Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 712746, 11 pages
doi:10.1155/2011/712746

Research Article

On Diffraction Fresnel Transforms for Boehmians

S. K. Q. Al-Omari[1] and A. Kılıçman[2]

[1] *Department of Applied Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan*
[2] *Department of Mathematics and Institute of Mathematical Research, Universiti Putra Malaysia (UPM), 43400 Serdang, Selangor, Malaysia*

Correspondence should be addressed to A. Kılıçman, akilicman@putra.upm.edu.my

Received 12 September 2011; Revised 27 October 2011; Accepted 11 November 2011

Academic Editor: Natig Atakishiyev

The theory of the diffraction Fresnel transform is extended to certain spaces of Schwartz distributions. In the context of Boehmian spaces, the diffraction Fresnel transform is obtained as a continuous function. Convergence with respect to δ and Δ is also defined.

1. Introduction

The integral transforms play important role in the various fields of optics. One of great importance in many applications is the Fourier transform, where the kernel takes the form of a complex exponential function. The generalization of the Fourier transform is known as the fractional Fourier transform which was introduced by Namias in [1] and, has recently attracted considerable attention in optics and the light propagation in gradient-index media; see, for example, [2, 3], similarly in some lens systems see [4, 5]. Another well-known linear transform is the Fresnel transform; see [4–7], where the complex version of kernel having a quadratic combination of t and ξ in the exponent, see [8]. Recently, much attention has been paid to the diffraction Fresnel transform

$$\mathfrak{F}_d f(\xi) =_{\mathbf{R}} K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t) f(t) dt, \tag{1.1}$$

where

$$K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t) = \frac{1}{\sqrt{2\pi i \gamma_1}} \exp\left(\frac{i}{2\gamma_1}\left(\alpha_1 t^2 - 2\xi t + \alpha_2 \xi^2\right)\right) \tag{1.2}$$

is the transform kernel with the real parameters and α_1, γ_1, and γ_2 satisfy the following relation:

$$\alpha_1\alpha_2 - \gamma_1\gamma_2 = 1 \tag{1.3}$$

holds; see [9].

Many familiar transforms can be considered as special cases of the generalized Fresnel transform. For example, if the parameters α_1, γ_1, γ_2 and α_2 satisfy the matrix

$$\begin{pmatrix} \alpha_1 & \gamma_1 \\ \gamma_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.4}$$

then the generalized Fresnel transform becomes a fractional Fourier transform.

In particular, when $\theta = \pi/2$, one obtains the standard Fourier transform. Further, if $\alpha_1 = \alpha_2 = 1$, the generalized Fresnel transform reduces to the complex form of the Fresnel transform.

In the present paper, we show that the diffraction Fresnel transform can be extended to certain spaces generalized functions. In Section 2, we extend the diffraction Fresnel transform to a space of tempered distributions and further, by the aid of the Parseval's equation, to a space of distributions of compact support. In Section 3, we define the diffraction Fresnel transform of a Boehmian and discuss its continuity with respect to δ and Δ convergence.

2. The Distributional Diffraction Fresnel Transform

Let S denote the space of all complex valued functions $\phi(t)$ that are infinitely smooth and are such that, as $|t| \to \infty$, they and their partial derivatives decrease to zero faster than every power of $1/|t|$. When t is one dimensional, every function $\phi(t)$ in S satisfies the infinite set of inequalities

$$\left| t^m \phi^{(k)}(t) \right| \leq C_{m,k}, \quad \text{where } t \in \mathbf{R}, \tag{2.1}$$

where m and k run through all nonnegative integers. The above expression can be interpreted as

$$\lim_{|t| \to \infty} t^m \phi^{(k)}(t) = 0. \tag{2.2}$$

Members of S are the so-called testing functions of rapid descent, then S is naturally a linear space. The dual space $\acute{S}$ of S is the space of distributions of slow growth (the space of tempered distributions). See [2, 10, 11].

Theorem 2.1. *If $\phi(t)$ is in S, then its diffraction Fresnel transform*

$$\mathfrak{F}_d(\phi)(\xi) = \frac{1}{\sqrt{2\pi i \gamma_1}} \int_{\mathbf{R}} \phi(t) \exp\left(\frac{i(\alpha_1 t^2 - 2t\xi + \alpha_2 \xi^2)}{2\gamma_1} \right) dt \tag{2.3}$$

exists and further also in S.

Proof. Let ξ be fixed. If $\phi(t)$ is in S, then its diffraction Fresnel transform certainly exists. Moreover, differentiating the right-hand side of (2.3) with respect to ξ, under the integral sign, k-times, yields a sum of polynomials, $p_k(t+\xi)$, say of combinations of t and ξ. That is,

$$\left|\frac{d^k}{dt^k}\mathfrak{F}_d(\phi)(\xi)\right| = \left|p_k(t+\xi)\phi(t)\exp\left(\frac{i(\alpha_1 t^2 - 2t\xi + \alpha_2\xi^2)}{2\gamma_1}\right)\right| \le \left|p_k(t+\xi)\phi(t)\right|, \tag{2.4}$$

which is also in S, since ϕ in S and S is a linear space. Hence,

$$\left|\xi^m \frac{d^k}{dt^k}\mathfrak{F}_d(\phi)(\xi)\right| \le \int_{\mathbf{R}} \left|\xi^m p_k(t+\xi)\phi(t)\right| dt. \tag{2.5}$$

Once again, since $\phi \in S$, the integral on the right-hand side of (2.5) is bounded by a constant $C_{m,k}$, for every pair of nonnegative integers m and k. Hence, we have the following theorem. □

Theorem 2.2 (Parseval's Equation for the diffraction transform). *If $f(x)$ and $g(x)$ are absolutely integrable, over $x \in \mathbf{R}$, then*

$$\int_{\mathbf{R}} f(x)\mathfrak{F}_d g(x)dx = \int_{\mathbf{R}} \mathfrak{F}_d f(x) g(x) dx, \tag{2.6}$$

where $\mathfrak{F}_d f$ and $\mathfrak{F}_d g$ are the corresponding diffraction Fresnel transforms of f and g, respectively.

Proof. The diffraction Fresnel transforms $\mathfrak{F}_d f(\xi)$ and $\mathfrak{F}_d g(\xi)$ are indeed bounded and continuous for all ξ. This ensure the convergence of the integrals in (2.6). Moreover,

$$\int_{\mathbf{R}} f(x)\mathfrak{F}_d g(x)dx = \int_{\mathbf{R}} dx_{\mathbf{R}} f(x) g(y)\exp\left(\frac{i(\alpha_1 y^2 - 2xy + \alpha_2 x^2)}{2\gamma_1}\right)dy, \quad \alpha_1\alpha_2 - \gamma_1\gamma_2 = 1. \tag{2.7}$$

Since the integral (2.7) is absolutely integrable over the entire (x, y)-plane, Fubini's theorem allows us to interchange the order of integration. Hence, (2.7) can be written as

$$\int_{\mathbf{R}} f(x)_{\mathbf{R}} g(y)\exp\left(\frac{i(\alpha_2 x^2 - 2xy + \alpha_1 y^2)}{2\gamma_1}\right)dy\,dx = \int_{\mathbf{R}} \mathfrak{F}_d f(y) g(y) dy, \tag{2.8}$$

where $\alpha_2\alpha_1 - \gamma_1\gamma_2 = 1$. This completes the proof of the theorem. □

Parseval's relation can be interpreted as

$$\langle \mathfrak{F}_d f, \phi\rangle = \langle f, \mathfrak{F}_d \phi\rangle. \tag{2.9}$$

Therefore, from the above relation, we state the *diffraction Fresnel transform* of a distribution f of slow growth ($f \in \acute{S}$) as

$$\langle \mathfrak{F}_d f, \phi \rangle = \langle f, \mathfrak{F}_d \phi \rangle, \quad \forall \phi \in S, \tag{2.10}$$

and it is well defined by Theorem 2.1.

Theorem 2.3. *If f is a distribution of slow growth, then its diffraction Fresnel transform $\mathfrak{F}_d f$ is also a distribution of slow growth.*

Proof. Linearity of $\mathfrak{F}_d f$ is obvious. To show continuity of $\mathfrak{F}_d f$, let $(\phi_n)_{n=1}^{\infty} \to 0$, in S, then also $(\mathfrak{F}_d \phi_n)_{n=1}^{\infty} \to 0$ in S as $n \to \infty$. Hence,

$$\langle \mathfrak{F}_d f, \phi_n \rangle = \langle f, \mathfrak{F}_d \phi_n \rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.11}$$

Hence $\mathfrak{F}_d f \in \acute{S}$. This completes the proof of the theorem. □

Theorem 2.4. *Let f be a distribution of compact support ($f \in \acute{E}$). Then, we define the Fresnel transform of f as*

$$\mathfrak{F}_d f(\xi) = \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), \exp\left(\frac{i(\alpha_1 t^2 - 2t\xi + \alpha_2 \xi^2)}{2\gamma_1} \right) \right\rangle. \tag{2.12}$$

Proof. Let $\phi \in S(\mathbb{R})$ be arbitrary. From (2.10), we read

$$\begin{aligned} \langle \mathfrak{F}_d f(\xi), \phi(\xi) \rangle &= \langle f(t), \mathfrak{F}_d \phi(t) \rangle \\ &= \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), \int_{\mathbf{R}} \phi(\xi) \exp\left(\frac{i(\alpha_1 \xi^2 - 2t\xi + \alpha_2 t^2)}{2\gamma_1} \right) d\xi \right\rangle \\ &= \frac{1}{\sqrt{2\pi i \gamma_1}} \int_{\mathbf{R}} \left\langle f(t), \exp\left(\frac{i(\alpha_2 t^2 + -2t\xi + \alpha_1 \xi^2)}{2\gamma_1} \right) \right\rangle \phi(\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle \left\langle f(t), \exp\left(\frac{i(\alpha_2 t^2 + -2t\xi + \alpha_1 \xi^2)}{2\gamma_1} \right) \right\rangle, \phi(\xi) \right\rangle. \end{aligned} \tag{2.13}$$

But since $\langle f(t), \exp(i(\alpha_2 t^2 + -2t\xi + \alpha_1 \xi^2)/2\gamma_1) \rangle$ is an infinitely smooth function, we get

$$\mathfrak{F}_d f(\xi) = \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), \exp\left(\frac{i(\alpha_2 t^2 + -2t\xi + \alpha_1 \xi^2)}{2\gamma_1} \right) \right\rangle. \tag{2.14}$$

This completes the proof of the theorem. □

Now, for distributions f and $g \in \acute{E}(\mathbf{R})$, we define the convolution product as

$$\langle (f * 9)(t), \phi(t) \rangle = \langle f(t), \langle g(\tau), \phi(t + \tau) \rangle \rangle, \tag{2.15}$$

for every $\phi \in E(\mathbf{R})$. This definition makes sense, since $\langle g(\tau), \phi(t+\tau)\rangle$ belongs to $\mathfrak{D}$, and hence a member of $E(\mathbf{R})$. With this definition, we are allowed to write the following theorem.

Theorem 2.5. *For every $f \in \acute{E}(\mathbf{R})$, the function $\psi(t) = \langle f(\tau), \phi(t+\tau)\rangle$ is infinitely smooth and satisfies the relation*

$$D_t^k \psi(t) = \left\langle f(\tau), D_t^k \phi(t+\tau) \right\rangle, \tag{2.16}$$

for all $k \in \mathbf{N}$.

Proof (see page 26 in [12]). A direct result of the convolution product is the following theorem. □

Theorem 2.6 (Convolution Theorem). *Let f and g be distributions of compact support and $\mathfrak{F}_d f(\xi) = \mathfrak{F}_d(f(t);\xi), \mathfrak{F}_d g(\xi) = \mathfrak{F}_d(g(\tau);\xi)$ their respective diffraction Fresnel transforms, then*

$$\mathfrak{F}_d((f*g)(t);\xi) = \sqrt{2\pi i\gamma_1}\exp\left(\frac{i(2\alpha_1 t\tau - \alpha_2\xi^2)}{2\gamma_1}\right)\mathfrak{F}_d(f(t);\xi)\mathfrak{F}_d(g(\tau);\xi). \tag{2.17}$$

Proof. Let $f, g \in \acute{E}(\mathbf{R})$, then by using (2.12), we get

$$\begin{aligned}
\mathfrak{F}_d((f*g)(t);\xi) &= \frac{1}{\sqrt{2\pi i\gamma_1}}\left\langle (f*g)(t), \exp\left(\frac{i(\alpha_1 t^2 - 2t\xi + \alpha_2\xi^2)}{2\gamma_1}\right)\right\rangle \\
\text{i.e.} &= \frac{1}{\sqrt{2\pi i\gamma_1}}\left\langle f(t), \left\langle g(\tau), \exp\left(\frac{i\left(\alpha_1(t+\tau)^2 - (t+\tau)\xi + \alpha_2\xi^2\right)}{2\gamma_1}\right)\right\rangle\right\rangle \\
&= \frac{1}{\sqrt{2\pi i\gamma_1}}\left\langle f(t), \left\langle g(\tau), \exp\left(\frac{i(\alpha_1 t^2 + \alpha_1\tau^2 + \alpha_1 t\tau - 2t\xi - 2\tau\xi + \alpha_2\xi^2)}{2\gamma_1}\right)\right\rangle\right\rangle.
\end{aligned} \tag{2.18}$$

Properties of distributions together with simple calculations on the exponent yield

$$\mathfrak{F}_d((f*g)(t);\xi) = \sqrt{2\pi i\gamma_1}\exp\left(\frac{i(2\alpha_1 t\tau - \alpha_2\xi^2)}{2\gamma_1}\right)\mathfrak{F}_d(f(t);\xi)\mathfrak{F}_d(g(\tau);\xi). \tag{2.19}$$

This completes the proof of the theorem. □

Corollary 2.7. *Let $f, g \in \acute{E}(\mathbf{R})$, then*

$$\begin{aligned}
&(1) \quad \mathfrak{F}_d(f*\delta_n(t);\xi) = \sqrt{2\pi i\gamma_1}\exp\left(-\frac{\alpha_2\xi^2}{2\gamma_1}\right)\mathfrak{F}_d(f)(\xi), \\
&(2) \quad \mathfrak{F}_d(\delta_n*g(t);\xi) = \sqrt{2\pi i\gamma_1}\exp\left(-\frac{\alpha_2\xi^2}{2\gamma_1}\right)\mathfrak{F}_d(g)(\xi),
\end{aligned} \tag{2.20}$$

where $\mathfrak{F}_d f(\xi) = \mathfrak{F}_d(f(t);\xi), \mathfrak{F}_d g(\xi) = \mathfrak{F}_d(g(\tau);\xi)$.

The following is a theorem which can be directly established from (2.12) *and the fact that [11]*

$$D^k(f * g) = D^k f * g = f * D^k g. \tag{2.21}$$

Theorem 2.8. *Let f and g be distributions of compact support and $\mathfrak{F}_d f(\xi) = \mathfrak{F}_d(f(t);\xi), \mathfrak{F}_d g(\xi) = \mathfrak{F}_d(g(\tau);\xi)$ their respective diffraction Fresnel transforms, then*

$$\begin{aligned}
&(1)\quad \mathfrak{F}_d\left(D_t^k(f * g)(t);\xi\right) = \sqrt{2\pi i\gamma_1}\exp\left(\frac{i(2\alpha_1 t\tau - \alpha_2\xi^2)}{2\gamma_1}\right)\mathfrak{F}_d\left(f^{(k)}(t);\xi\right)\mathfrak{F}_d g(\xi),\\
&(2)\quad \mathfrak{F}_d\left(D_t^k(f * g)(t);\xi\right) = \sqrt{2\pi i\gamma_1}\exp\left(\frac{i(2\alpha_1 t\tau - \alpha_2\xi^2)}{2\gamma_1}\right)\mathfrak{F}_d f(\xi)\mathfrak{F}_d\left(g^{(k)}(\tau);\xi\right).
\end{aligned} \tag{2.22}$$

3. Diffraction Fresnel Transform of Boehmians

Let $\mathfrak{X}$ be a linear space and $\mathfrak{I}$ a subspace of $\mathfrak{X}$. To each pair of elements $f \in \mathfrak{X}$ and $\phi \in \mathfrak{I}$, we assign a product $f \cdot g$ such that the following conditions are satisfied:

(i) if $\phi, \psi \in \mathfrak{I}$, then $\phi \cdot \psi \in \mathfrak{I}$ and $\phi \cdot \psi = \psi \cdot \phi$,

(ii) if $f \in \mathfrak{X}$ and $\phi, \psi \in \mathfrak{I}$, then $(f \cdot \phi) \cdot \psi = f \cdot (\phi \cdot \psi)$,

(iii) if $f, g \in \mathfrak{X}, \phi \in \mathfrak{I}$ and $\lambda \in \mathbf{R}$, then $(f + g) \cdot \phi = f \cdot \phi + g \cdot \phi$ and $\lambda(f \cdot \phi) = (\lambda f) \cdot \phi$. Let Δ be a family of sequences from $\mathfrak{I}$ such that

(a) if $f, g \in \mathfrak{X}, (\delta_n) \in \Delta$ and $f \cdot \delta_n = g \cdot \delta_n (n = 1, 2, \ldots)$, then $f = g$,

(b) if $(\phi_n), (\delta_n) \in \Delta$, then $(\phi_n \cdot \psi_n) \in \Delta$.

Elements of Δ will be called *delta sequences.* Consider the class $\mathbf{U}$ of pair of sequences defined by

$$\mathbf{U} = \left\{((f_n), (\phi_n)) : (f_n) \subseteq \mathfrak{X}^{\mathbf{N}}, (\phi_n) \in \Delta\right\}, \tag{3.1}$$

for each $n \in \mathbf{N}$. An element $((f_n), (\phi_n)) \in \mathbf{U}$ is called a quotient of sequences, denoted by f_n/ϕ_n, or $[f_n/\phi_n]$ if $f_i \cdot \phi_j = f_j \cdot \phi_i$, for all $i, j \in \mathbf{N}$.

Similarly, two quotients of sequences f_n/ϕ_n and g_n/ψ_n are said to be *equivalent*, $f_n/\phi_n \sim g_n/\psi_n$, if $f_i \cdot \psi_j = g_j \cdot \phi_i$, for all $i, j \in \mathbf{N}$. The relation $\sim$ is an equivalent relation on $\mathbf{U}$, and hence splits $\mathbf{U}$ into equivalence classes. The equivalence class containing f_n/ϕ_n is denoted by $[f_n/\phi_n]$. These equivalence classes are called *Boehmians*, and the *space of all Boehmians* is denoted by $\mathfrak{B}$.

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way

$$\begin{aligned}
\left[\frac{f_n}{\phi_n}\right] + \left[\frac{g_n}{\psi_n}\right] &= \left[\frac{((f_n \cdot \psi_n) + (g_n \cdot \phi_n))}{\phi_n \cdot \psi_n}\right],\\
\alpha\left[\frac{f_n}{\phi_n}\right] &= \left[\frac{\alpha f_n}{\phi_n}\right], \quad \alpha \in \mathbb{C}.
\end{aligned} \tag{3.2}$$

The operation $\cdot$ and the differentiation are defined by

$$\left[\frac{f_n}{\phi_n}\right]\cdot\left[\frac{g_n}{\psi_n}\right]=\left[\frac{(f_n\cdot g_n)}{(\phi_n\cdot\psi_n)}\right],$$
$$\mathfrak{D}^{\alpha}\left[\frac{f_n}{\phi_n}\right]=\left[\frac{\mathfrak{D}^{\alpha}f_n}{\phi_n}\right]. \tag{3.3}$$

The relationship between the notion of convergence and the product $\cdot$ are given by the following:

(i) if $f_n \to f$ as $n \to \infty$ in $\mathfrak{X}$ and, $\phi \in \mathfrak{I}$ is any fixed element, then $f_n \cdot \phi \to f \cdot \phi$ in $\mathfrak{X}$(as $n \to \infty$),

(ii) if $f_n \to f$ as $n \to \infty$ in $\mathfrak{X}$ and $(\delta_n) \in \Delta$, then $f_n \cdot \delta_n \to f$ in $\mathfrak{X}$ (as $n \to \infty$).

The operation $\cdot$ can be extended to $\mathfrak{B} \times \mathfrak{I}$ by

$$\text{If } \left[\frac{f_n}{\delta_n}\right] \in \mathfrak{B} \text{ and } \phi \in \mathfrak{I}, \text{ then } \left[\frac{f_n}{\delta_n}\right]\cdot\phi=\left[\frac{f_n\cdot\phi}{\delta_n}\right]. \tag{3.4}$$

In $\mathfrak{B}$, one can define two types of convergence as follows:

(i) (δ-convergence) *a sequence* (β_n) *in* $\mathfrak{B}$ *is said to be* δ*-convergent to* β *in* $\mathfrak{B}$, *denoted by* $\beta_n \xrightarrow{\delta} \beta$, *if there exists a delta sequence* (δ_n) *such that* $(\beta_n \cdot \delta_n), (\beta \cdot \delta_n) \in \mathfrak{X}$, for all $k, n \in \mathbf{N}$, *and* $(\beta_n \cdot \delta_k) \to (\beta \cdot \delta_k)$ *as* $n \to \infty$, *in* $\mathfrak{X}$, *for every* $k \in \mathbf{N}$,

(ii) (Δ-convergence) *a sequence* (β_n) *in* $\mathfrak{B}$ *is said to be* Δ*-convergent to* β *in* $\mathfrak{B}$, *denoted by* $\beta_n \xrightarrow{\Delta} \beta$, *if there exists a* $(\delta_n) \in \Delta$ *such that* $(\beta_n - \beta)\cdot\delta_n \in \mathfrak{X}$, for all $n \in \mathbf{N}$, *and* $(\beta_n - \beta)\cdot\delta_n \to 0$ *as* $n \to \infty$ *in* $\mathfrak{X}$.

For further analysis we refer, for example, to [10, 13–19]. Now we let L^1 be the space of Lebesgue integrable functions on $\mathbf{R}$ and $\mathfrak{B}_{L^1}$ the space of Lebesgue integrable Boehmians [17] with the set Δ of all delta sequence (δ_n) from $\mathfrak{D}$ (the test function space of compact support) such that

(1) $\int_{\mathbf{R}} \delta_n = 1$ for all $n \in \mathbf{N}$,

(2) $\int_{\mathbf{R}} |\delta_n| < M$ for certain positive number M and $n \in \mathbf{N}$,

(3) $\int_{|t|>\varepsilon} |\delta_n(t)|dt \to 0$ as $n \to \infty$ for every $\varepsilon > 0$.

Then, $\mathfrak{B}_{L^1}$ is a convolution algebra with the pointwise operations

(i) $\lambda[f_n/\delta_n] = [\lambda f_n/\delta_n]$,

(ii) $[f_n/\delta_n] + [g_n/\phi_n] = [(f_n * \phi_n + g_n * \delta_n)/(\delta_n * \phi_n)]$,

(iii) and the convolution

$$\left[\frac{f_n}{\delta_n}\right]*\left[\frac{g_n}{\phi_n}\right]=\left[\frac{f_n*g_n}{\delta_n*\phi_n}\right]. \tag{3.5}$$

Lemma 3.1. *Let* $[f_n/\delta_n] \in \mathfrak{B}_{L^1}$*, then the sequence*

$$\mathfrak{F}_d\left(f_n(t);\xi\right) = \frac{1}{\sqrt{2\pi i\gamma_1}}\int_{\mathbf{R}} f_n(t)\exp\left(\frac{i}{2\gamma_1}\left(\alpha_1 t^2 - 2t\xi + \alpha_2\xi^2\right)\right)dt \tag{3.6}$$

converges uniformly on each compact set K *in* $\mathbf{R}$.

Proof. Let $\tilde{f}_n = \mathfrak{F}_d f$. For each compact set K, $\tilde{\delta}_n(\tilde{\delta}_n = \mathfrak{F}_d\delta_n)$ converges uniformly to the function $\exp(-(i\alpha_2/2\gamma_1)\xi^2)$. Hence, by Corollary 2.7,

$$\mathfrak{F}_d\left(f_n(t);\xi\right) = \tilde{f}_n\frac{\tilde{\delta}_k}{\tilde{\delta}_k} = \frac{e^{(i\alpha_2/2\gamma_1)\xi^2}}{\sqrt{2\pi i\gamma_1}}\frac{\mathfrak{F}_d\left(f_n * \delta_k\right)}{\tilde{\delta}_k}. \tag{3.7}$$

Using the choice f_n/δ_n that is quotient of sequences and upon employing Corollary 2.7, we have

$$\mathfrak{F}_d\left(f_n(t);\xi\right) = \frac{e^{(i\alpha_2/2\gamma_1)\xi^2}}{\sqrt{2\pi i\gamma_1}}\frac{\mathfrak{F}_d\left(f_k * \delta_n\right)}{\tilde{\delta}_k} = \frac{\tilde{f}_k}{\tilde{\delta}_k}\tilde{\delta}_n = \frac{\tilde{f}_k}{\tilde{\delta}_k}\sqrt{2\pi i\gamma_1}e^{-(i\alpha_2/2\gamma_1)\xi^2}. \tag{3.8}$$

This completes the proof of the Lemma. □

By using this Lemma, we are able to define the diffractional Fresnel transform of a Boehmian as follows: $[f_n/\delta_n]$ *in* $\mathfrak{B}_{L^1}$ as

$$\mathcal{R}\left[\frac{f_n}{\delta_n}\right] = \lim_{n\to\infty}\tilde{f}_n, \tag{3.9}$$

where the limit ranges over compact subsets of $\mathbf{R}$. Now, let $[X_n/\delta_n] = [Y_n/\gamma_n]$ in $\mathfrak{B}_{L^1}$, then

$$X_n * \gamma_m = Y_m * \delta_n, \text{ for every } m, n \in \mathbf{N}. \tag{3.10}$$

Hence, employing the Fresnel transform to both sides of above equation implies

$$\mathfrak{F}_d(X_n * \gamma_m) = \mathfrak{F}_d(Y_m * \delta_n) = \mathfrak{F}_d(Y_n * \delta_m). \tag{3.11}$$

Thus, using Theorem 2.6 and the fact that

$$\widetilde{\delta}_n \text{ and } \widetilde{\delta}_m \longrightarrow \sqrt{2\pi i\gamma_1}e^{-(i\alpha_2/2\gamma_1)\xi^2}, \tag{3.12}$$

on compact subsets of $\mathbf{R}$, we get

$$\lim_{n\to\infty}\mathfrak{F}_d X_n = \lim_{n\to\infty}\mathfrak{F}_d Y_n. \tag{3.13}$$

Hence,

$$\mathcal{R}\left[\frac{X_n}{\delta_n}\right] = \mathcal{R}\left[\frac{Y_n}{\gamma_n}\right]. \tag{3.14}$$

The definition is therefore well defined.

Theorem 3.2. *Let B_1 and B_2 be in $\mathfrak{B}_{L^1}$ and $\alpha \in \mathbb{C}$, then*

(i) $\mathcal{R}(\alpha B_1) = \alpha\mathcal{R}B_1$,

(ii) $\mathcal{R}(B_1 + B_2) = \mathcal{R}B_1 + \mathcal{R}B_2$,

(iii) $\mathcal{R}(B_1 * \delta_n) = \sqrt{2\pi i\gamma_1}e^{-(i\alpha_2/2\gamma_1)\xi^2}\mathcal{R}B_1 = \mathcal{R}(\delta_n * B_1)$,

(iv) *if* $\mathcal{R}B_1 = 0$, *then* $B_1 = 0$,

(v) *if* $B_n \xrightarrow{\Delta} B$ *as* $n \to \infty$ *in* $\mathcal{B}_{L^1}$, *then* $\mathcal{R}B_n \xrightarrow{\Delta} \mathcal{R}B$ *as* $n \to \infty$ *in* $\mathfrak{B}_{L^1}$ *on compact subsets.*

Proof. The proof of (i), (ii), and (iv) follows from the corresponding properties of the distributional Fresnel transform. Since each $f \in \acute{E}$ has a representative

$$f \longrightarrow \left[\frac{f * \phi_n}{\phi_n}\right], \tag{3.15}$$

in the space $\mathfrak{B}_{L^1}$, Part (iii) follows from Corollary 2.7. Finally, the proof of Part (v) is analogous to that employed for the proof of Part (f) of [17, Theorem 2]. This completes the proof of the theorem. □

Theorem 3.3. *The Fresnel transform $\mathcal{R}$ is continuous with respect to the δ-convergence.*

Proof. Let $B_n \xrightarrow{\delta} B$ in B_{L^1} as $n \to \infty$, then we show that $\mathcal{R}B_n \xrightarrow{\delta} \mathcal{R}B$ as $n \to \infty$. Using [17, Theorem 2.6], we find $[f_{n,k}/\delta_k] = B_n$ and $[f_k/\delta_k] = B$ such that $f_{n,k} \to f_k$ as $n \to \infty, k \in \mathbf{N}$. Applying the Fresnel transform for both sides implies $\widetilde{f}_{n,k} \to \widetilde{f}_k$ in the space of continuous functions. Therefore, considering limits, we get

$$\mathcal{R}\left[\frac{f_{n,k}}{\delta_k}\right] \longrightarrow \mathcal{R}\left[\frac{f_k}{\delta_k}\right]. \tag{3.16}$$

This completes the proof of the theorem. □

Theorem 3.4. *The diffraction Fresnel transform $\mathcal{R}$ is continuous with respect to the Δ-convergence.*

Proof. Let $B_n \xrightarrow{\Delta} B$ as $n \to \infty$ in $n \to \infty$, then there is $f_n \in L^1$ and $\delta_n \in \Delta$ such that

$$(B_n - B) * \delta_n = \left[\frac{f_n * \delta_n}{\delta_k}\right], \quad f_n \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Thus

$$\begin{aligned}\mathcal{R}((B_n - B) * \delta_n) &= \mathcal{R}\left[\frac{f_n * \delta_n}{\delta_k}\right] \\ &\longrightarrow \mathfrak{F}_d(f_n * \delta_n) \quad \text{as } n \to \infty \\ &\longrightarrow \sqrt{2\pi i\gamma_1} e^{-(i\alpha_2/2\gamma_1)\xi^2} \mathfrak{F}_d f_n \quad \text{as } n \longrightarrow \infty \text{ by Corollary 2.7} \\ &\longrightarrow 0 \quad \text{by the linearity of } \mathfrak{F}_d f_n.\end{aligned} \tag{3.18}$$

Therefore, $\mathcal{R}(B_n - B) \to 0$ as $n \to \infty$. Thus, $\mathcal{R}B_n \xrightarrow{\Delta} \mathcal{R}B$ as $n \to \infty$. This completes the proof. □

Lemma 3.5. *Let $[f_n/\phi_n] \in \mathcal{B}_{L^1}$ and $\phi \in \mathfrak{D}(\mathbf{R})$, then*

$$\mathcal{R}\left(\left[\frac{f_n}{\phi_n}\right] * \phi\right) = \sqrt{2\pi i\gamma_1} e^{i(2\alpha_1 t\tau - \alpha_2\xi^2)/2\gamma_1} \mathcal{R}\left[\frac{f_n}{\phi_n}\right] * \mathfrak{F}_d\phi. \tag{3.19}$$

Proof. Let $[f_n/\phi_n] \in \mathfrak{B}_{L^1}$, then using (3.9), we have

$$\mathcal{R}\left(\left[\frac{f_n}{\phi_n}\right] * \phi\right) = \mathcal{R}\left[\frac{f_n * \phi}{\phi_n}\right] = \lim_{n\to\infty} \mathfrak{F}_d(f_n * \phi), \tag{3.20}$$

on compact subsets of $\mathbf{R}$. By applying Theorem 2.6, it yields

$$\mathcal{R}\left(\left[\frac{f_n}{\phi_n}\right] * \phi\right) = \sqrt{2\pi i\gamma_1} e^{i(2\alpha_1 t\tau - \alpha_2\xi^2)/2\gamma_1} \lim_{n\to\infty} \mathfrak{F}_d(f(t);\xi)\mathfrak{F}_d(\phi(\tau);\xi). \tag{3.21}$$

Hence, $\mathcal{R}([f_n/\phi_n] * \phi) = \sqrt{2\pi i\gamma_1} e^{i(2\alpha_1 t\tau - \alpha_2\xi^2)/2\gamma_1} \mathcal{R}[f_n/\phi_n]\mathfrak{F}_d(\phi(\tau);\xi)$. This completes the proof of the lemma. □

Acknowledgments

The authors would like to thank the referee for valuable remarks and suggestions on the previous version of the paper. The second author gratefully acknowledges that this research was partially supported by the University Putra Malaysia under the Research University Grant Scheme no. 05-01-09-0720RU.

References

[1] V. Namias, "The fractional order fourier transform and its application to quantum mechanics," *IMA Journal of Applied Mathematics*, vol. 25, no. 3, pp. 241–265, 1980.

[2] D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transforms and their optical implementation: I," *Journal of the Optical Society of America A*, vol. 10, no. 9, pp. 1875–1881, 1993.

[3] H. Ozaktas and D. Mendlovic, "Fractional Fourier transforms and their optical implementation. II," *Journal of the Optical Society of America A*, vol. 10, no. 12, pp. 2522–2531, 1993.

[4] L. M. Bernardo and O. D. D. Soares, "Fractional Fourier transforms and optical systems," *Optics Communications*, vol. 110, no. 5-6, pp. 517–522, 1994.

[5] A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," *Journal of the Optical Society of America A*, vol. 10, no. 10, pp. 2181–2186, 1993.

[6] A. Kılıçman, "On the fresnel sine integral and the convolution," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 37, pp. 2327–2333, 2003.

[7] A. Kılıçman and B. Fisher, "On the fresnel integrals and the convolution," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 41, pp. 2635–2643, 2003.

[8] L. Mertz, *Transformations in Optics*, Wiley, New York, NY, USA, 1965.

[9] H. Y. Fan and H. L. Lu, "Wave-function transformations by general SU(1, 1) single-mode squeezing and analogy to fresnel transformations in wave optics," *Optics Communications*, vol. 258, no. 1, pp. 51–58, 2006.

[10] S. K. Q. Al-Omari, D. Loonker, P. K. Banerji, and S. L. Kalla, "Fourier sine (cosine) transform for ultradistributions and their extensions to tempered and ultraBoehmian spaces," *Integral Transforms and Special Functions*, vol. 19, no. 6, pp. 453–462, 2008.

[11] R. S. Pathak, *Integral Transforms of Generalized Functions and Their Applications*, Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 1997.

[12] A. H. Zemanian, *Generalized Integral Transformations*, Dover Publications, New York, NY, USA, 2nd edition, 1987.

[13] S. K. Q. Al-Omari, "The generalized stieltjes and Fourier transforms of certain spaces of generalized functions," *Jordan Journal of Mathematics and Statistics*, vol. 2, no. 2, pp. 55–66, 2009.

[14] S. K. Q. Al-Omari, "On the distributional Mellin transformation and its extension to Boehmian spaces," *International Journal of Contemporary Mathematical Sciences*, vol. 6, no. 17, pp. 801–810, 2011.

[15] S. K. Q. Al-Omari, "A Mellin transform for a space of lebesgue integrable Boehmians," *International Journal of Contemporary Mathematical Sciences*, vol. 6, no. 32, pp. 1597–1606, 2011.

[16] T. K. Boehme, "The support of Mikusinski operators," *Transactions of the American Mathematical Society*, vol. 176, pp. 319–334, 1973.

[17] P. Mikusiński, "Fourier transform for integrable Boehmians," *Rocky Mountain Journal of Mathematics*, vol. 17, no. 3, pp. 577–582, 1987.

[18] P. Mikusiński, "Convergence of Boehmians," *Japanese Journal of Mathematics*, vol. 9, no. 1, pp. 159–179, 1983.

[19] R. Roopkumar, "Mellin transform for Boehmians," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 4, no. 1, pp. 75–96, 2009.