(s, \tilde{u}_1) = b_1(s)$ or

$u_1(s, \tilde{u}_1) = c_1(s)$ and, due to (2.13) and (2.14), $u_1(s+1, \tilde{u}_1) < b_1(s+1)$ or $u_1(s+1, \tilde{u}_1) > c_1(s+1)$, respectively.

(II) If $\tilde{u}_1 \in \partial I$, then $(a, u(a, \tilde{u}_1)) \notin \Omega(a)$. Thus, for this case, we could put $s = a$. Further, because of the strict egress property of Ω_B^1 and Ω_C^1, either $u_1(a+1, \tilde{u}_1) < b_1(a+1)$ (if $\tilde{u}_1 = b_1(a)$) or $u_1(a+1, \tilde{u}_1) > c_1(a+1)$ (if $\tilde{u}_1 = c_1(a)$) and thus $(a+1, u(a+1, \tilde{u}_1)) \notin \overline{\Omega(a+1)}$.

Unfortunately, for the next consideration the value s (the first value of the independent variable for which the graph of the solution is out of Ω) would be of little use. What we will need is the last value for which the graph of the solution stays in $\overline{\Omega}$. We will denote this value as n^* and will call it the *leaving value*. We can define n^* as

$$n^* = s - 1 \quad \text{if } (s, u(s, \tilde{u}_1)) \notin \overline{\Omega(s)}, \qquad n^* = s \quad \text{if } (s, u(s, \tilde{u}_1)) \in \partial\Omega(s). \tag{2.22}$$

As the value of n^* depends on the chosen initial point $\tilde{u}_1$, we could write $n^* = n^*(\tilde{u}_1)$ but we will mostly omit the argument $\tilde{u}_1$, unless it is necessary. From the above considerations it follows that

$$\begin{gathered} b_1(n^*) \le u_1(n^*, \tilde{u}_1) \le c_1(n^*), \\ u_1(n^*+1, \tilde{u}_1) < b_1(n^*+1) \quad \text{or} \quad u_1(n^*+1, \tilde{u}_1) > c_1(n^*+1). \end{gathered} \tag{2.23}$$

Auxiliary Mapping $\mathcal{R}_1$

Now we construct the auxiliary mapping $\mathcal{R}_1 : I \to \mathbb{R} \times \mathbb{R}$. First extend the discrete functions b_1, c_1 onto the whole interval $[a, \infty)$:

$$\begin{aligned} b_1(t) &:= b_1(\lfloor t \rfloor) + (b_1(\lfloor t \rfloor + 1) - b_1(\lfloor t \rfloor))(t - \lfloor t \rfloor), \\ c_1(t) &:= c_1(\lfloor t \rfloor) + (c_1(\lfloor t \rfloor + 1) - c_1(\lfloor t \rfloor))(t - \lfloor t \rfloor), \end{aligned} \tag{2.24}$$

$\lfloor t \rfloor$ being the integer part of t (the floor function). Note that b_1, c_1 are now piecewise linear continuous functions of a real variable t such that $b_1(t) < c_1(t)$ for every t and that the original values of $b_1(n), c_1(n)$ for $n \in \mathbb{Z}_a^\infty$ are preserved. This means that the graphs of these functions connect the points $(n, b_1(n))$ or $(n, c_1(n))$ for $n \in \mathbb{Z}_a^\infty$, respectively. Denote V the set

$$V := \{(t, u_1) : t \in [a, \infty),\ b_1(t) \le u_1 \le c_1(t)\}. \tag{2.25}$$

The boundary of V consists of three mutually disjoint parts V_a, V_b, and V_c:

$$\partial V = V_a \cup V_b \cup V_c, \tag{2.26}$$

where

$$\begin{aligned} V_a &:= \{(a, u_1) : b_1(a) < u_1 < c_1(a)\}, \\ V_b &:= \{(t, u_1) : t \in [a, \infty), u_1 = b_1(t)\}, \\ V_c &:= \{(t, u_1) : t \in [a, \infty), u_1 = c_1(t)\}. \end{aligned} \tag{2.27}$$

Define the mapping $\mathcal{R}_1 : I \to V_b \cup V_c$ as follows: let $\mathcal{R}_1(\tilde{u}_1)$ be the point of intersection of the line segment defined by its end points $(n^*, u_1(n^*, \tilde{u}_1))$, $(n^* + 1, u_1(n^* + 1, \tilde{u}_1))$ with $V_b \cup V_c$ (see Figure 1). The mapping $\mathcal{R}_1$ is obviously well defined on I and $\mathcal{R}_1(b_1(a)) = (a, b_1(a))$, $\mathcal{R}_1(c_1(a)) = (a, c_1(a))$.

Prove that the mapping $\mathcal{R}_1$ is continuous. The point $\mathcal{R}_1(\tilde{u}_1) = (t(\tilde{u}_1), u_1(\tilde{u}_1))$ lies either on V_b or on V_c. Without the loss of generality, consider the second case (the first one is analogical). The relevant boundary line segment for $t \in [n^*, n^* + 1]$, which is a part of V_c, is described by (see (2.24))

$$u_1 = c(n^*) + (c(n^* + 1) - c(n^*))(t - n^*), \tag{2.28}$$

and the line segment joining the points $(n^*, u_1(n^*, \tilde{u}_1))$, $(n^* + 1, u_1(n^* + 1, \tilde{u}_1))$ by the equation

$$u_1 = u_1(n^*, \tilde{u}_1) + (u_1(n^* + 1, \tilde{u}_1) - u_1(n^*, \tilde{u}_1))(t - n^*), \quad t \in [n^*, n^* + 1]. \tag{2.29}$$

The coordinates of the point $\mathcal{R}_1(\tilde{u}_1) = (t(\tilde{u}_1), u_1(\tilde{u}_1))$, which is the intersection of both these line segments, can be obtained as the solution of the system consisting of (2.28) and (2.29). Solving this system with respect to t and u_1, we get

$$t(\tilde{u}_1) = n^* + \frac{u_1(n^*, \tilde{u}_1) - c_1(n^*)}{c_1(n^* + 1) - u_1(n^* + 1, \tilde{u}_1) + u_1(n^*, \tilde{u}_1) - c_1(n^*)}, \tag{2.30}$$

$$u_1(\tilde{u}_1) = c_1(n^*) + \frac{(u_1(n^*, \tilde{u}_1) - c_1(n^*))(c_1(n^* + 1) - c_1(n^*))}{c_1(n^* + 1) - u_1(n^* + 1, \tilde{u}_1) + u_1(n^*, \tilde{u}_1) - c_1(n^*)}. \tag{2.31}$$

Let $\{v_k\}_{k=1}^{\infty}$ be any sequence with $v_k \in I$ such that $v_k \to \tilde{u}_1$. We will show that $\mathcal{R}_1(v_k) \to \mathcal{R}_1(\tilde{u}_1)$. Because of the continuity of the functions F_i, $i = 1, \dots, m$,

$$u_1(n, v_k) \to u_1(n, \tilde{u}_1) \quad \text{for every fixed } n \in \mathbb{Z}_a^{\infty}. \tag{2.32}$$

We have to consider two cases:

(I) $(n^*, u(n^*, \tilde{u}_1)) \in \Omega(n^*)$, that is, $b_1(n^*) < u_1(n^*, \tilde{u}_1) < c_1(n^*)$,

(II) $(n^*, u(n^*, \tilde{u}_1)) \in \partial\Omega(n^*)$, that is, $u_1(n^*, \tilde{u}_1) = c_1(n^*)$.

Recall that (due to our agreement) in both cases $u_1(n^* + 1, \tilde{u}_1) > c_1(n^* + 1)$.

(I) In this case also $u_1(n^*, v_k) < c_1(n^*)$ and $u_1(n^* + 1, v_k) > c_1(n^* + 1)$ for k sufficiently large. That means that the leaving value $n^*(v_k)$ is the same as n^* given by $\tilde{u}_1$ and thus the point $\mathcal{R}_1(v_k) = (t(v_k), u_1(v_k))$ is given by

$$t(v_k) = n^* + \frac{u_1(n^*, v_k) - c_1(n^*)}{c_1(n^* + 1) - u_1(n^* + 1, v_k) + u_1(n^*, v_k) - c_1(n^*)}, \tag{2.33}$$

$$u_1(v_k) = c_1(n^*) + \frac{(u_1(n^*, v_k) - c_1(n^*))(c_1(n^* + 1) - c_1(n^*))}{c_1(n^* + 1) - u_1(n^* + 1, v_k) + u_1(n^*, v_k) - c_1(n^*)}. \tag{2.34}$$

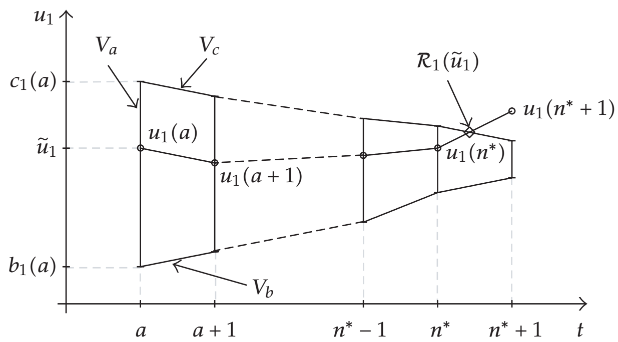

Figure 1: Construction of the mapping $\mathcal{R}_1$.

The desired convergence $\mathcal{R}_1(v_k) \to \mathcal{R}_1(\tilde{u}_1)$ is implied by equations (2.30) to (2.34).

(II) Suppose $n^* = a$. Then $\tilde{u}_1 = c_1(a)$, $v_k = u_1(a, v_k) < c_1(a)$ for all k and as $k \to \infty$, $u_1(a+1, v_k) > c_1(a+1)$. A minor edit of the text in the case (I) proof provides the continuity proof.

Suppose $n^* > a$. In this case there can be $u_1(n^*, v_k) \le c_1(n^*)$ for some members of the sequence $\{v_k\}$ and $u_1(n^*, v_k) > c_1(n^*)$ for the others. Without the loss of generality, we can suppose that $\{v_k\}$ splits into two infinite subsequences $\{v_{q_k}\}$ and $\{v_{r_k}\}$ such that

$$u_1(n^*, v_{q_k}) \le c_1(n^*), \qquad u_1(n^*+1, v_{q_k}) > c_1(n^*+1) \\ u_1(n^*, v_{r_k}) > c_1(n^*). \tag{2.35}$$

For the subsequence $\{v_{q_k}\}$, the text of the proof of (I) can be subjected to a minor edit to provide the proof of continuity. As for the subsequence $\{v_{r_k}\}$, the leaving value $n^*(v_{r_k})$ is different from n^* given by $\tilde{u}_1$ because $(n^*, u_1(n^*, v_{r_k}))$ is already out of $\overline{\Omega}$. For k sufficiently large,

$$n^*(v_{r_k}) = n^* - 1 \tag{2.36}$$

because $u_1(n^*-1, \tilde{u}_1) < c_1(n^*-1)$ and thus, as $k \to \infty$, $u_1(n^*-1, v_{r_k}) < c_1(n^*-1)$. Hence, the value of the mapping $\mathcal{R}_1$ for v_{r_k} is (in (2.33), (2.34) we replace n^* by n^*-1)

$$t(v_{r_k}) = n^* - 1 + \frac{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)}{c_1(n^*) - u_1(n^*, v_{r_k}) + u_1(n^*-1, v_{r_k}) - c_1(n^*-1)}, \\ u_1(v_{r_k}) = c(n^*-1) + \frac{(u_1(n^*-1, v_{r_k}) - c_1(n^*-1))(c_1(n^*) - c_1(n^*-1))}{c_1(n^*) - u_1(n^*, v_{r_k}) + u_1(n^*-1, v_{r_k}) - c_1(n^*-1)}. \tag{2.37}$$

Due to (2.32), $u_1(n^*, v_{r_k}) \to u_1(n^*, \tilde{u}_1) = c_1(n^*)$ and thus

$$t(v_{r_k}) \longrightarrow n^* - 1 + \frac{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)}{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)} = n^*,$$
$$u_1(v_{r_k}) \longrightarrow c(n^*-1) + \frac{(u_1(n^*-1, v_{r_k}) - c_1(n^*-1))(c_1(n^*) - c_1(n^*-1))}{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)} = c_1(n^*), \tag{2.38}$$
$$\mathcal{R}_1(v_{r_k}) = (t(v_{r_k}), u_1(v_{r_k})) \longrightarrow (n^*, c_1(n^*)) = \mathcal{R}_1(\tilde{u}_1).$$

We have shown that $\mathcal{R}_1(v_{q_k}) \to \mathcal{R}_1(\tilde{u}_1)$ and $\mathcal{R}_1(v_{r_k}) \to \mathcal{R}_1(\tilde{u}_1)$ and thus $\mathcal{R}_1(v_k) \to \mathcal{R}_1(\tilde{u}_1)$.

Auxiliary Mapping $\mathcal{R}_2$

Define $\mathcal{R}_2 : V_b \cup V_c \to \{b_1(a), c_1(a)\}$ as

$$\mathcal{R}_2(P) = \begin{cases} b_1(a) & \text{if } P \in V_b, \\ c_1(a) & \text{if } P \in V_c. \end{cases} \tag{2.39}$$

The mapping $\mathcal{R}_2$ is obviously continuous.

Resulting Mapping $\mathcal{R}$ *and Its Properties*

Define $\mathcal{R} := \mathcal{R}_2 \circ \mathcal{R}_1$. Due to construction we have

$$\mathcal{R}(b_1(a)) = b_1(a), \ \mathcal{R}(c_1(a)) = c_1(a), \tag{2.40}$$

and $\mathcal{R}(I) = \partial I$. The mapping $\mathcal{R}$ is continuous because of the continuity of the two mappings $\mathcal{R}_1$ and $\mathcal{R}_2$. Hence, it is the sought retraction of I onto ∂I. But such a retraction cannot exist and thus we get a contradiction and the proof is complete. □

3. Application of Theorem 2.1 to the Delayed Discrete Equation

Now, let us return to the original delayed discrete equation (1.2), that is,

$$\Delta v(n) = f(n, v(n), v(n-1), \dots, v(n-k)). \tag{3.1}$$

As it was said in Section 1, this equation will be transformed to a system of $k+1$ first-order discrete equations. Then we will apply Theorem 2.1 to this system and prove that under certain conditions there exists a solution of delayed equation (1.2) that stays in the prescribed domain. In the end, we will study a special case of (1.2).

3.1. Transformation of (1.2) to the System of First-Order Equations

We will proceed in accordance with the well-known scheme similarly as when constructing the system of first-order differential equations from a differential equation of a higher order. Put

$$\begin{aligned} u_1(n) &:= v(n), \\ u_2(n) &:= v(n-1), \\ &\cdots \\ u_{k+1}(n) &:= v(n-k), \end{aligned} \tag{3.2}$$

where $u_1, u_2, \ldots, u_{k+1}$ are new unknown functions. From (1.2) we get $\Delta u_1(n) = f(n, u_1(n), u_2(n), \ldots, u_{k+1}(n))$. Obviously $u_2(n+1) = u_1(n), \ldots, u_{k+1}(n+1) = u_k(n)$. Rewriting these equalities in terms of differences, we have $\Delta u_2(n) = u_1(n) - u_2(n), \ldots, \Delta u_{k+1}(n) = u_k(n) - u_{k+1}(n)$. Altogether, we get the system

$$\begin{aligned} \Delta u_1(n) &= f(n, u_1(n), \ldots, u_{k+1}(n)), \\ \Delta u_2(n) &= u_1(n) - u_2(n), \\ &\cdots \\ \Delta u_{k+1}(n) &= u_k(n) - u_{k+1}(n) \end{aligned} \tag{3.3}$$

which is equivalent to (1.2).

3.2. The Existence Theorem for the Delayed Equation (1.2) (Solution of Problem 1)

The following theorem is a consequence of Theorem 2.1. In fact, this theorem has been already proved in [12]. There, the proof is based upon a modification of the retract method for delayed equations. Our method (rearranging a delayed equation to a system of first-order equations) is, by its principle, more general than that used in [12].

Theorem 3.1. *Let $b(n)$, $c(n)$, $b(n) < c(n)$, be real functions defined on $\mathbb{Z}_{a-k}^{\infty}$. Further, let $f : \mathbb{Z}_a^{\infty} \times \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function and let the inequalities*

$$b(n) + f(n, b(n), v_2, \ldots, v_{k+1}) < b(n+1), \tag{3.4}$$

$$c(n) + f(n, c(n), v_2, \ldots, v_{k+1}) > c(n+1) \tag{3.5}$$

hold for every $n \in \mathbb{Z}_a^{\infty}$ and every $v_2, \ldots, v_{k+1}$ such that

$$b(n-i+1) < v_i < c(n-i+1), \quad i = 2, \ldots, k+1. \tag{3.6}$$

Then there exists a solution $v = v^*(n)$ of (1.2) *satisfying the inequalities*

$$b(n) < v^*(n) < c(n) \tag{3.7}$$

for every $n \in \mathbb{Z}_{a-k}^{\infty}$.

Proof. We have shown that (1.2) is equivalent to system (3.3) which can be seen as a special case of system (2.1) with $m = k + 1$ and $F = (F_1, \dots F_{k+1})$ where

$$\begin{gathered} F_1(n, u_1, \dots, u_{k+1}) := f(n, u_1, \dots, u_{k+1}), \\ F_2(n, u_1, \dots, u_{k+1}) := u_1 - u_2, \\ \cdots \\ F_k(n, u_1, \dots, u_{k+1}) := u_{k-1} - u_k, \\ F_{k+1}(n, u_1, \dots, u_{k+1}) := u_k - u_{k+1}. \end{gathered} \tag{3.8}$$

Define the polyfacial set Ω as

$$\Omega := \{(n, u) : n \in \mathbb{Z}_a^{\infty},\ b_i(n) < u_i < c_i(n),\ i = 1, \dots, k + 1\} \tag{3.9}$$

with

$$b_i(n) := b(n - i + 1), \qquad c_i(n) := c(n - i + 1), \quad i = 1, \dots, k + 1. \tag{3.10}$$

We will show that for system (3.3) and the set Ω, all the assumptions of Theorem 2.1 are satisfied.

As the function f is supposed to be continuous, the mapping F is continuous, too. Put the index j from Theorem 2.1, characterizing the points of egress, equal to 1. We will verify that the set Ω is of Liapunov-type with respect to the ith variable for any $i = 2, \dots, k + 1$, that is, (see (2.8)) that for every $(n, u) \in \Omega$

$$b_i(n + 1) < u_i + F_i(n, u) < c_i(n + 1) \quad \text{for } i = 2, \dots, k + 1. \tag{3.11}$$

First, we compute

$$u_i + F_i(n, u) = u_i + u_{i-1} - u_i = u_{i-1} \quad \text{for } i = 2, \dots, k + 1. \tag{3.12}$$

Thus we have to show that for $i = 2, \dots, k + 1$

$$b_i(n + 1) < u_{i-1} < c_i(n + 1). \tag{3.13}$$

Because $(n, u) \in \Omega$, then $b_p(n) < u_p < c_p(n)$ for any $p \in \{1, \dots, k + 1\}$, and therefore

$$b_{i-1}(n) < u_{i-1} < c_{i-1}(n) \quad \text{for } i = 2, \dots, k + 1. \tag{3.14}$$

But, by (3.10), we have

$$b_{i-1}(n) = b(n-i+1+1) = b(n-i+2), \tag{3.15}$$

meanwhile

$$b_i(n+1) = b(n+1-i+1) = b(n-i+2), \tag{3.16}$$

and thus $b_{i-1}(n) = b_i(n+1)$. Analogously we get that $c_{i-1}(n) = c_i(n+1)$. Thus inequalities (3.11) are fulfilled.

Further we will show that all the boundary points $M \in \Omega_B^1 \cup \Omega_C^1$ are points of strict egress for the set Ω with respect to system (3.3). According to (2.11), we have to show that if $u_1 = b_1(n)$ and $b_i(n) < u_i < c_i(n)$ for $i = 2, \dots, k+1$, then

$$b_1(n) + F_1(n,u) < b_1(n+1), \tag{3.17}$$

that is,

$$b_1(n) + f(n, b_1(n), u_2, \dots, u_{k+1}) < b_1(n+1). \tag{3.18}$$

Notice that the condition $b_i(n) < u_i < c_i(n)$ for $i = 2, \dots, k+1$ is equivalent with condition $b(n-i+1) < u_i < c(n-i+1)$ (see (3.10)). Looking at the supposed inequality (3.4) and realizing that $b_1(n) = b(n)$ and $b_1(n+1) = b(n+1)$, we can see that inequality (3.18) is fulfilled.

Analogously, according to (2.12), we have to prove that for $u_1 = c_1(n)$ and $b_i(n) < u_i < c_i(n)$ for $i = 2, \dots, k+1$ the inequality

$$c_1(n) + F_1(n,u) > c_1(n+1), \tag{3.19}$$

that is,

$$c_1(n) + f(n, c_1(n), u_2, \dots, u_{k+1}) > c_1(n+1) \tag{3.20}$$

holds.

Again, considering (3.5) and the fact that $c_1(n) = c(n)$ and $c_1(n+1) = c(n+1)$, we can see that this inequality really holds.

Thus, by the assertion of Theorem 2.1, there exists a solution $u = u^*(n)$ of system (3.3) such that for every $n \in \mathbb{Z}_a^\infty$

$$b_i(n) < u_i^*(n) < c_i(n) \quad \text{for } i = 1, \dots, k+1. \tag{3.21}$$

In our case, $v = v^*(n) = u_1^*(n)$ is the solution of the original equation (1.2). Further, $b_1(n) = b(n)$ and $c_1(n) = c(n)$, and thus the existence of a solution of the delayed equation (1.2) such that inequalities (3.7) are satisfied is guaranteed. □

3.3. Asymptotic Solution Estimates for Delayed Difference Equations

Let us suppose that two functions $u, w : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ are given such that

$$u(n) < w(n), \quad n \in \mathbb{Z}_{a-k}^{\infty}, \tag{3.22}$$

$$\Delta u(n) \geq f(n, u(n), u(n-1), \ldots, u(n-k)), \quad n \in \mathbb{Z}_{a}^{\infty}, \tag{3.23}$$

$$\Delta w(n) \leq f(n, w(n), w(n-1), \ldots, w(n-k)), \quad n \in \mathbb{Z}_{a}^{\infty}. \tag{3.24}$$

Consider the problem of whether there exists a solution $v = v^*(n)$, $n \in \mathbb{Z}_{a-k}^{\infty}$ of (1.2) such that

$$u(n) < v^*(n) < w(n), \quad n \in \mathbb{Z}_{a-k}^{\infty}. \tag{3.25}$$

The following corollary of Theorem 3.1 presents sufficient conditions for the existence of a solution of this problem.

Corollary 3.2. *Let functions $u, w : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ satisfy inequalities* (3.22)–(3.24). *Let $f : \mathbb{Z}_{a}^{\infty} \times \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function such that*

$$f(n, u(n), y_2, \ldots, y_{k+1}) > f(n, u(n), z_2, \ldots, z_{k+1}), \tag{3.26}$$

$$f(n, w(n), y_2, \ldots, y_{k+1}) > f(n, w(n), z_2, \ldots, z_{k+1}) \tag{3.27}$$

for every $n \in \mathbb{Z}_{a}^{\infty}$ and every $y_2, \ldots, y_{k+1}, z_2, \ldots, z_{k+1} \in \mathbb{R}$ such that

$$y_i < z_i, \quad i = 2, \ldots, k+1. \tag{3.28}$$

Then there exists a solution $v = v^*(n)$ of (1.2) *satisfying inequalities* (3.25) *for every $n \in \mathbb{Z}_{a-k}^{\infty}$.*

Proof. This assertion is an easy consequence of Theorem 3.1.

Put $b(n) := u(n)$, $c(n) := w(n)$. Considering inequalities (3.23) and (3.26), we can see that

$$\Delta u(n) > f(n, u(n), v_2, \ldots, v_{k+1}) \tag{3.29}$$

for every $n \in \mathbb{Z}_{a}^{\infty}$ and every $v_2, \ldots, v_{k+1}$ such that

$$b(n-i+1) < v_i < c(n-i+1), \quad i = 2, \ldots, k+1. \tag{3.30}$$

Similarly,

$$\Delta w(n) < f(n, w(n), v_2, \ldots, v_{k+1}) \tag{3.31}$$

for every $n \in \mathbb{Z}_{a}^{\infty}$ and every $b(n-i+1) < v_i < c(n-i+1)$, $i = 2, \ldots, k+1$.

Obviously, inequalities (3.29) and (3.31) are equivalent with inequalities (3.4) and (3.5), respectively. Thus, all the assumptions of Theorem 3.1 are satisfied and there exists a solution $v = v^*(n)$ of (1.2) satisfying inequalities (3.25) for every $n \in \mathbb{Z}_{a-k}^{\infty}$. □

Example 3.3. Consider the equation

$$\Delta v(n) = v^2(n) - v(n-1) \tag{3.32}$$

for $n \in \mathbb{Z}_3^{\infty}$ which is a second-order delayed discrete equation with delay $k = 1$. We will show that there exists a solution $v = v^*(n)$ of (3.32) that satisfies the inequalities

$$1 < v^*(n) < n \tag{3.33}$$

for $n \in \mathbb{Z}_2^{\infty}$.

We will prove that for the functions

$$u(n) := 1, \quad w(n) := n, \quad f(n, v_1, v_2) := v_1^2 - v_2 \tag{3.34}$$

all the assumptions of Corollary 3.2 are satisfied. Inequality (3.22) is obviously fulfilled for $n \in \mathbb{Z}_2^{\infty}$. Inequality (3.23) can be also proved very easily:

$$\Delta u(n) = 0, \qquad f(n, u(n), u(n-1)) = 1^2 - 1 = 0, \tag{3.35}$$

and thus for every $n \in \mathbb{Z}_3^{\infty}$, $\Delta u(n) \geq f(n, u(n), u(n-1))$.

As for inequality (3.24), we get

$$\Delta w(n) = 1, \qquad f(n, w(n), w(n-1)) = n^2 - n + 1 \tag{3.36}$$

and thus $\Delta w(n) \leq f(n, w(n), w(n-1))$ for $n \in \mathbb{Z}_3^{\infty}$.

Finally, the functions

$$f(n, u(n), v_2) = 1 - v_2, \qquad f(n, w(n), v_2) = n^2 - v_2 \tag{3.37}$$

are decreasing with respect to v_2. Therefore, conditions (3.26) and (3.27) are satisfied, too. Hence, due to Corollary 3.2, there exists a solution of (3.32) satisfying (3.33).

3.4. Positive Solutions of a Linear Equation with a Single Delay

We will apply the result of Theorem 3.1 to the investigation of a simple linear difference equation of the $(k+1)$-st order with only one delay, namely, the equation

$$\Delta v(n) = -p(n)v(n-k), \tag{3.38}$$

where, again, $n \in \mathbb{Z}_a^{\infty}$ is the independent variable and $k \in \mathbb{N}$, $k \geq 1$, is the fixed delay. The function $p : \mathbb{Z}_a^{\infty} \to \mathbb{R}$ is assumed to be positive. Our goal is to give sharp sufficient conditions for the existence of positive solutions. The existence of such solutions is very often substantial for a concrete model considered. For example, in biology, when a model of population dynamics is described by an equation, the positivity of a solution may mean that the studied biological species can survive in the supposed environment.

For its simple form, (3.38) often serves for testing new results and is very frequently investigated. It was analyzed, for example, in papers [10, 11, 26]. A sharp result on existence of positive solutions given in [26] is proved by a comparison method [8, 9]. Here we will use Theorem 3.1 to generalize this result.

For the purposes of this section, define the expression $\ln_q t$, where $q \in \mathbb{N}$, as

$$\begin{aligned} \ln_q t &:= \ln(\ln_{q-1} t) \\ \ln_0 t &:= t. \end{aligned} \tag{3.39}$$

We will write only $\ln t$ instead of $\ln_1 t$. Further, for a fixed integer $\ell \geq 0$ define auxiliary functions

$$\begin{aligned} \mu_\ell(n) &:= \frac{1}{8n^2} + \frac{1}{8(n\ln n)^2} + \cdots + \frac{1}{8(n\ln n \cdots \ln_\ell n)^2}, \\ p_\ell(n) &:= \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + k\mu_\ell(n)\right), \end{aligned} \tag{3.40}$$

$$\nu_\ell(n) := \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln n \ln_2 n \cdots \ln_\ell n}. \tag{3.41}$$

In [26], it was proved that if $p(n)$ in (3.38) is a positive function bounded by $p_\ell(n)$ for some $\ell \geq 0$, then there exists a positive solution of (3.38) bounded by the function $\nu_\ell(n)$ for n sufficiently large. Since $\lim_{n\to\infty} \nu_\ell(n) = 0$, such solution will vanish for $n \to \infty$. Here we show that (3.38) has a positive solution bounded by $\nu_\ell(n)$ even if the coefficient $p(n)$ satisfies a less restrictive inequality (see inequality (3.58) below). The proof of this statement will be based on the following four lemmas. The symbols "o" and "O" stand for the Landau order symbols and are used for $n \to \infty$.

Lemma 3.4. *The formula*

$$\ln(y - z) = \ln y - \sum_{i=1}^{\infty} \frac{z^i}{i y^i} \tag{3.42}$$

holds for any numbers $y, z \in \mathbb{R}$ such that $y > 0$ and $|z| < y$.

Proof. The assertion is a simple consequence of the well-known Maclaurin expansion

$$\ln(1 - x) = -\sum_{i=1}^{\infty} \frac{1}{i} x^i \quad \text{for } -1 \leq x < 1. \tag{3.43}$$

As $\ln(y-z)-\ln y=\ln(1-z/y)$, substituting $x=z/y$ we get

$$\ln(y-z)-\ln y=-\sum_{i=1}^{\infty}\frac{z^i}{iy^i}\quad \text{for } -y\leq z<y \tag{3.44}$$

and adding $\ln y$ to both sides of this equality, we get (3.42). □

Lemma 3.5. *For fixed $r\in\mathbb{R}\setminus\{0\}$ and fixed $q\in\mathbb{N}$, the asymptotic representation*

$$\begin{aligned}\ln_q(n-r)=\ln_q n&-\frac{r}{n\ln n\cdots\ln_{q-1}n}-\frac{r^2}{2n^2\ln n\cdots\ln_{q-1}n}\\&-\frac{r^2}{2(n\ln n)^2\ln_2 n\cdots\ln_{q-1}n}-\cdots-\frac{r^2}{2(n\ln n\cdots\ln_{q-1}n)^2}\\&-\frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_{q-1}n}\end{aligned} \tag{3.45}$$

holds for $n\to\infty$.

Proof. We will prove relation (3.45) by induction with respect to q. For $q=1$, (3.45) reduces to

$$\ln(n-r)=\ln n-\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3(1+o(1))}{3n^3} \tag{3.46}$$

which holds due to Lemma 3.4. Suppose that relation (3.45) holds for some q. We can write $\ln_q(n-r)=y-z$ with $y=\ln_q n$ and

$$\begin{aligned}z=&\frac{r}{n\ln n\cdots\ln_{q-1}n}+\frac{r^2}{2n^2\ln n\cdots\ln_{q-1}n}+\frac{r^2}{2(n\ln n)^2\ln_2 n\cdots\ln_{q-1}n}\\&+\cdots+\frac{r^2}{2(n\ln n\cdots\ln_{q-1}n)^2}+\frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_{q-1}n}.\end{aligned} \tag{3.47}$$

Now we will show that (3.45) holds for $q+1$. Notice that in our case, the condition $|z|<y$ from Lemma 3.4 is fulfilled for n sufficiently large because $z\to 0$ for $n\to\infty$, meanwhile $y\to\infty$ for $n\to\infty$. Thus we are justified to use Lemma 3.4 and doing so, we get

$$\begin{aligned}
\ln_{q+1}(n-r) &= \ln\left(\ln_q(n-r)\right) \\
&= \ln(y-z) = \ln y - \frac{1}{y}\, z - \frac{1}{2y^2}\, z^2 - \cdots \\
&= \ln(\ln_q n) - \frac{1}{\ln_q n}\cdot\left(\frac{r}{n\ln n\cdots\ln_{q-1}n} + \frac{r^2}{2n^2\ln n\cdots\ln_{q-1}n} + \cdots \right.\\
&\qquad\left. + \frac{r^2}{2\left(n\ln n\cdots\ln_{q-1}n\right)^2} + \frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_{q-1}n}\right) \\
&\quad - \frac{1}{2\left(\ln_q n\right)^2}\cdot\left(\frac{r^2}{\left(n\ln n\cdots\ln_{q-1}n\right)^2} + O\left(\frac{1}{n^3\left(\ln n\cdots\ln_{q-1}n\right)^2}\right)\right) \\
&\quad + O\left(\frac{1}{\left(n\ln n\cdots\ln_q n\right)^3}\right) \\
&= \ln_{q+1}n - \frac{r}{n\ln n\cdots\ln_q n} - \frac{r^2}{2n^2\ln n\cdots\ln_q n} - \frac{r^2}{2(n\ln n)^2\ln_2 n\cdots\ln_q n} \\
&\quad - \cdots - \frac{r^2}{2\left(n\ln n\cdots\ln_q n\right)^2} - \frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_q n}.
\end{aligned} \tag{3.48}$$

Thus, formula (3.45) holds for $q+1$, too, which ends the proof. □

Lemma 3.6. *For fixed $r \in \mathbb{R}\setminus\{0\}$ and fixed $q \in \mathbb{N}$, the asymptotic representations*

$$\begin{aligned}
\sqrt{\frac{\ln_q(n-r)}{\ln_q n}} &= 1 - \frac{r}{2n\ln n\cdots\ln_q n} - \frac{r^2}{4n^2\ln n\cdots\ln_q n} - \frac{r^2}{4(n\ln n)^2\ln_2 n\cdots\ln_q n} - \cdots \\
&\quad - \frac{r^2}{4\left(n\ln n\cdots\ln_{q-1}n\right)^2\ln_q n} - \frac{r^2}{8\left(n\ln n\cdots\ln_q n\right)^2} - \frac{r^3(1+o(1))}{6n^3\ln n\cdots\ln_q n},
\end{aligned} \tag{3.49}$$

$$\sqrt{\frac{n-r}{n}} = 1 - \frac{r}{2n} - \frac{r^2}{8n^2} - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right) \tag{3.50}$$

hold for $n \to \infty$.

Proof. Both these relations are simple consequences of the asymptotic formula

$$\sqrt{1-x} = 1 - \frac{1}{2}\, x - \frac{1}{8}\, x^2 - \frac{1}{16}\, x^3 + o\left(x^3\right) \quad \text{for } x \longrightarrow 0 \tag{3.51}$$

and of Lemma 3.5 (for formula (3.49)). In the case of relation (3.49), we put

$$x = \frac{r}{n\ln n\cdots\ln_q n} + \frac{r^2}{2n^2\ln n\cdots\ln_q n} + \cdots + \frac{r^2}{2\left(n\ln n\cdots\ln_{q-1}n\right)^2\ln_q n} + \frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_q n} \tag{3.52}$$

and in the case of relation (3.50), we put $x = r/n$. □

Lemma 3.7. *For fixed $r \in \mathbb{R} \setminus \{0\}$ and fixed $q \in \mathbb{N}$, the asymptotic representation*

$$\begin{aligned}&\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}\cdots\frac{\ln_q(n-r)}{\ln_q n}}\\&\quad=1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_q n}\right)-r^2\mu_q(n)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right)\end{aligned}\tag{3.53}$$

holds for $n \to \infty$.

Proof. We will prove relation (3.53) by induction with respect to q. For $q = 1$, (3.53) reduces to

$$\begin{aligned}\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}}&=1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}\right)-r^2\mu_1(n)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right)\\&=1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}\right)-r^2\left(\frac{1}{8n^2}+\frac{1}{8(n\ln n)^2}\right)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right).\end{aligned}\tag{3.54}$$

On the other hand, using Lemma 3.6, we get

$$\begin{aligned}&\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}}\\&\quad=\left(1-\frac{r}{2n}-\frac{r^2}{8n^2}-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right)\right)\\&\qquad\times\left(1-\frac{r}{2n\ln n}-\frac{r^2}{4n^2\ln n}-\frac{r^2}{8(n\ln n)^2}-\frac{r^3(1+o(1))}{6n^3\ln n}\right)\\&\quad=1-\frac{r}{2n\ln n}-\frac{r^2}{4n^2\ln n}-\frac{r^2}{8(n\ln n)^2}-\frac{r}{2n}+\frac{r^2}{4n^2\ln n}-\frac{r^2}{8n^2}-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right)\\&\quad=1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}\right)-r^2\left(\frac{1}{8n^2}+\frac{1}{8(n\ln n)^2}\right)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right).\end{aligned}\tag{3.55}$$

Thus, for $q = 1$, relation (3.53) holds. Now suppose that (3.53) holds for some q and prove that it holds for $q + 1$. In the following calculations, we use Lemma 3.6 and we skip some tedious expressions handling.

$$\begin{aligned}
&\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}\cdots\frac{\ln_{q+1}(n-r)}{\ln_{q+1}n}}\\
&\quad=\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}\cdots\frac{\ln_{q}(n-r)}{\ln_{q}n}}\cdot\sqrt{\frac{\ln_{q+1}(n-r)}{\ln_{q+1}n}}\\
&\quad=\left(1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_q n}\right)-r^2\mu_q(n)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right)\right)\\
&\qquad\times\left(1-\frac{r}{2n\ln n\cdots\ln_{q+1}n}-\frac{r^2}{4n^2\ln n\cdots\ln_{q+1}n}-\cdots\right.\\
&\qquad\left.-\frac{r^2}{4(n\ln n\cdots\ln_q n)^2\ln_{q+1}n}-\frac{r^2}{8(n\ln n\cdots\ln_{q+1}n)^2}+o\left(\frac{1}{n^3}\right)\right)\\
&\quad=1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{q+1} n}\right)-r^2\mu_{q+1}(n)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right).
\end{aligned}\tag{3.56}$$

We can see that formula (3.53) holds for $q+1$, too, which ends the proof. □

Now we are ready to prove that there exists a bounded positive solution of (3.38). Remind that functions p_ℓ and v_ℓ were defined by (3.40) and (3.41), respectively.

Theorem 3.8. *Let $\omega:\mathbb{Z}_a^\infty\to\mathbb{R}$ satisfy the inequality*

$$|\omega(n)|\le\varepsilon\left(\frac{k}{k+1}\right)^k\cdot\frac{k(2k^2+k-1)}{16n^3(k+1)},\quad n\in\mathbb{Z}_a^\infty,\tag{3.57}$$

for a fixed $\varepsilon\in(0,1)$. Suppose that there exists an integer $\ell\ge 0$ such that the function p satisfies the inequalities

$$0<p(n)\le p_\ell(n)+\omega(n)\tag{3.58}$$

for every $n\in\mathbb{Z}_a^\infty$. Then there exists a solution $v=v^*(n)$, $n\in\mathbb{Z}_{a-k}^\infty$ of (3.38) such that for n sufficiently large the inequalities

$$0<v^*(n)<v_\ell(n)\tag{3.59}$$

hold.

Proof. Show that all the assumptions of Theorem 3.1 are fulfilled. For (3.38), $f(n,v_1,\dots,v_{k+1})=-p(n)v_{k+1}$. This is a continuous function. Put

$$b(n):=0,\qquad c(n):=v_\ell(n).\tag{3.60}$$

We have to prove that for every $v_2, \ldots, v_{k+1}$ such that $b(n-i+1) < v_i < c(n-i+1)$, $i = 2, \ldots, k+1$, the inequalities (3.4) and (3.5) hold for n sufficiently large. Start with (3.4). That gives that for $b(n-k) < v_{k+1} < c(n-k)$, it has to be

$$0 - p(n) \cdot v_{k+1} < 0. \tag{3.61}$$

This certainly holds, because the function p is positive and so is v_{k+1}.

Next, according to (3.5), we have to prove that

$$\nu_\ell(n) - p(n) v_{k+1} > \nu_\ell(n+1) \tag{3.62}$$

which is equivalent to the inequality

$$-p(n) v_{k+1} > \nu_\ell(n+1) - \nu_\ell(n). \tag{3.63}$$

Denote the left-hand side of (3.63) as $L_{(3.63)}$. As $v_{k+1} < c(n-k) = \nu_\ell(n-k)$ and as by (3.40), (3.58), and (3.57)

$$p(n) \leq \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + k\mu_\ell(n)\right) + \varepsilon \left(\frac{k}{k+1}\right)^k \cdot \frac{k(2k^2+k-1)}{16n^3(k+1)}, \tag{3.64}$$

we have

$$\begin{aligned} L_{(3.63)} &> -\left(\frac{k}{k+1}\right)^k \left(\frac{1}{k+1} + k\mu_\ell(n) + \varepsilon \cdot \frac{k(2k^2+k-1)}{16n^3(k+1)}\right) \\ &\quad \times \left(\frac{k}{k+1}\right)^{n-k} \sqrt{(n-k)\ln(n-k)\cdots\ln_\ell(n-k)} \\ &= -\left(\frac{k}{k+1}\right)^n \left(\frac{1}{k+1} + k\mu_\ell(n) + \varepsilon \cdot \frac{k(2k^2+k-1)}{16n^3(k+1)}\right) \cdot \sqrt{(n-k)\ln(n-k)\cdots\ln_\ell(n-k)}. \end{aligned} \tag{3.65}$$

Further, we can easily see that

$$\nu_\ell(n+1) - \nu_\ell(n) = \left(\frac{k}{k+1}\right)^n \sqrt{n \ln n \cdots \ln_\ell n} \left(\frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_\ell(n+1)}{\ln_\ell n}} - 1\right). \tag{3.66}$$

Thus, to prove (3.63), it suffices to show that for n sufficiently large,

$$-\left(\frac{1}{k+1}+k\mu_\ell(n)+\varepsilon\cdot\frac{k(2k^2+k-1)}{16n^3(k+1)}\right)\sqrt{\frac{(n-k)}{n}\frac{\ln(n-k)}{\ln n}\cdots\frac{\ln_\ell(n-k)}{\ln_\ell n}}$$
$$>\frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_\ell(n+1)}{\ln_\ell n}}-1. \tag{3.67}$$

Denote the left-hand side of inequality (3.67) as $L_{(3.67)}$ and the right-hand side as $R_{(3.67)}$. Using Lemma 3.7 with $r=k$ and $q=\ell$, we can write

$$\begin{aligned}
L_{(3.67)} &= -\left(\frac{1}{k+1}+k\mu_\ell(n)+\varepsilon\cdot\frac{k(2k^2+k-1)}{16n^3(k+1)}\right)\\
&\quad\times\left(1-k\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_\ell n}\right)-k^2\mu_\ell(n)-\frac{k^3}{16n^3}+o\left(\frac{1}{n^3}\right)\right)\\
&= -\frac{1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_\ell n}\right)\\
&\quad+\frac{k^2}{k+1}\mu_\ell(n)+\frac{k^3}{16n^3(k+1)}-k\mu_\ell(n)+\frac{k^2}{16n^3}-\varepsilon\cdot\frac{k(2k^2+k-1)}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right)\\
&= -\frac{1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_\ell n}\right)\\
&\quad-\frac{k}{k+1}\mu_\ell(n)+\frac{2k^3(1-\varepsilon)+k^2(1-\varepsilon)+k\varepsilon}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right).
\end{aligned} \tag{3.68}$$

Using Lemma 3.7 with $r=-1$ and $q=\ell$, we get for $R_{(3.67)}$

$$\begin{aligned}
R_{(3.67)} &= \frac{k}{k+1}\left(1+\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_\ell n}-\mu_\ell(n)+\frac{1}{16n^3}+o\left(\frac{1}{n^3}\right)\right)-1\\
&= \frac{-1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_\ell n}\right)\\
&\quad-\frac{k}{k+1}\cdot\mu_\ell(n)+\frac{k}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right).
\end{aligned} \tag{3.69}$$

It is easy to see that the inequality (3.67) reduces to

$$\frac{2k^3(1-\varepsilon)+k^2(1-\varepsilon)+k\varepsilon}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right)>\frac{k}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right). \tag{3.70}$$

This inequality is equivalent to

$$\frac{k\left(2k^2(1-\varepsilon)+k(1-\varepsilon)-(1-\varepsilon)\right)}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right)>0. \tag{3.71}$$

The last inequality holds for n sufficiently large because $k \geq 1$ and $1-\varepsilon \in (0,1)$. We have proved that all the assumptions of Theorem 3.1 are fulfilled and hence there exists a solution of (3.38) satisfying conditions (3.59). □

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