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Research Article

Arens Regularity of Certain Class of Banach Algebras

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We study Arens regularity of the left and right module actions of A on $A^{(n)}$, where $A^{(n)}$ is the nth dual space of a Banach algebra A, and then investigate (quotient) Arens regularity of $U = A \oplus A^{(n)}$ as a module extension of Banach algebras.

1. Introduction and Preliminaries

In 1951, Arens showed that every bounded bilinear map $m : X \times Y \to Z$ on normed spaces has two natural but different extensions m''' and $m^{t'''t}$ from $X'' \times Y''$ to Z'' [1]. The first extension m''' of m is constructed by forming in turn the following bilinear maps:

$$\begin{aligned} m' : Z' \times X \longrightarrow Y', \qquad & \langle m'(z', x), y\rangle = \langle z', m(x, y)\rangle, \\ m'' : Y'' \times Z' \longrightarrow X', \qquad & \langle m''(y'', z'), x\rangle = \langle y'', m'(z', x)\rangle, \\ m''' : X'' \times Y'' \longrightarrow Z'', \qquad & \langle m'''(x'', y''), z'\rangle = \langle x'', m''(y'', z')\rangle. \end{aligned} \tag{1.1}$$

The bilinear map m''' is the unique extension of m which is w^*-separately continuous on $X \times Y''$. The second extension $m^{t'''t}$ of m can be made in the same way if we start by transpose map $m^t : Y \times X \to Z$ instead of m, which is defined by $m^t(y, x) = m(x, y)$. Similarly, it is the unique extension of m that is w^*-separately continuous on $X'' \times Y$. It is easy to check that

$$m'''(x'', y'') = w^* - \lim_i \lim_j m(x_i, y_j), \qquad m^{t'''t}(x'', y'') = w^* - \lim_j \lim_i m(x_i, y_j), \tag{1.2}$$

where (x_i) and (y_j) are nets in X and Y that converge, in w^*-topologies, to x'' and y'', respectively. According to [1], m is said to be Arens regular if $m''' = m^{t'''t}$.

For the product map $\pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ of a Banach algebra $\mathcal{A}$, we denote $\pi'''(\Phi, \Psi)$ and $\pi^{t'''t}(\Phi, \Psi)$ by the symbols $\Phi \square \Psi$ and $\Phi \lozenge \Psi$, respectively. These are called the first and second Arens products on $\mathcal{A}''$. The Banach algebra $\mathcal{A}$ is said to be Arens regular if $\Phi \square \Psi = \Phi \lozenge \Psi$ on the whole of $\mathcal{A}''$. The higher extensions $\pi^{(3n)}$ and $\pi^{t(3n)t}$ of π and Arens products on $\mathcal{A}^{(2n)}$ can be defined similarly. For any fixed $\Phi \in \mathcal{A}''$, the maps $\Psi \mapsto \Psi \square \Phi$ and $\Psi \mapsto \Phi \lozenge \Psi$ are w^*-w^* continuous on $\mathcal{A}''$. Thus with the w^*-topology, $(\mathcal{A}'', \square)$ is a right topological semigroup and $(\mathcal{A}'', \lozenge)$ is a left topological semigroup. The following sets

$$Z_t^1(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \longmapsto \Phi \square \Psi \text{ is } w^*\text{-}w^* \text{ continuous on } \mathcal{A}''\},$$
$$Z_t^2(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \longmapsto \Psi \lozenge \Phi \text{ is } w^*\text{-}w^* \text{ continuous on } \mathcal{A}''\} \tag{1.3}$$

are called the first and the second topological centres of $\mathcal{A}''$, respectively. One can verify that $\mathcal{A}$ is Arens regular if and only if $Z_t^1(\mathcal{A}'') = Z_t^2(\mathcal{A}'') = \mathcal{A}''$. For example, the group algebra $L^1(G)$ for locally compact group G is Arens regular if and only if G is finite [2]. The reader is referred to [3, 4] for more information on Arens products and topological centres.

Throughout the paper we identify an element of a Banach space X with its canonical image in X''. Also for closed linear subspace E of X we write $E^{\perp} = \{f \in X' : f|_E = 0\}$.

In [5], Eshaghi Gordji and Filali obtained significant results related to the topological centres of Banach module actions and regularity of bilinear maps. They showed that if $\mathcal{A}$ enjoys a bounded approximate identity, then the left (right) module action of $\mathcal{A}$ on $\mathcal{A}'$ is regular if and only if $\mathcal{A}$ is reflexive; see also [6].

In this paper, under certain conditions we prove that the left and right module actions of $\mathcal{A}$ on $\mathcal{A}^{(n)}$ are regular, where $\mathcal{A}$ has not bounded approximate identity. Then we apply this fact to determine Arens regularity and quotient Arens regularity of certain class of Banach algebras.

2. Arens Regularity of Module Extension Banach Algebras

Suppose that X is a Banach $\mathcal{A}$-bimodule with the left and right module actions $\pi_1 : \mathcal{A} \times X \to X$ and $\pi_2 : X \times \mathcal{A} \to X$, respectively. According to [7], X'' is a Banach $\mathcal{A}''$-bimodule, where $\mathcal{A}''$ is equipped with the first Arens product. The module actions are defined by

$$\Phi \cdot \nu = w^* - \lim_i \lim_j \widehat{a_i \cdot x_j}, \qquad \nu \cdot \Phi = w^* - \lim_j \lim_i \widehat{x_j \cdot a_i}, \tag{2.1}$$

where (a_i) and (x_j) are nets in $\mathcal{A}$ and X that converge, in w^*-topologies, to Φ and ν, respectively.

Now suppose that $\mathcal{U} = \mathcal{A} \oplus X$. Then $\mathcal{U}$ with norm $\|(a, x)\| = \|a\| + \|x\|$ and product

$$(a, x)(b, y) = (ab, a \cdot y + x \cdot b) \quad (a, b \in \mathcal{A}, x, y \in X) \tag{2.2}$$

is a Banach algebra which is known as a module extension Banach algebra. The second dual $\mathcal{U}''$ of $\mathcal{U}$ is identified with $\mathcal{A}'' \oplus X''$, as a Banach space. Also the first Arens product $\square$ on $\mathcal{U}''$ is specified by

$$(\Phi, \mu)\square(\Psi, \nu) = \left(\Phi\square\Psi, \Phi \cdot \nu + \mu \cdot \Psi\right). \tag{2.3}$$

It is straightforward to check that $(\Phi, \mu) \in Z_t^1(\mathcal{U}'')$ if and only if

(a) $\Phi \in Z_t^1(\mathcal{A}'')$,

(b) $\nu \mapsto \Phi \cdot \nu : X'' \longrightarrow X''$ is w^*-w^* continuous,

(c) $\Psi \mapsto \mu \cdot \Psi : \mathcal{A}'' \longrightarrow X''$ is w^*-w^* continuous, (see [5, 8]).

If $\mathcal{A}''$ has the second Arens product $\Diamond$, then X'' is an $\mathcal{A}''$-bimodule in the same way. We denote this module action by the symbol "$\bullet$". The second Arens product $\Diamond$ on $\mathcal{U}''$ and second topological centre $Z_t^2(\mathcal{U}'')$ of $\mathcal{U}''$ can be defined analogously. Thus, the Banach algebra $\mathcal{U}$ is Arens regular if and only if $\mathcal{A}$ is Arens regular and

$$\Phi \cdot \nu = \Phi \bullet \nu, \qquad \nu \cdot \Phi = \nu \bullet \Phi \quad \left(\Phi \in \mathcal{A}'', \nu \in X''\right). \tag{2.4}$$

We consider $\mathcal{A}$ as a Banach $\mathcal{A}$-bimodule equipped with its own multiplication. Then $\mathcal{A} = \mathcal{A}^{(0)}, \mathcal{A}', \mathcal{A}'', \ldots, \mathcal{A}^{(n)}$ can be made into a Banach $\mathcal{A}$-bimodule in a natural fashion [4]. Clearly, regularity of $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ implies that of $\mathcal{A}$ but the converse is not true in general. For example, let $\mathcal{A}$ be a nonreflexive Banach space and let φ be a nonzero element of $\mathcal{A}'$ such that $\|\varphi\| \le 1$. Then the product $a \cdot b = \varphi(a)b$ turns $\mathcal{A}$ into a Banach algebra [6], such that $\mathcal{A}^{(2n)}$ is Arens regular for all $n \in \mathbb{N}$.

Now we consider the bilinear mappings

$$\pi_1 : \mathcal{A} \times \mathcal{A}^{(2n-1)} \longrightarrow \mathcal{A}^{(2n-1)}, \qquad \pi_2 : \mathcal{A}^{(2n-1)} \times \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}. \tag{2.5}$$

One can verify that π_2 is Arens regular for all $n \in \mathbb{N}$ but π_1 is not regular for each $n \in \mathbb{N}$. This shows that $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n-1)}$ is not regular. However, $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n)}$ is Arens regular.

We commence with the next result which studies Arens regularity of the left and right module actions of $\mathcal{A}$ on $\mathcal{A}^{(2n-1)}$.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra and $n \in \mathbb{N}$.*

(i) *If $\pi^{(3n)}(\mathcal{A}^{(2n)}, \mathcal{A}^{(2n)}) \subseteq \mathcal{A}^{(2n-2)}$, then the right module action of $\mathcal{A}$ on $\mathcal{A}^{(2n-1)}$ is Arens regular.*

(ii) *If $\pi^{t(3n)t}(\mathcal{A}^{(2n)}, \mathcal{A}^{(2n)}) \subseteq \mathcal{A}^{(2n-2)}$, then the left module action of $\mathcal{A}$ on $\mathcal{A}^{(2n-1)}$ is Arens regular.*

Proof. We prove (i) that the assertion (ii) can be proved similarly.

Since $\mathcal{A}^{(n+2)} = \mathcal{A}^{(n)} \oplus (\mathcal{A}^{(n-1)})^{\perp}$ [7], as a direct sum of $\mathcal{A}$-bimodules, it is enough to show that the result is valid for $n = 1$, and it can be deduced for $n \ge 2$, analogously. To this end let $\Phi \in \mathcal{A}''$ and let (a_i) be bounded net in $\mathcal{A}$ that is w^*-convergent to Φ. Since $\mathcal{A}''' = \mathcal{A}' \oplus \mathcal{A}^{\perp}$, for each $\mu \in \mathcal{A}'''$ there exist $f \in \mathcal{A}'$ and $\rho \in \mathcal{A}^{\perp}$ such that $\mu = \widehat{f} + \rho$. It follows that; for each $\Psi \in \mathcal{A}''$, $\pi'''(a_i, \Psi) \to \pi'''(\Phi, \Psi)$ in the weak topology. So we have that

$$\begin{aligned}\langle \mu \bullet \Phi, \Psi \rangle = \langle \Phi, \Psi \cdot \mu \rangle &= \lim_i \langle \widehat{a}_i, \Psi \cdot \mu \rangle \\ &= \lim_i \langle \mu, \pi'''(a_i, \Psi) \rangle \\ &= \langle \mu, \pi'''(\Phi, \Psi) \rangle \\ &= \langle \mu \cdot \Phi, \Psi \rangle.\end{aligned} \tag{2.6}$$

Therefore the right module action of $\mathcal{A}$ on $\mathcal{A}'$ is regular, as required. □

The corollary below follows from Theorem 3.1 of [5] and Theorem 2.1.

Corollary 2.2. *Let π_1 be the left module action of a Banach algebra $\mathcal{A}$ on $\mathcal{A}'$. If π_1'' is onto and $\mathcal{A}'' \Diamond \mathcal{A}'' \subseteq \mathcal{A}$, then $\mathcal{A}$ is Arens regular.*

The following theorem, which is the main one in the paper, characterizes Arens regularity of $\mathcal{A}^{(2n)}$.

Theorem 2.3. *Let $\mathcal{A}$ be an Arens regular Banach algebra. If $\mathcal{A}'' \square \mathcal{A}'' \subseteq \mathcal{A}$, then, for all $n \in \mathbb{N}$, $\mathcal{A}^{(2n)}$ is Arens regular and*

$$\pi^{(3n+3)}\left(\mathcal{A}^{(2n+2)}, \mathcal{A}^{(2n+2)}\right) \subseteq \mathcal{A}^{(2n)}. \tag{2.7}$$

Proof. Since $\mathcal{A}$ is Arens regular, $\mathcal{A}''$ is a dual Banach algebra with predual space $E = \mathcal{A}'$ [4]. Let $\mu \in \mathcal{A}'''$ and $\Phi \in \mathcal{A}''$. Then the inclusion $\mathcal{A}'' \square \mathcal{A}'' \subseteq \mathcal{A}$ shows that $\mu \cdot \Phi$ is w^*-continuous linear functional on $\mathcal{A}''$ and so it must be in $\mathcal{A}'$. It follows that $\beta \cdot \mu = 0$ for all $\beta \in E^{\perp}$, and hence $\pi^{(6)}(\alpha, \beta) = 0$ for each $\alpha \in E^{\perp}$. Similarly, we obtain $\pi^{t(6)t}(\alpha, \beta) = 0\ (\alpha, \beta \in E^{\perp})$. Then by Proposition 2.16 of [4] $\mathcal{A}''$ is Arens regular and

$$\pi^{(6)}((\Phi, \alpha), (\Psi, \beta)) = (\Phi \square \Psi, \Phi \cdot \beta + \alpha \cdot \Psi), \quad \left(\Phi, \Psi \in \mathcal{A}'', \alpha, \beta \in E^{\perp}\right). \tag{2.8}$$

One may verify that $\Phi \cdot \beta = \alpha \cdot \Psi = 0$ and, since $\mathcal{A}^{(4)} = \mathcal{A}'' \oplus E^{\perp}$, that we have $\pi^{(6)}(\mathcal{A}^{(4)}, \mathcal{A}^{(4)}) \subseteq \mathcal{A}''$. Thus the result is established for $n = 1$. An easy induction argument now finishes the proof. □

As a consequence of Theorems 2.1 and 2.3, we have the next result.

Corollary 2.4. *Let $\mathcal{A}$ be an Arens regular Banach algebra. If $\mathcal{A}'' \square \mathcal{A}'' \subseteq \mathcal{A}$, then the following assertions hold for all $n \in \mathbb{N}$.*

(i) $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ *is Arens regular.*

(ii) $\mathcal{A}^{(2n-1)}$ *is an* $\mathcal{A}^{(2n)}$*-submodule of* $\mathcal{A}^{(2n+1)}$.

Let $\mathcal{A} = l^1$, with pointwise product. Then $\mathcal{A}$ is an Arens regular Banach algebra which is not reflexive but satisfies $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$ [4]. Therefore by the preceding corollary $\mathcal{U} = \mathcal{A}\oplus\mathcal{A}^{(n)}$ is Arens regular.

It is easy to verify that regularity of the left and right module actions of $\mathcal{A}$ on $\mathcal{A}^{(2n-1)}$ are equivalent for each Arens regular Banach algebra $\mathcal{A}$ which is commutative.

Remark 2.5. It is well known that each C^*-algebra $\mathcal{A}$ is Arens regular and $\mathcal{A}''$ is also a C^*-algebra [3], and therefore $\mathcal{A}''$ itself are Arens regular. This shows that for each $n \in \mathbb{N}$, $\mathcal{A}^{(2n)}$ and hence $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n)}$ is Arens regular. But in general, $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n-1)}$ is not Arens regular. Indeed, it is Arens regular if and only if $\mathcal{A}$ is reflexive [5].

3. Quotient Arens Regularity of Module Extension Banach Algebras

Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and let $X = \mathcal{A}' \cdot \mathcal{A}$, the subspace of $\mathcal{A}'$ consisting of the functionals of the form $f \cdot a$, for all $f \in \mathcal{A}'$ and $a \in \mathcal{A}$. By Cohen's factorization theorem [9], X is a closed $\mathcal{A}$-submodule of $\mathcal{A}'$. It is also left introverted in $\mathcal{A}'$; that is, $\Phi \cdot \lambda \in X$ for each $\lambda \in X$ and $\Phi \in \mathcal{A}''$. Then X' is a Banach algebra by the following (first Arens type) product:

$$\langle \Phi\Box\Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle \quad (\Phi, \Psi \in X', \lambda \in X). \tag{3.1}$$

As in [10], the Banach algebra $\mathcal{A}$ is said to be left quotient Arens regular if $Z_t(X') = X'$, where

$$Z_t(X') = \{\Phi \in X' : \Psi \longmapsto \Phi\Box\Psi \text{ is } w^*\text{-}w^* \text{ continuous on } X'\}. \tag{3.2}$$

Similarly, $X = \mathcal{A} \cdot \mathcal{A}'$ is an $\mathcal{A}$-module and is right introverted in $\mathcal{A}'$. As mentioned above, the second Arens product on $\mathcal{A}''$ induces naturally a Banach algebra product on X', which is denoted by $\Diamond$. The topological centre $Z_t(X')$ and right quotient Arens regularity can be defined analogously. Obviously, every Arens regular Banach algebra is quotient Arens regular but the converse does not hold; see example 38 of [10]. Also a direct proof shows that, if $\mathcal{A}$ is an ideal in $\mathcal{A}''$, then $\mathcal{A}$ is quotient Arens regular.

Proposition 3.1. *Suppose that the Banach algebra $\mathcal{A}$ is a left ideal in $\mathcal{A}''$. Then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n)}$ is a left ideal in $\mathcal{U}''$ for all $n \in \mathbb{N}$.*

Proof. We first show that, if $\mathcal{A}$ is a left ideal in $\mathcal{A}''$, then it is also a left ideal in $\mathcal{A}^{(2n)}$ for each $n \in \mathbb{N}$. So let $a \in \mathcal{A}$ and $\alpha \in \mathcal{A}^{(4)}$. Then, for all $\mu \in \mathcal{A}'''$, there exist $f \in \mathcal{A}'$ and $\rho \in \mathcal{A}^{\perp}$ such that $\mu = \widehat{f} + \rho$. By assumption $\widehat{a} \cdot \rho = 0$, and therefore $\widehat{a} \cdot \mu = \widehat{a} \cdot \widehat{f}$. This shows that $\widehat{a} \cdot \mu$ is w^*-continuous linear functional on $\mathcal{A}''$ and so $\widehat{a} \cdot \mu \in \mathcal{A}'$. Since $\mathcal{A}^{(4)} = \mathcal{A}'' \oplus (\mathcal{A}')^{\perp}$, $\alpha = \Phi + \sigma$ for some $\Phi \in \mathcal{A}''$ and $\sigma \in (\mathcal{A}')^{\perp}$. Then we have that

$$\langle \alpha \cdot \widehat{a}, \mu \rangle = \langle \alpha, \widehat{a} \cdot \mu \rangle = \langle \Phi + \sigma, \widehat{a} \cdot \mu \rangle = \langle \Phi, \widehat{a} \cdot \mu \rangle = \langle \Phi \cdot \widehat{a}, \mu \rangle. \tag{3.3}$$

It follows that $\alpha \cdot \widehat{a} = \Phi \cdot \widehat{a}$, and thus $\mathcal{A}$ is a left ideal in $\mathcal{A}^{(4)}$. An easy induction argument now finishes our claim. Therefore by definition $\mathcal{U}$ is a left ideal in $\mathcal{U}''$ for each $n \in \mathbb{N}$. □

In general, the above result is not valid if we replace $2n$ with $2n-1$. For example, let $\mathcal{A}$ be the group algebra of an infinite compact group G. Then $\mathcal{A}$ is an ideal in $\mathcal{A}''$, as is well known, but $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n-1)}$ is not ideal in $\mathcal{U}''$. By additional hypothesis we have the next result.

Theorem 3.2. *If the Banach algebra $\mathcal{A}$ is a left ideal in $\mathcal{A}''$ and the right module action of $\mathcal{A}$ on $\mathcal{A}^{(2n-2)}$ is regular, then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n-1)}$ is a left ideal in $\mathcal{U}''$.*

Proof. The result is straightforward for the case $n = 1$. So we give the proof for $n = 2$. Let $(a, \mu) \in \mathcal{U}$ and $(\Phi, \Lambda) \in \mathcal{U}''$. Then a similar argument to what has been used in the proof of the preceding proposition shows that $\Lambda \cdot \widehat{a} \in \mathcal{A}'''$. On the other hand, regularity of the right module action of $\mathcal{A}$ on $\mathcal{A}''$ implies that $\Phi \cdot \widehat{\mu}$ is w^*-continuous linear functional on $\mathcal{A}^{(4)}$ and so it must be in $\mathcal{A}'''$. Thus, $\Phi \cdot \widehat{\mu} + \Lambda \cdot \widehat{a} \in \mathcal{A}'''$. Therefore by definition we have that $(\Phi, \Lambda) \square (a, \mu) \in \mathcal{U}$, and hence $\mathcal{U}$ is a left ideal in $\mathcal{U}''$. A similar discussion reveals that the result will be established for $n > 2$. □

Recall that the right version of Proposition 3.1 and Theorem 3.2 holds. Therefore, we have the following results.

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}$ is an ideal in $\mathcal{A}''$. Suppose that the left and right module actions of $\mathcal{A}$ on $\mathcal{A}^{(2n-2)}$ are regular. Then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ is quotient Arens regular.*

Corollary 3.4. *Let $\mathcal{A}$ be a C^*-algebra and $n \in \mathbb{N}$. If $\mathcal{A}$ is an ideal in $\mathcal{A}''$, then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ is quotient Arens regular.*

Example 3.5. Let $\mathcal{A} = c_0$, with pointwise product and $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$. Then $\mathcal{A}$ is a commutative C^*-algebra which is an ideal in $\mathcal{A}''$. Therefore by the above corollary $\mathcal{U}$ is quotient Arens regular for all $n \in \mathbb{N}$. Note that, by Remark 2.5, $\mathcal{U}$ is not Arens regular for the odd case n.

References

[1] R. Arens, "The adjoint of a bilinear operation," *Proceedings of the American Mathematical Society*, vol. 2, pp. 839–848, 1951.

[2] N. J. Young, "The irregularity of multiplication in group algebras," *The Quarterly Journal of Mathematics*, vol. 24, pp. 59–62, 1973.

[3] H. G. Dales, *Banach Algebras and Automatic Continuity*, vol. 24 of *London Mathematical Society Monographs*, The Clarendon Press, Oxford, UK, 2000.

[4] H. G. Dales and A. T. M. Lau, "The second duals of Beurling algebras," *Memoirs of the American Mathematical Society*, vol. 177, no. 836, 2005.

[5] M. Eshaghi Gordji and M. Filali, "Arens regularity of module actions," *Studia Mathematica*, vol. 181, no. 3, pp. 237–254, 2007.

[6] H. G. Dales, A. Rodríguez-Palacios, and M. V. Velasco, "The second transpose of a derivation," *Journal of the London Mathematical Society*, vol. 64, no. 3, pp. 707–721, 2001.

[7] H. G. Dales, F. Ghahramani, and N. Grønbæk, "Derivations into iterated duals of Banach algebras," *Studia Mathematica*, vol. 128, no. 1, pp. 19–54, 1998.

[8] F. Ghahramani, J. P. McClure, and M. Meng, "On asymmetry of topological centers of the second duals of Banach algebras," *Proceedings of the American Mathematical Society*, vol. 126, no. 6, pp. 1765–1768, 1998.

[9] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, vol. 2 of *Fundamental Principles of Mathematical Sciences*, Springer, Berlin, Germany, 1970.

[10] Z. Hu, M. Neufang, and Z. J. Ruan, "On topological centre problems and SIN quantum groups," *Journal of Functional Analysis*, vol. 257, no. 2, pp. 610–640, 2009.