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Research Article

Weak Convergence of the Projection Type Ishikawa Iteration Scheme for Two Asymptotically Nonexpansive Nonself-Mappings

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We study weak convergence of the projection type Ishikawa iteration scheme for two asymptotically nonexpansive nonself-mappings in a real uniformly convex Banach space E which has a Fréchet differentiable norm or its dual E^* has the Kadec-Klee property. Moreover, weak convergence of projection type Ishikawa iterates of two asymptotically nonexpansive nonself-mappings without any condition on the rate of convergence associated with the two maps in a uniformly convex Banach space is established. Weak convergence theorem without making use of any of the Opial's condition, Kadec-Klee property, or Fréchet differentiable norm is proved. Some results have been obtained which generalize and unify many important known results in recent literature.

1. Introduction and Preliminaries

Let C be a nonempty closed convex subset of real normed linear space X. Let $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of T if and only if $Tx = x$. The set of all fixed points of a mapping T is denoted by $F(T)$. A self-mapping $T : C \to C$ is said to be nonexpansive if $\|T(x) - T(y)\| \le \|x - y\|$ for all $x, y \in C$. A self-mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\left\|T^n(x) - T^n(y)\right\| \le k_n\left\|x - y\right\| \tag{1.1}$$

for all $x, y \in C$ and $n \ge 1$. A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\left\|T^n(x) - T^n(y)\right\| \le L\left\|x - y\right\| \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$. T is uniformly Hölder continuous if there exist positive constants L and α such that

$$\left\| T^n(x) - T^n(y) \right\| < L \|x - y\|^{\alpha} \tag{1.3}$$

for all $x, y \in C$ and $n \geq 1$. T is termed as uniformly equicontinuous if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| T^n(x) - T^n(y) \right\| \leq \varepsilon \tag{1.4}$$

whenever $\|x - y\| \leq \delta$ for all $x, y \in C$ and $n \geq 1$ or, equivalently, T is uniformly equicontinuous if and only if

$$\left\| T^n(x_n) - T^n(y_n) \right\| \longrightarrow 0 \tag{1.5}$$

whenever $\|x_n - y_n\| \to 0$ as $n \to \infty$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

Remark 1.1. It is clear that asymptotically nonexpansiveness $\Rightarrow$ uniformly L-Lipschitz $\Rightarrow$ uniformly Hölder continuous $\Rightarrow$ uniformly equicontinuous.

However, their converse fail in the presence of the following example.

Example 1.2 (see [1]). Define $T : [0,1] \to [0,1]$ by $Tx = (1 - x^{3/2})^{2/3}$ for all $x \in [0,1]$.

Fixed-point iteration process for nonexpansive self-mappings including Mann and Ishikawa iteration processes has been studied extensively by various authors [2–8]. For nonexpansive nonself-mappings, some authors (see [9–13]) have studied the strong and weak convergence theorems in Hilbert spaces or uniformly convex Banach spaces.

In [14], Tan and Xu introduced a modified Ishikawa iteration process:

$$x_{n+1} = (1 - b_n)x_n + b_n T\big((1 - \gamma_n)x_n + \gamma_n T x_n\big), \quad n \geq 1, \tag{1.6}$$

to approximate fixed points of nonexpansive self-mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space X. The mapping T remains self-mapping of a nonempty closed convex subset C of a uniformly convex Banach space. If, however, the domain C of T is a proper subset of X (and this is the case in several applications) and T maps C into X then, the sequence $\{x_n\}$ generated by (1.6) may not be well defined. More precisely, Tan and Xu [14] proved weak convergence of the sequences generated by (1.6) to some fixed point of T in a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm.

Note that each l^p $(1 \leq p < \infty)$ satisfies Opial's condition, while all L^p do not have the property unless $p = 2$ and the dual of reflexive Banach spaces with a Fréchet differentiable norm has the Kadec-Klee property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial property; however, their dual does have the Kadec-Klee property (see [15, 16]).

In 2005, Shahzad [11] extended Tan and Xu's result [14] to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. He studied weak convergence of the modified Ishikawa type iteration process:

$$x_{n+1} = P\big((1-b_n)x_n + b_n TP\big((1-\gamma_n)x_n + \gamma_n Tx_n\big)\big), \quad n \geq 1, \tag{1.7}$$

in a uniformly convex Banach space whose dual has the Kadec-Klee property. The result applies not only to L^p spaces with $(1 \leq p < \infty)$ but also to other spaces which do not satisfy Opial's condition or have a Fréchet differentiable norm. Meanwhile, the results of [11] generalized the results of [14].

The class of asymptotically nonexpansive self-mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [17] proved that if C is a nonempty closed convex and bounded subset of a real uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

In 1991, the modified Mann iteration which was introduced by Schu [18] generates a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$ and $T : C \to C$ is an asymptotically nonexpansive mapping. To be more precise, Schu [18] obtained the following weak convergence result for an asymptotically nonexpansive mapping in a uniformly convex Banach space which satisfies Opial's condition.

Theorem 1.3 (see [18]). *Let X be a uniformly convex Banach space satisfying Opial's condition, $\emptyset \neq C \subset X$ closed bounded and convex, and $T : C \to C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1,\infty)$ for which $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\} \in [0,1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in* (1.8). *Then, the sequence $\{x_n\}$ converges weakly to some fixed point of T.*

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces (see [6, 14, 19, 20]).

In 1994, Tan and Xu [21] obtained the following results.

Theorem 1.4 (see [21]). *Let X be a uniformly convex Banach space whose norm is Fréchet differentiable, C a nonempty closed and convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ such that $F(T)$ is nonempty. Let $\{x_n\}$ be sequence generated in* (1.8), *where $\{\alpha_n\}$ is a real sequence bounded away from 0 and 1. Then, the sequence $\{x_n\}$ converges weakly to some point in $F(T)$.*

In 2001, Khan and Takahashi [22] constructed and studied the following Ishikawa iteration process:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2^n x_n, \quad n \geq 1, \end{aligned} \tag{1.9}$$

where T_1, T_2 are asymptotically nonexpansive self-mappings on C with $\sum_{n=1}^{\infty}(k_n-1)<\infty$ (rate of convergence) and $0\le\alpha_n,\beta_n\le 1$.

Note that the rate of convergence condition, namely, $\sum_{n=1}^{\infty}(k_n-1)<\infty$ has remained in extensive use to prove both weak and strong convergence theorems to approximate fixed points of asymptotically nonexpansive maps. The conditions like Opial's condition, Kadec-Klee property, or Fréchet differentiable norm have remained key to prove weak convergence theorems.

In 2010, Khan and Fukhar-Ud-Din [23] established weak convergence of Ishikawa iterates of two asymptotically nonexpansive self-mappings without any condition on the rate of convergence associated with the two mappings. They got that the following new weak convergence theorem does not require any of Opial's condition, Kadec-Klee property or Fréchet differentiable norm.

Theorem 1.5 (see [23]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $T_1,T_2: C\to C$ be asymptotically nonexpansive maps with sequences $\{k_n\},\{l_n\}\subset[1,\infty)$ such that $\lim_{n\to\infty}k_n=1$, $\lim_{n\to\infty}l_n=1$, respectively. Let the sequence $\{x_n\}$ be as in* (1.9) *with $\delta\le\alpha_n,\beta_n\le 1-\delta$. for some $\delta\in(0,1/2)$. If $F(T_1)\cap F(T_2)\neq\emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume et al. [24] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows.

Definition 1.6 (see [24]). Let C be a nonempty subset of a real normed linear space X. Let $P: X\to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T: C\to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}\subset[1,\infty)$, $k_n\to 1$ as $n\to\infty$ such that

$$\left\|T(PT)^{n-1}x-T(PT)^{n-1}y\right\|\le k_n\|x-y\| \tag{1.10}$$

for all $x,y\in C$ and $n\ge 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L>0$ such that

$$\left\|T(PT)^{n-1}x-T(PT)^{n-1}y\right\|\le L\|x-y\| \tag{1.11}$$

for all $x,y\in C$ and $n\ge 1$.

By studying the following iteration process:

$$x_1\in C,\qquad x_{n+1}=P\Big((1-\alpha_n)x_n+\alpha_nT(PT)^{n-1}x_n\Big), \tag{1.12}$$

Chidume et al. [24] got the following weak convergence theorem for asymptotically nonexpansive nonself-mapping.

Theorem 1.7 (see [24]). *Let X be a real uniformly convex Banach space which has a Fréchet differentiable norm and C a nonempty closed convex subset of X. Let $T: C\to X$ be an asymptotically nonexpansive map with sequence $\{k_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$ and $F(T)\neq\emptyset$. Let*

$\{\alpha_n\} \subset (0,1)$ *be such that* $\epsilon \le 1-\alpha_n \le 1-\epsilon$, *for all* $n \ge 1$ *and some* $\epsilon > 0$. *From an arbitrary* $x_1 \in C$, *define the sequence* $\{x_n\}$ *by* (1.12). *Then,* $\{x_n\}$ *converges weakly to some fixed point of* T.

If T is a self-mapping, then P becomes the identity mapping so that (1.10) and (1.11) reduce to (1.1) and (1.2), respectively. Equation (1.12) reduces to (1.8).

In 2006, Wang [25] generalizes the iteration process (1.12) as follows: $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= P\Big((1-\alpha_n)x_n + \alpha_n T_1 (PT_1)^{n-1} y_n\Big), \\ y_n &= P\Big((1-\beta_n)x_n + \beta_n T_2 (PT_2)^{n-1} x_n\Big), \quad n \ge 1, \end{aligned} \tag{1.13}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$. He studied the strong and weak convergence of the iterative scheme (1.13) under proper conditions. Meanwhile, the results of [25] generalized the results of [24].

Recently, an iterative scheme which is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings was defined and constructed by Thianwan [26]. It is given as follows:

$$\begin{aligned} x_{n+1} &= P\Big((1-\alpha_n)y_n + \alpha_n T_1 (PT_1)^{n-1} y_n\Big), \\ y_n &= P\Big((1-\beta_n)x_n + \beta_n T_2 (PT_2)^{n-1} x_n\Big), \quad n \ge 1, \end{aligned} \tag{1.14}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0,1)$.

In [26], Thianwan gave the following weak convergence theorem.

Theorem 1.8. *Let* X *be a uniformly convex Banach space which satisfies Opial's condition and* C *a nonempty closed convex nonexpansive retract of* X *with* P *as a nonexpansive retraction. Let* $T_1, T_2 : C \to X$ *be two asymptotically nonexpansive nonself-mappings of* C *with sequences* $\{k_n\}, \{l_n\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n-1) < \infty$, $\sum_{n=1}^{\infty}(l_n-1) < \infty$, *respectively, and* $F(T_1) \cap F(T_2) \ne \emptyset$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[\epsilon, 1-\epsilon]$ *for some* $\epsilon \in (0,1)$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by* (1.14). *Then,* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to a common fixed point of* T_1 *and* T_2.

The iterative schemes (1.14) and (1.13) are independent: neither reduces to the other. If $T_1 = T_2$ and $\beta_n = 0$ for all $n \ge 1$, then (1.14) reduces to (1.12). It also can be reduces to Schu process (1.8).

Inspired and motivated by the recent works, we prove some new weak convergence theorems of the sequences generated by the projection type Ishikawa iteration scheme (1.14) for two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces.

Now, we recall some well-known concepts and results.

Let X be a Banach space with dimension $X \ge 2$. The modulus of X is the function $\delta_X : (0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x-y\|\right\}. \tag{1.15}$$

Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

Recall that a Banach space X is said to satisfy *Opial's condition* [27] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|. \tag{1.16}$$

The norm of X is said to be Fréchet differentiable if for each $x \in X$ with $\|x\| = 1$ the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{1.17}$$

exists and is attained uniformly for y, with $\|y\| = 1$. In the case of Fréchet differentiable norm, it has been obtained in [21] that

$$\begin{aligned}\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 &\leq \frac{1}{2}\|x+h\|^2 \\ &\leq \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|)\end{aligned} \tag{1.18}$$

for all x, h in E, where J is the normalized duality map from E to E^* defined by

$$J(x) = \left\{x^* \in E^* : \langle x, s^*\rangle = \|x\|^2 = \|x^*\|^2\right\}, \tag{1.19}$$

$\langle\cdot,\cdot\rangle$ is the duality pairing between E and E^* and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0} b(t)/t = 0$.

A subset C of X is said to be retract if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P. A set C is optimal if each point outside C can be moved to be closer to all points of C. It is well known (see [28]) that

(1) if X is a separable, strictly convex, smooth, reflexive Banach space, and if $C \subset X$ is an optimal set with interior, then C is a nonexpansive retract of X;

(2) a subset of l^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. Moreover, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Recall that weak convergence is defined in terms of bounded linear functionals on X as follows.

A sequence $\{x_n\}$ in a normed space X is said to be weakly convergent if there is an $x \in X$ such that $\lim_{n\to\infty} f(x_n) = f(x)$ for every bounded linear functional f on X. The element x is called the weak limit of $\{x_n\}$, and we say that $\{x_n\}$ converges weakly to x. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

A Banach space X is said to have the Kadec-Klee property if, for every sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$; for more details on Kadec-Klee property, the reader is referred to [29, 30] and the references therein.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.9 (see [31]). *Let $p > 1$, $r > 0$ be two fixed numbers. Then, a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|) \tag{1.20}$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0,1]$, where

$$w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda). \tag{1.21}$$

Lemma 1.10 (see [24]). *Let X be a uniformly convex Banach space and C a nonempty closed convex subset of X, and let $T : C \to X$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then, $I - T$ is demiclosed at zero; that is, if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$.*

Lemma 1.11 (see [26]). *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.14). If $q \in F(T_1) \cap F(T_2)$, then $\lim_{n\to\infty}\|x_n - q\|$ exists.*

Lemma 1.12 (see [26]). *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.14). Then, $\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = 0$.*

Lemma 1.13 (see [16]). *Let X be a real reflexive Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $x^*, y^* \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$. Suppose that $\lim_{n\to\infty}\|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0, 1]$. Then, $x^* = y^*$.*

We denote by Γ the set of strictly increasing, continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. Let C be a convex subset of the Banach space X. A mapping $T : C \to C$ is said to be type (γ) [32] if $\gamma \in \Gamma$ and $0 \leqslant \alpha \leqslant 1$,

$$\gamma(\|\alpha Tx + (1-\alpha)Ty - T(\alpha x + (1-\alpha)y)\|) \leqslant \|x-y\| - \|Tx - Ty\| \tag{1.22}$$

for all x, y in C. Obviously, every type (γ) mapping is nonexpansive. For more information about mappings of type (γ), see [33–35].

Lemma 1.14 (see [36, 37]). *Let X be a uniformly convex Banach space and C a convex subset of X. Then, there exists $\gamma \in \Gamma$ such that for each mapping $S : C \to C$ with* Lipschitz constant L,

$$\left\| \alpha Sx + (1-\alpha)Sy - S\left(\alpha x + (1-\alpha)y\right) \right\| \leqslant L\gamma^{-1}\left(\|x-y\| - \frac{1}{L}\|Sx - Sy\| \right) \tag{1.23}$$

for all $x, y \in C$ and $0 < \alpha < 1$.

2. Main Results

In this section, we prove weak convergence theorems of the projection type Ishikawa iteration scheme (1.14) for two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces.

Firstly, we deal with the weak convergence of the sequence $\{x_n\}$ defined by (1.14) in a real uniformly convex Banach space X whose dual X^* has the Kadec-Klee property. In order to prove our main results, the following lemma is needed.

Lemma 2.1. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by* (1.14)*. Then, for all $u, v \in F(T_1) \cap F(T_2)$, the limit $\lim_{n\to\infty} \|tx_n - (1-t)u - v\|$ exists for all $t \in [0,1]$.*

Proof. It follows from Lemma 1.11 that the sequence $\{x_n\}$ is bounded. Then, there exists $R > 0$ such that $\{x_n\} \subset B_R(0) \cap C$. Let $a_n(t) := \|tx_n + (1-t)u - v\|$ where $t \in [0,1]$. Then, $\lim_{n\to\infty} a_n(0) = \|u - v\|$ and, by Lemma 1.11, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - v\|$ exists. Without loss of the generality, we may assume that $\lim_{n\to\infty} \|x_n - v\| = r$ for some positive number r. Let $x \in C$ and $t \in (0,1)$. For each $n \geq 1$, define $A_n : C \to C$ by

$$A_n x = P\left((1-\alpha_n) y_n(x) + \alpha_n T_1 (PT_1)^{n-1} y_n(x) \right), \tag{2.1}$$

where

$$y_n(x) = P\left((1-\beta_n) x + \beta_n T_2 (PT_2)^{n-1} x \right). \tag{2.2}$$

Setting $k_n = 1 + s_n$ and $l_n = 1 + t_n$. For $x, z \in C$, we have

$$\begin{aligned}
\|A_n x - A_n z\| &= \Big\| P\left((1-\alpha_n) y_n(x) + \alpha_n T_1 (PT_1)^{n-1} y_n(x) \right) \\
&\qquad - P\left((1-\alpha_n) y_n(z) + \alpha_n T_1 (PT_1)^{n-1} y_n(z) \right) \Big\| \\
&\leq \Big\| (1-\alpha_n)\left(y_n(x) - y_n(z) \right) \\
&\qquad + \alpha_n \left(T_1 (PT_1)^{n-1} y_n(x) - T_1 (PT_1)^{n-1} y_n(z) \right) \Big\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\left\|y_n(x)-y_n(z)\right\| + \alpha_n k_n \left\|y_n(x)-y_n(z)\right\| \\
&\leq (1-\alpha_n)\left\|(1-\beta_n)(x-z) + \beta_n T_2 (PT_2)^{n-1}(x-z)\right\| \\
&\quad + \alpha_n k_n \left\|(1-\beta_n)(x-z) + \beta_n T_2 (PT_2)^{n-1}(x-z)\right\| \\
&\leq (1-\alpha_n)(1-\beta_n)\|x-z\| + (1-\alpha_n)\beta_n l_n \|x-z\| \\
&\quad + \alpha_n k_n (1-\beta_n)\|x-z\| + \alpha_n \beta_n k_n l_n \|x-z\| \\
&= \left(1-\alpha_n-\beta_n+\alpha_n\beta_n\right)\|x-z\| + (1-\alpha_n)\beta_n(1+t_n)\|x-z\| \\
&\quad + \alpha_n(1+s_n)(1-\beta_n)\|x-z\| + \alpha_n\beta_n(1+s_n)(1+t_n)\|x-z\| \\
&= \|x-z\| + \beta_n t_n \|x-z\| + \alpha_n s_n \|x-z\| + \alpha_n \beta_n t_n s_n \|x-z\| \\
&\leq (1+t_n+s_n+t_n s_n)\|x-z\|.
\end{aligned}
\tag{2.3}
$$

Set $S_{n,m} := A_{n+m-1}A_{n+m-2}\cdots A_n$, $n, m \geq 1$ and $b_{n,m} = \|S_{n,m}(tx_n + (1-t)u) - (tS_{n,m}x_n + (1-t)u)\|$, where $0 \leq t \leq 1$. Also,

$$
\begin{aligned}
\left\|S_{n,m}x - S_{n,m}y\right\| &\leq \left\|A_{n+m-1}(A_{n+m-2}\cdots A_n x) - A_{n+m-1}\left(A_{n+m-2}\cdots A_n y\right)\right\| \\
&\leq (1+t_{n+m-1}+s_{n+m-1}+t_{n+m-1}s_{n+m-1}) \\
&\quad \left\|A_{n+m-2}(A_{n+m-3}\cdots A_n x) - A_{n+m-2}\left(A_{n+m-3}\cdots A_n y\right)\right\| \\
&\quad \vdots \\
&\leq \prod_{j=n}^{n+m-1}\left(1+t_j+s_j+t_j s_j\right)\left\|x-y\right\|
\end{aligned}
\tag{2.4}
$$

for all $x, y \in C$ and $S_{n,m}\, x_n = x_{n+m}$, $S_{n,m}\, x^* = x^*$ for all $x^* \in F(T_1) \cap F(T_2)$.

Applying Lemma 1.14 with $x = x_n$, $y = u$, $S = S_{n,m}$ and using the facts that $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(l_n - 1) = 0$, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(k_n - 1) = 0$, and $\lim_{n\to\infty}\|x_n - x^*\|$ exist for all $x^* \in F(T_1) \cap F(T_2)$, we obtain $\lim_{n\to\infty} b_{n,m} = 0$. Observe that

$$
\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)u - v\| \\
&= \|tS_{n,m}x_n + (1-t)u - S_{n,m}v\| \\
&= \|S_{n,m}v - (tS_{n,m}x_n + (1-t)u)\| \\
&= \|S_{n,m}v - S_{n,m}(tx_n + (1-t)u) + S_{n,m}(tx_n + (1-t)u) \\
&\quad - (tS_{n,m}x_n + (1-t)u)\| \\
&\leq \|S_{n,m}v - S_{n,m}(tx_n + (1-t)u)\| + b_{n,m}
\end{aligned}
$$

$$
\begin{aligned}
&= \|S_{n,m}(tx_n + (1-t)u) - S_{n,m}v\| + b_{n,m} \\
&\leq \prod_{j=n}^{n+m-1} \left(1 + t_j + s_j + t_j s_j\right) \|tx_n + (1-t)u - v\| + b_{n,m} \\
&\leq \prod_{j=n}^{\infty} \left(1 + t_j + s_j + t_j s_j\right) a_n(t) + b_{n,m}.
\end{aligned}
\tag{2.5}
$$

Consequently,

$$
\limsup_{m\to\infty} a_m(t) = \limsup_{m\to\infty} a_{n+m}(t) \leq \limsup_{m\to\infty} \left(b_{n,m} + \prod_{j=n}^{\infty} \left(1 + t_j + s_j + t_j s_j\right) a_n(t) \right), \tag{2.6}
$$

$$
\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t). \tag{2.7}
$$

This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0,1]$. This completes the proof. □

Theorem 2.2. *Let X be a real uniformly convex Banach space which has a Fréchet differentiable norm and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty, \sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by* (1.14). *Then, $\{x_n\}$ converges weakly to a fixed point of T_1 and T_2.*

Proof. Set $x = p_1 - p_2$ and $h = t(x_n - p_1)$ in (1.18). By using Lemmas 1.11, and 2.1 and the same proof of Lemma 4 of Osilike and Udomene [7], we can show that, for every $p_1, p_2 \in F(T_1) \cap F(T_2)$,

$$
\langle p - q, J(p_1 - p_2) \rangle = 0, \tag{2.8}
$$

for all $p, q \in \omega_w(x_n)$. Since E is reflexive and $\{x_n\}$ is bounded, we from Lemma 1.13 conclude that $\omega_w(x_n) \subset F(T_i)$ for each $i = 1, 2$. Let $p, q \in \omega_w(x_n)$. It follows that $p, q \in F(T_1) \cap F(T_2)$; that is,

$$
\|p - q\|^2 = \langle p - q, J(p - q) \rangle = 0. \tag{2.9}
$$

Therefore, $p = q$. This completes the proof. □

Theorem 2.3. *Let X be a real uniformly convex Banach space such that its dual X^* has the Kadec-Klee property and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by* (1.14). *Then, $\{x_n\}$ converges weakly to a fixed point of T_1 and T_2.*

Proof. It follows from Lemma 1.11 that the sequence $\{x_n\}$ is bounded. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to a point $x^* \in C$. By Lemma 1.12, we have

$$\lim_{n\to\infty}\left\|x_{n_j} - T_1 x_{n_j}\right\| = 0 = \lim_{n\to\infty}\left\|x_{n_j} - T_2 x_{n_j}\right\|. \tag{2.10}$$

Now, using Lemma 1.10, we have $(I - T)x^* = 0$; that is, $Tx^* = x^*$. Thus, $x^* \in F(T_1) \cap F(T_2)$. It remains to show that $\{x_n\}$ converges weakly to x^*. Suppose that $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then, $y^* \in C$ and so $x^*, y^* \in \omega_w(x_n) \cap F(T_1) \cap F(T_2)$. By Lemma 2.1,

$$\lim_{n\to\infty}\left\|tx_n - (1-t)x^* - y^*\right\| \tag{2.11}$$

exists for all $t \in [0,1]$. It follows from Lemma 1.13 that $x^* = y^*$. As a result, $\omega_w(x_n)$ is a singleton, and so $\{x_n\}$ converges weakly to a fixed point of T. □

In the remainder of this section, we deal with the weak convergence of the sequences generated by the projection type Ishikawa iteration scheme (1.14) for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space without any of the Opial's condition, Kadec-Klee property, or Fréchet differentiable norm.

Let T_1 and T_2 be two asymptotically nonexpansive nonself-mappings of C with $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$, and $\{l_n\} \subset [1,\infty)$, $\lim_{n\to\infty} l_n = 1$, respectively. In the sequel, we take $\{t_n\} \subset [1,\infty)$, where $t_n = \max\{k_n, l_n\}$.

We start with proving the following lemma for later use.

Lemma 2.4. *Let X be a uniformly convex Banach space and C a nonempty bounded closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that $k_n \to 1$, $l_n \to 1$ as $n \to \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then, for the sequence $\{x_n\}$ given in* (1.14)*, we have that*

$$\lim_{n\to\infty}\|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty}\|x_n - T_2 x_n\|. \tag{2.12}$$

Proof. By setting $t_n = \max\{k_n, l_n\}$, then $\lim_{n\to\infty} t_n = 1$ if $\lim_{n\to\infty} k_n = 1 = \lim_{n\to\infty} l_n$. Let $p \in F(T_1) \cap F(T_2)$. Since C is bounded, there exists $B_r(0)$ such that $C \subset B_r(0)$ for some $r > 0$. Applying Lemma 1.9 for scheme (1.14), we have

$$\begin{aligned}
\left\|y_n - p\right\|^2 &= \left\|P\Big((1-\beta_n)x_n + \beta_n T_2 (PT_2)^{n-1} x_n\Big) - p\right\|^2 \\
&\le \left\|(1-\beta_n)(x_n - p) + \beta_n\Big(T_2 (PT_2)^{n-1} x_n - p\Big)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\beta_n)\left\|x_n-p\right\|^2+\beta_n l_n^2\left\|x_n-p\right\|^2 \\
&\quad -\beta_n(1-\beta_n)g\left(\left\|x_n-T_2(PT_2)^{n-1}x_n\right\|\right) \\
&= \left(1-\beta_n+\beta_n l_n^2\right)\left\|x_n-p\right\|^2-\beta_n(1-\beta_n)g\left(\left\|x_n-T_2(PT_2)^{n-1}x_n\right\|\right)
\end{aligned}
\tag{2.13}
$$

and so,

$$
\begin{aligned}
\left\|x_{n+1}-p\right\|^2 &= \left\|P\left((1-\alpha_n)y_n+\alpha_n T_1(PT_1)^{n-1}y_n\right)-p\right\|^2 \\
&\leq \left\|(1-\alpha_n)(y_n-p)+\alpha_n(T_1(PT_1)^{n-1}y_n-p)\right\|^2 \\
&= (1-\alpha_n)\left\|y_n-p\right\|^2+\alpha_n k_n^2\left\|y_n-p\right\|^2 \\
&\quad -\alpha_n(1-\alpha_n)g\left(\left\|y_n-T_1(PT_1)^{n-1}y_n\right\|\right) \\
&= \left(1-\alpha_n+\alpha_n k_n^2\right)\left\|y_n-p\right\|^2 \\
&\quad -\alpha_n(1-\alpha_n)g\left(\left\|y_n-T_1(PT_1)^{n-1}y_n\right\|\right) \\
&\leq \left(1-\alpha_n+\alpha_n k_n^2\right)\left(\left(1-\beta_n+\beta_n l_n^2\right)\left\|x_n-p\right\|^2-\beta_n(1-\beta_n)g\left(\left\|x_n-T_2(PT_2)^{n-1}x_n\right\|\right)\right) \\
&\quad -\alpha_n(1-\alpha_n)g\left(\left\|y_n-T_1(PT_1)^{n-1}y_n\right\|\right) \\
&= \left(1-\alpha_n+\alpha_n k_n^2\right)\left(1-\beta_n+\beta_n l_n^2\right)\left\|x_n-p\right\|^2 \\
&\quad -\left(1-\alpha_n+\alpha_n k_n^2\right)\beta_n(1-\beta_n)g\left(\left\|x_n-T_2(PT_2)^{n-1}x_n\right\|\right) \\
&\quad -\alpha_n(1-\alpha_n)g\left(\left\|y_n-T_1(PT_1)^{n-1}y_n\right\|\right) \\
&= \left((1-\alpha_n)(1-\beta_n)+(1-\alpha_n)\beta_n l_n^2+\ (1-\beta_n)\alpha_n k_n^2+\alpha_n k_n^2\beta_n l_n^2\right)\left\|x_n-p\right\|^2 \\
&\quad -\left(1-\alpha_n+\alpha_n k_n^2\right)\beta_n(1-\beta_n)g\left(\left\|x_n-T_2(PT_2)^{n-1}x_n\right\|\right) \\
&\quad -\alpha_n(1-\alpha_n)g\left(\left\|y_n-T_1(PT_1)^{n-1}y_n\right\|\right) \\
&\leq \left((1-\alpha_n)(1-\beta_n)+(1-\alpha_n)\beta_n t_n^2+\ (1-\beta_n)\alpha_n t_n^2+\alpha_n\beta_n t_n^4\right)\left\|x_n-p\right\|^2 \\
&\quad -\left(1-\alpha_n+\alpha_n k_n^2\right)\beta_n(1-\beta_n)g\left(\left\|x_n-T_2(PT_2)^{n-1}x_n\right\|\right) \\
&\quad -\alpha_n(1-\alpha_n)g\left(\left\|y_n-T_1(PT_1)^{n-1}y_n\right\|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big((1-\alpha_n)(1-\beta_n)t_n^4 + (1-\alpha_n)\beta_n t_n^4 + \ (1-\beta_n)\alpha_n t_n^4 + \alpha_n\beta_n t_n^4\Big)\|x_n - p\|^2 \\
&\quad - \Big(1+\alpha_n\Big(k_n^2 - 1\Big)\Big)\beta_n(1-\beta_n)g\Big(\Big\|x_n - T_2(PT_2)^{n-1}x_n\Big\|\Big) \\
&\quad - \alpha_n(1-\alpha_n)g\Big(\Big\|y_n - T_1(PT_1)^{n-1}y_n\Big\|\Big) \\
&\le \Big((1-\alpha_n)(1-\beta_n)t_n^4 + (1-\alpha_n)\beta_n t_n^4 + \ (1-\beta_n)\alpha_n t_n^4 + \alpha_n\beta_n t_n^4\Big)\|x_n - p\|^2 \\
&\quad - \beta_n(1-\beta_n)g\Big(\Big\|x_n - T_2(PT_2)^{n-1}x_n\Big\|\Big) \\
&\quad - \alpha_n(1-\alpha_n)g\Big(\Big\|y_n - T_1(PT_1)^{n-1}y_n\Big\|\Big) \\
&\le \|x_n - p\|^2 + r\Big(t_n^4 - 1\Big) - \varepsilon^2 g\Big(\Big\|x_n - T_2(PT_2)^{n-1}x_n\Big\|\Big) \\
&\quad - \varepsilon^2 g\Big(\Big\|y_n - T_1(PT_1)^{n-1}y_n\Big\|\Big).
\end{aligned}
\tag{2.14}
$$

From (2.14), we obtain the following two important inequalities:

$$
\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + r\Big(t_n^4 - 1\Big) - \varepsilon^2 g\Big(\Big\|x_n - T_2(PT_2)^{n-1}x_n\Big\|\Big), \tag{2.15}
$$

$$
\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + r\Big(t_n^4 - 1\Big) - \varepsilon^2 g\Big(\Big\|y_n - T_1(PT_1)^{n-1}y_n\Big\|\Big). \tag{2.16}
$$

Now, we prove that

$$
\lim_{n\to\infty}\Big\|x_n - T_2(PT_2)^{n-1}x_n\Big\| = 0 = \lim_{n\to\infty}\Big\|y_n - T_1(PT_1)^{n-1}y_n\Big\|. \tag{2.17}
$$

Assume that $\limsup_{n\to\infty}\|x_n - T_2(PT_2)^{n-1}x_n\| > 0$. Then, there exists a subsequence (use the same notation for subsequence as for the sequence) of $\{x_n\}$ and $\mu > 0$ such that

$$
\Big\|x_n - T_2(PT_2)^{n-1}x_n\Big\| \ge \mu > 0. \tag{2.18}
$$

By definition of g, we have

$$g\left(\left\|x_n - T_2(PT_2)^{n-1}x_n\right\|\right) \geq g(\mu) > 0. \tag{2.19}$$

From (2.15), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + r\left(t_n^4 - 1\right) - \varepsilon^2 g(\mu) \\ &= \|x_n - p\|^2 + r\left(\left(t_n^4 - 1\right) - \frac{\varepsilon^2}{2r}g(\mu)\right) - \frac{\varepsilon^2}{2}g(\mu). \end{aligned} \tag{2.20}$$

In addition, $t_n^4 \to 1$ and $(\varepsilon^2/2r)g(\mu) > 0$; there exists $n_0 \geq 1$ such that $(t_n^4 - 1) < (\varepsilon^2/2r)g(\mu)$ for all $n \geq n_0$. From (2.20), we obtain

$$\frac{\varepsilon^2}{2}g(\mu) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{2.21}$$

for all $n \geq n_0$.

Let $m \geq n_0$. It follows from (2.21) that

$$\begin{aligned} \frac{\varepsilon^2}{2}\sum_{n=n_0}^{m} g(\mu) &\leq \sum_{n=n_0}^{m}\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right) \\ &= \|x_{n_0} - p\|^2. \end{aligned} \tag{2.22}$$

By letting $m \to \infty$ in (2.22), we obtain

$$\infty = \|x_{n_0} - p\|^2 < \infty \tag{2.23}$$

which contradicts the reality. This proves that $\mu = 0$. Thus, $\limsup_{n\to\infty}\|x_n - T_2(PT_2)^{n-1}x_n\| \leq 0$. Consequently, we have

$$\lim_{n\to\infty}\left\|x_n - T_2(PT_2)^{n-1}x_n\right\| = 0. \tag{2.24}$$

Similarly, using (2.16), we may show that

$$\lim_{n\to\infty}\left\|y_n - T_1(PT_1)^{n-1}y_n\right\| = 0. \tag{2.25}$$

Using (2.24), we have

$$\|x_n - y_n\| \leq \beta_n\left\|x_n - T_2(PT_2)^{n-1}x_n\right\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{2.26}$$

From (2.25), (2.26), and the uniform equicontinuous of T_1 (see Remark 1.1), we have

$$\begin{aligned}\left\|x_n - T_1(PT_1)^{n-1}x_n\right\| &\le \|x_n - y_n\| + \left\|y_n - T_1(PT_1)^{n-1}x_n\right\| \\ &\le \|x_n - y_n\| + \left\|y_n - T_1(PT_1)^{n-1}y_n\right\| \\ &\quad + \left\|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\right\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\end{aligned} \tag{2.27}$$

Since

$$\begin{aligned}\|x_n - x_{n+1}\| &\le (1-\alpha_n)\|y_n - x_n\| + \alpha_n\left\|T_1(PT_1)^{n-1}y_n - x_n\right\| \\ &= (1-\alpha_n)\|y_n - x_n\| + \alpha_n\left\|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n + T_1(PT_1)^{n-1}x_n - x_n\right\| \\ &\le \|y_n - x_n\| + \left\|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\right\| + \left\|T_1(PT_1)^{n-1}x_n - x_n\right\|,\end{aligned} \tag{2.28}$$

it follows from (2.26), (2.27), and the uniform equi-continuity of T_1 (see Remark 1.1) that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{2.29}$$

Since $\lim_{n\to\infty}\|x_n - T_1(PT_1)^{n-1}x_n\| = 0$ and again from the fact that T_1 is uniformly equicontinuous mapping, by Using (2.29), we have

$$\begin{aligned}\left\|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\right\| &= \left\|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-1}x_n + T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}x_{n+1}\right\| \\ &\le \|x_{n+1} - x_n\| + \left\|T_1(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{n-1}x_n\right\| \\ &\quad + \left\|T_1(PT_1)^{n-1}x_n - x_n\right\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\end{aligned} \tag{2.30}$$

In addition,

$$\begin{aligned}&\left\|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\right\| \\ &\quad = \left\|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-2}x_n + T_1(PT_1)^{n-2}x_n - T_1(PT_1)^{n-2}x_{n+1}\right\| \\ &\quad \le \|x_{n+1} - x_n\| + \left\|T_1(PT_1)^{n-2}x_n - x_n\right\| + \left\|T_1(PT_1)^{n-2}x_{n+1} - T_1(PT_1)^{n-2}x_n\right\| \\ &\quad \le \|x_{n+1} - x_n\| + \left\|T_1(PT_1)^{n-2}x_n - x_n\right\| + L\|x_{n+1} - x_n\|,\end{aligned} \tag{2.31}$$

where $L = \sup\{k_n : n \geq 1\}$. It follows from (2.29) and (2.30) that

$$\lim_{\to\infty} \left\| x_{n+1} - T_1 (PT_1)^{n-2} x_{n+1} \right\| = 0. \tag{2.32}$$

We denote $(PT_1)^{1-1}$ to be the identity maps from C onto itself. Thus, by the inequality (2.30) and (2.32), we have

$$\begin{aligned}
\|x_{n+1} - T_1 x_{n+1}\| &= \left\| x_{n+1} - T_1 (PT_1)^{n-1} x_{n+1} + T_1 (PT_1)^{n-1} x_{n+1} - T_1 x_{n+1} \right\| \\
&\leq \left\| x_{n+1} - T_1 (PT_1)^{n-1} x_{n+1} \right\| + \left\| T_1 (PT_1)^{n-1} x_{n+1} - T_1 x_{n+1} \right\| \\
&= \left\| x_{n+1} - T_1 (PT_1)^{n-1} x_{n+1} \right\| + \left\| T_1 (PT_1)^{1-1} (PT_1)^{n-1} x_{n+1} - T_1 (PT_1)^{1-1} x_{n+1} \right\| \\
&\leq \left\| x_{n+1} - T_1 (PT_1)^{n-1} x_{n+1} \right\| + L \left\| (PT_1)^{n-1} x_{n+1} - x_{n+1} \right\| \\
&= \left\| x_{n+1} - T_1 (PT_1)^{n-1} x_{n+1} \right\| + L \left\| (PT_1)(PT_1)^{n-2} x_{n+1} - P(x_{n+1}) \right\| \\
&\leq \left\| x_{n+1} - T_1 (PT_1)^{n-1} x_{n+1} \right\| \\
&\quad + L \left\| T_1 (PT_1)^{n-2} x_{n+1} - x_{n+1} \right\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty),
\end{aligned} \tag{2.33}$$

which implies that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Similarly, we may show that $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$. The proof is completed. □

Our weak convergence theorem is as follows. We do not use the rate of convergence conditions, namely, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (l_n - 1) < \infty$ in its proof.

Theorem 2.5. *Let X be a uniformly convex Banach space and C a nonempty bounded closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $k_n \to 1$, $l_n \to 1$ as $n \to \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then, the sequence $\{x_n\}$ given in* (1.14) *converges weakly to a common fixed point of T_1 and T_2.*

Proof. Since C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to $q \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$. This show that $\omega_w(x_n) \neq \emptyset$ and, by Lemma 2.4, $\lim_{n\to\infty} \|x_{n_j} - T_1 x_{n_j}\| = \lim_{n\to\infty} \|x_{n_j} - T_2 x_{n_j}\| = 0$. Since $I - T_1$ and $I - T_2$ are demiclosed at zero, using Lemma 1.10, we have $T_1 q = q = T_2 q$. Therefore, $\omega_w(x_n) \subset F(T_1) \cap F(T_2)$. For any $q \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup q \quad (\text{as } i \longrightarrow \infty). \tag{2.34}$$

It follows from (2.24) and (2.34) that

$$T_2(PT_2)^{n_j-1}x_{n_j} = \left(T_2(PT_2)^{n_j-1}x_{n_j} - x_{n_j}\right) + x_{n_j} \rightharpoonup q. \tag{2.35}$$

Now, from (1.14), (2.34), and (2.35),

$$y_{n_j} = P\left(\left(1-\beta_{n_j}\right)x_{n_j} + \beta_{n_j}T_2(PT_2)^{n_j-1}x_{n_j}\right) \rightharpoonup q. \tag{2.36}$$

Also, from (2.25) and (2.36), we have

$$T_1(PT_1)^{n_j-1}y_{n_j} = \left(T_1(PT_1)^{n_j-1}y_{n_j} - y_{n_j}\right) + y_{n_j} \rightharpoonup q. \tag{2.37}$$

It follows from (2.36) and (2.37) that

$$x_{n_j+1} = P\left(\left(1-\alpha_{n_j}\right)y_{n_j} + \alpha_{n_j}T_1(PT_1)^{n_j-1}y_{n_j}\right) \rightharpoonup q. \tag{2.38}$$

Continuing in this way, by induction, we can prove that, for any $m \geq 0$,

$$x_{n_j+m} \rightharpoonup q. \tag{2.39}$$

By induction, one can prove that $\cup_{m=0}^{\infty}\{x_{n_j+m}\}$ converges weakly to q as $j \to \infty$; in fact, $\{x_n\}_{n=n_1}^{\infty} = \cup_{m=0}^{\infty}\{x_{n_j+m}\}_{j=1}^{\infty}$ gives that $x_n \rightharpoonup q$ as $n \to \infty$. This completes the prove. □

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