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Research Article

A-Sequence Spaces in 2-Normed Space Defined by Ideal Convergence and an Orlicz Function

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We study some new A-sequence spaces using ideal convergence and an Orlicz function in 2-normed space and we give some relations related to these sequence spaces.

1. Introduction

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A = (a_{nk})$, $(n, k = 1, 2, \ldots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each n. If $x = (x_k) \in X \Rightarrow Ax = (A_n(x)) \in Y$ we say that A defines a (matrix) transformation from X to Y, and we denote it by $A : X \to Y$.

The notion of ideal convergence was introduced first by Kostyrko et al. [1] as a generalization of statistical convergence. More applications of ideals can be seen in [2–5].

The concept of 2-normed space was initially introduced by Gähler [6] as an interesting nonlinear generalization of a normed linear space which was subsequently studied by many authors (see, [7, 8]). Recently a lot of activities have started to study summability, sequence spaces, and related topics in these nonlinear spaces (see, [9–12]).

Let $(X, \|\cdot\|)$ be a normed space. Recall that a sequence (x_n) of elements of X is called statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

A family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if

(i) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;

(ii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$, (see [7, 13]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$, (see, [1, 3]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;

(ii) $\|x, y\| = \|y, x\|$;

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$;

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space [7]. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$:= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x_1, x_2\|_E = \text{abs}\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\right). \tag{1.1}$$

Recall that $(X, \|\cdot,\cdot\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some x in X.

Recall in [14] that an Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, convex, nondecreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently Orlicz function was used to define sequence spaces by Parashar and Choudhary [15] and others [16, 17].

If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called modulus function, which was presented and discussed by Ruckle [18] and Maddox [19]. It should be mentioned that notable works involving Orlicz function and modulus function were done in [16, 18–23].

In this article, we define some new sequence spaces in 2-normed spaces by using Orlicz function, infinite matrix, generalized difference sequences, and ideals. We introduce and examine certain new sequence spaces using the above tools as also the 2-norm.

2. Main Results

Let I be an admissible ideal of $\mathbb{N}$, M be an Orlicz function, $(X, \|\cdot,\cdot\|)$ be a 2-normed space, and $A = (a_{n,k})$ be a nonnegative matrix method. Further, let $p = (p_k)$ be a bounded sequence of positive real numbers. By $S(2-X)$, we denote the space of all sequences defined over $(X,\|\cdot,\cdot\|)$. Now we define the following sequence spaces:

$$\begin{aligned}
&W^I(M,\Delta^m,p,\|,\cdot,\|)\\
&\quad=\left\{\begin{array}{c}x\in S(2-X):\forall\varepsilon>0\quad\left\{n\in\mathbb{N}:\displaystyle\sum_{k=1}^{\infty}a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k-L}{\rho},z\right\|\right)\right]^{p_k}\geq\varepsilon\right\}\in I\\ \text{for some }\rho>0,\ L\in X\text{ and each }z\in X\end{array}\right\},\\
&W_0^I(A,M,\Delta^m,p,\|,\cdot,\|)\\
&\quad=\left\{\begin{array}{c}x\in S(2-X):\forall\varepsilon>0\quad\left\{n\in\mathbb{N}:\displaystyle\sum_{k=1}^{\infty}a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k}{\rho},z\right\|\right)\right]^{p_k}\geq\varepsilon\right\}\in I\\ \text{for some }\rho>0,\text{ and each }z\in X\end{array}\right\},\\
&W_\infty(A,M,\Delta^m,p,\|,\cdot,\|)\\
&\quad=\left\{\begin{array}{c}x\in S(2-X):\exists K>0\text{ s.t. }\displaystyle\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k}{\rho},z\right\|\right)\right]^{p_k}\leq K\\ \text{for some }\rho>0,\text{ and each }z\in X\end{array}\right\},\\
&W_\infty^I(A,M,\Delta^m,p,\|,\cdot,\|)\\
&\quad=\left\{x\in S(2-X):\exists K>0,\text{ s.t.}\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k}{\rho},z\right\|\right)\right]^{p_k}\geq K\right\}\right.\\
&\qquad\left.\in I\text{ for some }\rho>0,\text{and each }z\in X\right\},
\end{aligned}\tag{2.1}$$

where $\Delta^m x_k=\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}$.

Let us consider a few special cases of the above sets.

(1) If $M(x)=x$, for all $x\in[0,\infty)$, then the above classes of sequences are denoted by $W^I(A,\Delta^m,p,\|,\cdot,\|)$, $W_0^I(A,\Delta^m,p,\|,\cdot,\|)$, $W_\infty(A,\Delta^m,p,\|,\cdot,\|)$, and $W_\infty^I(A,\Delta^m,p,\|,\cdot,\|)$, respectively.

(2) If $p_k=1$ for all $k\in N$, then we denote the above classes of sequences by $W^I(A,M,\Delta^m,\|,\cdot,\|)$, $W_0^I(A,\Delta^m,\|,\cdot,\|)$, $W_\infty(A,\Delta^m,\|,\cdot,\|)$, and $W_\infty^I(A,\Delta^m,\|,\cdot,\|)$, respectively.

(3) If $M(x)=x$, for all $x\in[0,\infty)$, and $p_k=1$ for all $k\in N$, then we denote the above spaces by $W^I(A,\Delta^m,\|,\cdot,\|)$, $W_0^I(A,\Delta^m,\|,\cdot,\|)$, $W_\infty(A,\Delta^m,\|,\cdot,\|)$, and $W_\infty^I(A,\Delta^m,\|,\cdot,\|)$, respectively.

(4) If we take $A=(a_{nk})$ as

$$a_{nk}=\begin{cases}\dfrac{1}{n}, & \text{if } n\geq k,\\ 0, & \text{otherwise,}\end{cases}\tag{2.2}$$

then the above classes of sequences are denoted by $W^I(C, M, \Delta^m, p, \|, \cdot, \|)$, $W_0^I(C, M, \Delta^m, p, \|, \cdot, \|)$, $W_\infty(C, M, \Delta^m, p, \|, \cdot, \|)$, and $W_\infty^I(C, M, \Delta^m, p, \|, \cdot, \|)$ respectively, which were defined and studied by Savaş [24]

(5) If we take $A = (a_{nk})$ is a de la Vallée poussin mean, that is,

$$a_{nk} = \begin{cases} \dfrac{1}{\lambda_n}, & \text{if } k \in I_n = [n - \lambda_n + 1, n], \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

where (λ_n) is a nondecreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, then the above classes of sequences are denoted by $W^I(M, \Delta^m, \lambda, p, \|, \cdot, \|)$, $W_0^I(M, \Delta^m, \lambda, p, \|, \cdot, \|)$, $W_\infty(M, \Delta^m, \lambda, p, \|, \cdot, \|)$, and $W_\infty^I(M, \Delta^m, \lambda, p, \|, \cdot, \|)$.

(6) By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$ where $k_0 = 0$, we will mean an increasing sequence of nonnegative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. As a final illustration let

$$a_{nk} = \begin{cases} \dfrac{1}{h_r}, & \text{if } k_{r-1} < k \leq k_r, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then we denote the above classes of sequences by $W^I(M, \Delta^m, \theta, p, \|, \cdot, \|)$, $W_0^I(M, \Delta^m, \theta, p, \|, \cdot, \|)$, $W_\infty(M, \Delta^m, \theta, p, \|, \cdot, \|)$, and $W_\infty^I(M, \Delta^m, \theta, p, \|, \cdot, \|)$.

The following well-known inequality (see [25, p. 190]) will be used in the study.

If

$$0 \leq p_k \leq \sup p_k = H, \qquad D = \max\left(1, 2^{H-1}\right), \tag{2.5}$$

then

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}, \tag{2.6}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

Theorem 2.1. *$W^I(A, M, \Delta^m, p, \|, \cdot, \|)$, $W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$, and $W_\infty^I(A, M, \Delta^m, p, \|, \cdot, \|)$ are linear spaces.*

Proof. We will prove the assertion for $W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$ only, and the others can be proved similarly. Assume that $x, y \in W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$ and $\alpha, \beta \in \mathbb{R}$. In order to prove the result we need to find some ρ_3 such that

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\alpha\Delta^m x_k + \beta\Delta^m x_k}{\rho_3}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \in I \quad \text{for some } \rho_3 > 0. \tag{2.7}$$

Since $x, y \in W_0^I(A, M, \Delta^m, p, \|,\cdot,\|)$, there exist some positive ρ_1 and ρ_2 such that

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m x_k}{\rho_1}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \in I \quad \text{for some } \rho_1 > 0, \\
&\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m x_k}{\rho_2}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \in I \quad \text{for some } \rho_2 > 0.
\end{aligned} \tag{2.8}$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is nondecreasing and convex and also $\|,\cdot,\|$ is a 2-norm, Δ^m is linear

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m(\alpha x_k + \beta y_k)}{\rho_3}, z\right\|\right)\right]^{p_k} &\leq \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\alpha \Delta^m x_k}{\rho_3}, z\right\| + \left\|\frac{\beta \Delta^m x_k}{\rho_3}, z\right\|\right)\right]^{p_k} \\
&\leq \sum_{k=1}^{\infty} a_{nk} \frac{1}{2^{p_k}} \left[M\left(\left\|\frac{\Delta^m x_k}{\rho_1}, z\right\| + \left\|\frac{\Delta^m x_k}{\rho_2}, z\right\|\right)\right]^{p_k} \\
&\leq \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m x_k}{\rho_1}, z\right\| + \left\|\frac{\Delta^m x_k}{\rho_2}, z\right\|\right)\right]^{p_k} \\
&\leq D \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m x_k}{\rho_1}, z\right\|\right)\right]^{p_k} \\
&\quad + D \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m x_k}{\rho_2}, z\right\|\right)\right]^{p_k},
\end{aligned} \tag{2.9}$$

where $D = \max(1, 2^{H-1})$. From the above inequality we get

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m(\alpha x_k + \beta y_k)}{\rho_3}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \\
&\quad \subseteq \left\{n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m x_k}{\rho_1}, z\right\|\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\} \\
&\quad \cup \left\{n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\|\frac{\Delta^m y_k}{\rho_2}, z\right\|\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}.
\end{aligned} \tag{2.10}$$

Two sets on the right-hand side belong to I, and this completes the proof. □

It is also easy to verify that the space $W_\infty(A, M, \Delta^m, p, \|,\cdot,\|)$ is also a linear space and moreover we have the following.

Theorem 2.2. *For any fixed $n \in \mathbb{N}$, $W_\infty(A, M, \Delta^m, p, \|,\cdot,\|)$ is paranormed space with respect to the paranorm defined by*

$$g_n(x) = \inf_{z \in X} \left\{ \rho^{p_n/H} : \left(\sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^m x_k}{\rho}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1,\ \forall z \in X \right\}. \tag{2.11}$$

Proof. The proof is parallel to the proof of the Theorem 2 in [24] and so is omitted. □

Theorem 2.3. *Let $X(A, \Delta^{m-1})$ stand for $W_0^I(A, \Delta^{m-1}, M, p, \|,\cdot,\|)$, $W^I(A, \Delta^{m-1}, M, p, \|,\cdot,\|)$, or $W_\infty^I(A, \Delta^{m-1}, M, p, \|,\cdot,\|)$ and $m \geq 1$. Then the inclusion $X(A, \Delta^{m-1}) \subset X(A, \Delta^m)$ is strict. In general $X(A, \Delta^i) \subset X(A, \Delta^m)$ for all $i = 1, 2, 3, \ldots, m-1$ and the inclusion is strict.*

Proof. We shall give the proof for $W_0^I(A, \Delta^{m-1}, M, p, \|,\cdot,\|)$ only. It can be proved in a similar way for $W_\infty^I(A, \Delta^{m-1}, M, p, \|,\cdot,\|)$, and $W^I(A, \Delta^{m-1}, M, p, \|,\cdot,\|)$. Let $x = (x_k) \in W_0^I(A, \Delta^{m-1}, M, p, \|,\cdot,\|)$. Then given $\varepsilon > 0$ we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^{m-1} x_k}{\rho}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I \quad \text{for some } \rho > 0. \tag{2.12}$$

Since M is nondecreasing and convex it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^m x_k}{2\rho}, z \right\| \right) \right]^{p_k} \\ &\quad = \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^{m-1} x_{k+1} - \Delta^{m-1} x_k}{2\rho}, z \right\| \right) \right]^{p_k} \\ &\quad \leq D \sum_{k=1}^{\infty} a_{nk} \left(\left[\frac{1}{2} M\left(\left\| \frac{\Delta^{m-1} x_{k+1}}{\rho}, z \right\| \right) \right]^{p_k} + \left[\frac{1}{2} M\left(\left\| \frac{\Delta^{m-1} x_k}{\rho}, z \right\| \right) \right]^{p_k} \right) \\ &\quad \leq D \sum_{k=1}^{\infty} a_{nk} \left(\left[M\left(\left\| \frac{\Delta^{m-1} x_{k+1}}{\rho}, z \right\| \right) \right]^{p_k} + \left[M\left(\left\| \frac{\Delta^{m-1} x_k}{\rho}, z \right\| \right) \right]^{p_k} \right). \end{aligned} \tag{2.13}$$

Hence we have

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^m x_k}{2\rho}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\quad \subseteq \left\{ n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^{m-1} x_{k+1}}{\rho}, z \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \\ &\quad \cup \left\{ n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^{m-1} x_k}{\rho}, z \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}. \end{aligned} \tag{2.14}$$

Since the set on the right hand side belongs to I, so does the left hand side. The inclusion is strict as the sequence $x = (k^r)$, for example, belongs to $W_0^I(\Delta^m, M, \|, \cdot, \|)$ but does not belong to $W_0^I(\Delta^{m-1}, M, \|, \cdot, \|)$ for $M(x) = x$, $A = (a_{nk}) = (C, 1)$ Cesàro matrix and $p_k = 1$ for all k. □

Theorem 2.4. (i) *Let* $0 < \inf p_k \le p_k \le 1$. *Then* $W^I(A, \Delta^m, M, p, \|, \cdot, \|) \subset W^I(A, \Delta^m, M, \|, \cdot, \|)$.

(ii) $1 < p_k \le \sup p_k \le \infty$. *Then* $W^I(A, \Delta^m, M, \|, \cdot, \|) \subset W^I(A, \Delta^m, M, p\|, \cdot, \|)$.

Proof. (i) Let $(x_k) \in W^I(A, M, \Delta^m, p, \|, \cdot, \|)$. Since $0 < \inf p_k \le p_k \le 1$, we have

$$\sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right] \le \sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right]^{p_k}. \tag{2.15}$$

So

$$\begin{aligned}&\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right] \ge \varepsilon\right\}\\ &\quad\subseteq \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right]^{p_k} \ge \varepsilon\right\} \in I.\end{aligned} \tag{2.16}$$

(ii) Let $p_k \ge 1$ for each k, and $\sup p_k \le \infty$. Let $(x_k) \in W^I(A, M, \Delta^m, p, \|, \cdot, \|)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer N such that

$$\sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right] \le \varepsilon < 1, \tag{2.17}$$

for all $n \ge N$. This implies that

$$\sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right]^{p_k} \le \sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right]. \tag{2.18}$$

So we have

$$\begin{aligned}&\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\left[M\left(\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right]^{p_k} \ge \varepsilon\right\}\\ &\quad\subseteq \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\left[\left(M\left\|\frac{\Delta^m x_k - L}{\rho}, z\right\|\right)\right] \ge \varepsilon\right\} \in I.\end{aligned} \tag{2.19}$$

This completes the proof. □

The following corollary follows immediately from the above theorem.

Corollary 2.5. *Let* $A = (C, 1)$ *Cesàro matrix and let* M *be an Orlicz function.*

(1) *If* $0 < \inf p_k \le p_k < 1$, *then* $W^I(\Delta^m, M, p, \|, \cdot, \|) \subset W^I(\Delta^m, M, \|, \cdot, \|)$.

(2) *If* $1 \le p_k \le \sup p_k < \infty$, *then* $W^I(\Delta^m, M, \|, \cdot, \|) \subset W^I(\Delta^m, M, p\|, \cdot, \|)$.

Definition 2.6. Let X be a sequence space. Then X is called solid if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.

Theorem 2.7. *The sequence spaces $W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$ and $W_\infty^I(A, M, \Delta^m, p, \|, \cdot, \|)$ are solid.*

Proof. We give the proof for $W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$ only. Let $(x_k) \in W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$, and let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then we have

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\left\| \frac{\Delta^m(\alpha_k x_k)}{\rho}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\quad \subseteq \left\{ n \in \mathbb{N} : C \sum_{k=1}^{\infty} a_{nk} \left[\left(M \left\| \frac{\Delta^m x_k}{\rho}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \end{aligned} \tag{2.20}$$

where $C = \max_k \{1, |\alpha_k|^H\}$. Hence $(\alpha_k x_k) \in W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$ whenever $(x_k) \in W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$. □

Remark 2.8. In general it is difficult to predict the solidity of $W_0^I(A, M, \Delta^m, p, \|, \cdot, \|)$ and $W_\infty^I(A, M, \Delta^m, p, \|, \cdot, \|)$ when $m > 0$. For this, consider the following example.

Example 2.9. Let $m = 2$, $p_k = 1$ for all k, $A = (C, 1)$ Cesàro matrix and $M(x) = x$. Then $(x_k) = (k) \in W_0^I(M, \Delta^2, p, \|, \cdot, \|)$ but $(\alpha_k x_k) \notin W_0^I(M, \Delta^2, p, \|, \cdot, \|)$ when $\alpha_k = (-1)^k$ for all $k \in N$. Hence $W_0^I(M, \Delta^2, p, \|, \cdot, \|)$ is not solid.

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