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Research Article

Existence of Positive Solutions for a Fourth-Order Periodic Boundary Value Problem

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The existence results of positive solutions are obtained for the fourth-order periodic boundary value problem $u^{(4)} - \beta u'' + \alpha u = f(t, u, u'')$, $0 \le t \le 1$, $u^{(i)}(0) = u^{(i)}(1)$, $i = 0, 1, 2, 3$, where $f : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous, $\alpha, \beta \in \mathbb{R}$, and satisfy $0 < \alpha < ((\beta/2) + 2\pi^2)^2$, $\beta > -2\pi^2$, $(\alpha/\pi^4) + (\beta/\pi^2) + 1 > 0$. The discussion is based on the fixed point index theory in cones.

1. Introduction

This paper concerns the existence of positive solutions for the fourth-order periodic boundary value problem (PBVP)

$$\begin{gathered} u^{(4)}(t) - \beta u''(t) + \alpha u(t) = f\left(t,\ u(t),\ u''(t)\right), \quad 0 \le t \le 1, \\ u^{(i)}(0) = u^{(i)}(1)\ , \quad i = 0, 1, 2, 3, \end{gathered} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$ and $f : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous, $\mathbb{R}^+ = [0, \infty)$. PBVP (1.1) describes the deformations of an elastic beam in equilibrium state with periodic boundary condition. In the equation, the u'' denotes the bending moment term which represents bending effect. Owing to its importance in physics, the existence of solutions to this problem has been studied by some authors, see [1–6]. In practice, only its positive solutions are significant. In this paper, we discuss the existence of positive solutions of PBVP (1.1).

In [1, 2], Cabada and Lois obtained the maximum principles for fourth-order operator $L_{4,\alpha}\, u = u^{(4)} + \alpha u$ in periodic boundary condition and then they proved the existence of solutions and the validity of the monotone method in the presence of lower and upper solutions for the periodic boundary problem

$$\begin{aligned} &u^{(4)}(t) = g(t, u(t)), \quad 0 \leq t \leq 1, \\ &u^{(i)}(0) = u^{(i)}(1)\ , \quad i = 0, 1, 2, 3. \end{aligned} \tag{1.2}$$

In [3], the present author established a strongly maximum principle for operator $L_4\ u = u^{(4)} - \beta u'' + \alpha u$ in periodic boundary condition, and showed that if α, β satisfy the assumption

$$0 < \alpha < \left(\frac{\beta}{2} + 2\pi^2\right)^2, \qquad \beta > -2\pi^2, \qquad \frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} + 1 > 0, \tag{1.3}$$

then L_4 is strongly inverse positive in space

$$F_4 = \left\{u \in C^4[0,1] \mid u^{(i)}(0) = u^{(i)}(1),\ i = 0, 1, 2;\ u^{(3)}(0) \geq u^{(3)}(1)\right\}. \tag{1.4}$$

As an application of this strongly maximum principle, the author considered the existence of positive solutions for the special fourth-order periodic boundary problem

$$\begin{aligned} u^{(4)}(t) - \beta u''(t) + \alpha u(t) &= g(t, u(t)), \quad 0 \leq t \leq 1, \\ u^{(i)}(0) &= u^{(i)}(1), \quad i = 0, 1, 2, 3, \end{aligned} \tag{1.5}$$

and obtained the following result.

Theorem A. *Let* $g : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous and the assumption* (1.3) *hold. If* g *satisfies one of the following conditions*

(G1) $g^0 < \alpha$, $g_\infty > \alpha$;

(G2) $g_0 > \alpha$, $g^\infty < \alpha$,

where

$$\begin{aligned} g_0 &= \liminf_{u \to 0^+} \min_{t \in [0,\,1]} \left(\frac{f(t,u)}{u}\right), \qquad g^0 = \limsup_{u \to 0^+} \max_{t \in [0,\,1]} \left(\frac{f(t,u)}{u}\right), \\ g_\infty &= \liminf_{u \to +\infty} \min_{t \in [0,\,1]} \left(\frac{f(t,u)}{u}\right), \qquad g^\infty = \limsup_{u \to +\infty} \max_{t \in [0,\,1]} \left(\frac{f(t,u)}{u}\right), \end{aligned} \tag{1.6}$$

then PBVP (1.5) *has at least one positive solution.*

Based upon this strongly maximum principle, the authors of [4, 5] further consider the existence and multiplicity of positive solutions of PBVP (1.5). In [6], Bereanu obtained existence results for PBVP (1.5) by using the method of topological degree. However, all of these works are on the special equation (1.5), and few people consider the existence of positive solutions of PBVP (1.1) that nonlinearity f contains the bending moment term u''. The purpose of this paper is to discuss the existence of positive solutions of PBVP (1.1).

The strongly maximum principle implies that the fourth-order linear boundary value problem (LBVP)

$$L_4\ u := u^{(4)} - \beta u'' + \alpha u = 0, \quad 0 \le t \le 1,$$
$$u^{(i)}(0) - u^{(i)}(1) = 0, \quad i = 0, 1, 2, \tag{1.7}$$
$$u^{(3)}(0) - u^{(3)}(1) = 1$$

has a unique positive solution $\Phi : [0,1] \to (0,\infty)$, see [3, Lemma 3]. This function has been introduced in [2, Lemma 2.1 and Remark 2.1]. Let $I = [0,1]$, and set

$$\sigma = \frac{\min_{t\in I}\ \Phi(t)}{\max_{t\in I}\ \Phi(t)}, \qquad M = \frac{\max_{t\in I}\ |\Phi''(t)|}{\min_{t\in I}\ \Phi(t)}. \tag{1.8}$$

Let $f : I \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous. To be convenient, we introduce the notations

$$f_0 = \liminf_{u\to 0^+} \min_{|v|\le Mu, t\in I} \left(\frac{f(t,u,v)}{u}\right),$$
$$f^0 = \limsup_{u\to 0^+} \max_{|v|\le Mu, t\in I} \left(\frac{f(t,u,v)}{u}\right),$$
$$f_\infty = \liminf_{u\to +\infty} \min_{|v|\le Mu, t\in I} \left(\frac{f(t,u,v)}{u}\right), \tag{1.9}$$
$$f^\infty = \limsup_{u\to +\infty} \max_{|v|\le Mu, t\in I} \left(\frac{f(t,u,v)}{u}\right).$$

Our main result is as follows.

Theorem 1.1. *Let* $f : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ *be continuous, and let the assumption* (1.3) *hold. If* f *satisfies one of the following conditions:*

(F1) $f^0 < \alpha,\ f_\infty > \alpha$,

(F2) $f_0 > \alpha,\ f^\infty < \alpha$,

then PBVP (1.1) *has at least one positive solution.*

Clearly, Theorem 1.1 is an extension of Theorem A. Since that α is an eigenvalue of linear eigenvalue problem

$$u^{(4)} - \beta u'' + \alpha u = \lambda u, \tag{1.10}$$

with periodic boundary condition, if one inequality in (F1) or (F2) of Theorem 1.1 is not true, the existence of solution to PBVP (1.1) cannot be guaranteed. Hence, (F1) and (F2) are the optimal conditions for the existence of the positive of PBVP (1.1).

In Theorem 1.1, the condition (F1) allows that $f(t,u,v)$ may be superlinear growth on u and v, for example, $f(t,u,v) = u^2 + v^2$, and the condition (F2) allows that $f(t,u,v)$ may be sublinear growth on u and v, for example, $f(t,u,v) = \sqrt[3]{u^2 + v^2}$.

The proof of Theorem 1.1 is based on the theory of the fixed point index in cones. Since the nonlinearity f of PBVP (1.1) contains u'', the argument of Theorem A in [3] is not applicable to Theorem 1.1. We will prove Theorem 1.1 by choosing a proper cone of $C^2(I)$ in Section 3. Some preliminaries to discuss PBVP (1.1) are presented in Section 2.

2. Preliminaries

Let $C(I)$ be the Banach space of all continuous functions on the unit interval $I = [0,1]$ with the norm $\|u\|_C = \max_{0\le t\le\omega} |u(t)|$. Let $C^+(I)$ denote the cone of all nonnegative functions in $C(I)$. Generaly, for $n \in \mathbb{N}$, we use $C^n(I)$ to denote the Banach space of the nth-order continuous differentiable functions on I with the norm $\|u\|_{C^n} = \sum_{k=1}^{n} \|u^{(k)}\|_C$. In $C^2(I)$, we define a new norm by

$$\|u\|_{C^{02}} = \|u\|_C + \left\|u''\right\|_C. \tag{2.1}$$

Then $\|u\|_{C^{02}}$ is equivalent to $\|u\|_{C^2}$. In fact, for every $u \in C^2(I)$, it is clear that $\|u\|_{C^{02}} \le \|u\|_{C^2}$. On the other hand, by the Lagrange mean-value theorem, there exists $\xi \in (0,1)$ such that $u(1) - u(0) = u'(\xi)$. For $t \in I$, we have

$$\begin{aligned}\left|u'(t)\right| &\le \left|u'(t) - u'(\xi)\right| + \left|u'(\xi)\right| = \left|\int_\xi^t u''(s)ds\right| + |u(1) - u(0)| \\ &\le \int_0^1 \left|u''(s)\right| \, ds + |u(1)| + |u(0)| \\ &\le \left\|u''\right\|_C + 2\|u\|_C \le 2\|u\|_{C^{02}}.\end{aligned} \tag{2.2}$$

Hence, $\|u'\|_C \le 2\|u\|_{C^{02}}$. By this, we have

$$\|u\|_{C^2} = \|u\|_C + \left\|u'\right\|_C + \left\|u''\right\|_C = \|u\|_{C^{02}} + \left\|u'\right\|_C \le 3\|u\|_{C^{02}}. \tag{2.3}$$

Therefore, the norms $\|u\|_{C^{02}}$ and $\|u\|_{C^2}$ are equivalent.

Let $\alpha, \beta \in \mathbb{R}$ satisfy the assumption (1.3). For $h \in C(I)$, we consider the fourth-order linear periodic boundary value problem (LPBVP)

$$\begin{gathered}u^{(4)}(t) - \beta u''(t) + \alpha u(t) = h(t), \quad 0 \le t \le 1, \\ u^{(i)}(0) = u^{(i)}(1), \quad i = 0,1,2,3.\end{gathered} \tag{2.4}$$

Let $\Phi(t)$ be the unique positive solution of LBVP(1.7), and set

$$G(t,s) = \begin{cases}\Phi(t-s), & 0 \le s \le t \le 1, \\ \Phi(1+t-s), & 0 \le t < s \le 1.\end{cases} \tag{2.5}$$

By [3, Lemma 1], we have the following result.

Lemma 2.1. *Let $\alpha, \beta \in \mathbb{R}$ satisfy the assumption* (1.3). *Then for every $h \in C(I)$, LPBVP* (2.4) *has a unique solution $u(t)$ which is given by*

$$u(t) = \int_0^1 G(t,s)h(s)ds := Sh(t), \quad t \in \mathbb{R}. \tag{2.6}$$

Moreover, $S : C(I) \to C^4(I)$ is a linear bounded operator.

Let σ and M be the positive constants given by (1.8). Choose a cone K in $C^2(I)$ by

$$K = \left\{ u \in C^2(I) \mid u(t) \geq \sigma \|u\|_C, \left|u''(t)\right| \leq M|u(t)|,\ t \in I \right\}. \tag{2.7}$$

We have the following.

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{R}$ satisfy the assumption* (1.3). *Then for every $h \in C^+(I)$, the solution of LPBVP* (2.4) *$u = Sh \in K$. Namely, $S(C^+(I)) \subset K$.*

Proof. Let $h \in C^+(I)$, $u = Sh$. For every $t \in I$, from (2.6) it follows that

$$0 \leq u(t) = \int_0^1 G(t,s)h(s)ds \leq \max_{t\in I} \Phi(t) \int_0^1 h(s)ds, \tag{2.8}$$

which implies that

$$\|u\|_C \leq \max_{t\in I} \Phi(t) \int_0^1 h(s)ds. \tag{2.9}$$

By this and (2.6), we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)h(s)ds \geq \min_{t\in I} \Phi(t) \int_0^1 h(s)ds \\ &= \sigma \max_{t\in I} \Phi(t) \int_0^1 h(s)ds \geq \sigma \|u\|_C. \end{aligned} \tag{2.10}$$

For $t \in I$, by the definition of G and Φ, we have

$$u(t) = \int_0^t \Phi(t-s)h(s)ds + \int_t^1 \Phi(1+t-s)h(s)ds. \tag{2.11}$$

Making derivation to both sides of this equality, we have

$$u''(t) = \int_0^t \Phi''(t-s)h(s)ds + \int_t^1 \Phi''(1+t-s)h(s)ds, \tag{2.12}$$

from which it follows that

$$\begin{aligned}
\left|u''(t)\right| &\le \max_{t\in I}\left|\Phi''(t)\right| \int_0^t h(s)ds + \max_{t\in I}\left|\Phi''(t)\right| \int_t^1 h(s)ds \\
&= \max_{t\in I}\left|\Phi''(t)\right| \int_0^1 h(s)ds \\
&= M\min_{t\in I}\Phi(t) \int_0^1 h(s)ds \le Mu(t), \quad t \in I.
\end{aligned} \tag{2.13}$$

Therefore, $u \in K$. This means that $S(C^+(I)) \subset K$. □

For every $u \in K$, since $f : I \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous, we see that $F(u) := f(\cdot, u(\cdot), u''(\cdot)) \in C^+(I)$. By Lemma 2.2, $w = S(F(u)) \in K$. Define an operator $A : K \to K$ by

$$Au(t) = S(F(u)) = \int_0^1 G(t,s) f\left(s, u(s), u''(s)\right) ds. \tag{2.14}$$

We have the following.

Lemma 2.3. *$A : K \to K$ is a completely continuous operator.*

Proof. Let $D \subset K$ be a bounded set in $C^2(I)$. By the continuity of $f : I\times\mathbb{R}^+\times\mathbb{R} \to \mathbb{R}^+$, $F(D)$ is a bounded set in $C(I)$. By the boundedness of the operator $S : C(I) \to C^4(I)$, $A(D) = S(F(D))$ is a bounded set in $C^4(I)$. By the compactness of the embedding $C^4(I) \hookrightarrow C^2(I)$, $A(D)$ is a precompact set in $C^2(I)$. So $A : K \to K$ is completely continuous. □

By the definition of S and K, the positive solution of PBVP (1.1) is equivalent to the nontrivial fixed point of A. We will find the nonzero fixed point of A by using the fixed point index theory in cones.

We recall some concepts and conclusions on the fixed point index in [7, 8]. Let E be a Banach space, and let $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K\cap\Omega \ne \emptyset$. Let $A : K\cap\overline{\Omega} \to K$ be a completely continuous mapping. If $Au \ne u$ for any $u \in K\cap\partial\Omega$, then the fixed point index $i(A, K\cap\Omega, K)$ has definition. One important fact is that if $i(A, K\cap\Omega, K) \ne 0$, then A has a fixed point in $K\cap\Omega$. The following two lemmas are needed in our argument.

Lemma 2.4 (see [8]). *Let Ω be a bounded open subset of E with $\theta \in \Omega$, and let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $\lambda Au \ne u$ for every $u \in K \cap \partial\Omega$ and $0 < \lambda \le 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.5 (see [8]). *Let Ω be a bounded open subset of E, and let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If there exists an $e \in K \setminus \{\theta\}$ such that $u - Au \ne \tau e$ for every $u \in K \cap \partial\Omega$ and $\tau \ge 0$, then $i(A, K \cap \Omega, K) = 0$.*

3. Proof of the Main Result

Proof of Theorem 1.1. Choose the working space $E = C^2(I)$ with the norm $\|u\|_{C^{02}}$. Let K be the closed convex cone in $C^2(I)$ defined by (2.7), and let $A : K \to K$ be the operator defined by (2.14). By Lemma 2.3 and the definition of K, the nonzero fixed of the operator A is the positive solution of PBVP (1.1). Let $0 < r < R < +\infty$, and set

$$\Omega_1 = \left\{u \in C^2(I) \mid \|u\|_{C^{02}} < r\right\}, \qquad \Omega_2 = \left\{u \in C^2(I) \mid \|u\|_{C^{02}} < R\right\}. \tag{3.1}$$

We show that, if r is small enough and R large enough, the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega_1})$ in either case that (F1) holds or (F2) holds.

Case 1. Assume that (F1) holds.

Since $f^0 < \alpha$, by the definition of f^0, we may choose $\varepsilon \in (0, \alpha)$ and $\delta > 0$, such that

$$f(t, u, v) \le (\alpha - \varepsilon)u, \quad t \in I,\ |v| \le Mu,\ 0 \le u \le \delta. \tag{3.2}$$

Let $r \in (0, \delta)$. We prove that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$; namely, $\lambda Au \neq u$, for every $u \in K \cap \partial\Omega_1$ and $0 < \lambda \le 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \lambda_0 \le 1$ such that $\lambda_0 Au_0 = u_0$, then by the definition of A and Lemma 2.1, $u_0 \in C^4(I)$ satisfies the differential equation

$$u_0^{(4)}(t) - \beta u_0''(t) + \alpha u_0(t) = \lambda_0 f\left(t, u_0(t), u_0''(t)\right), \quad t \in I, \tag{3.3}$$

and the periodic boundary condition

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, 2, 3. \tag{3.4}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$\left|u_0''(t)\right| \le Mu_0(t), \quad 0 < \sigma\|u_0\|_C \le u_0(t) \le \|u_0\|_{C^{02}} = r < \delta, \quad t \in \mathbb{R}. \tag{3.5}$$

From this and (3.2), it follows that

$$f\left(t,\ u_0(t), u_0''(t)\right) \le (\alpha - \varepsilon)u_0(t), \quad t \in \mathbb{R}. \tag{3.6}$$

By this inequality and (3.3), we have

$$u_0^{(4)}(t) - \beta u_0''(t) + \alpha u_0(t) \le \lambda_0(\alpha - \varepsilon)u_0(t) \le (\alpha - \varepsilon)\ u_0(t), \quad \in I. \tag{3.7}$$

Integrating this inequality from 0 to 1 and using the periodic boundary condition (3.4), we obtain that

$$\alpha \int_0^1 u_0(t)dt \leq (\alpha - \varepsilon) \int_0^1 u_0(t)dt. \tag{3.8}$$

Since $\int_0^1 u_0(t)dt \geq \sigma\|u_0\|_C > 0$, form this inequality it follows that $\alpha \leq \alpha - \varepsilon$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$. By Lemma 2.4 we have

$$i(A, K \cap \Omega_1, K) = 1. \tag{3.9}$$

On the other hand, since $f_\infty > \alpha$, by the definition of f_∞, there exist $\varepsilon_1 > 0$ and $H > 0$ such that

$$f(t, u, v) \geq (\alpha + \varepsilon_1)x, \quad t \in I, \ |v| \leq Mu, \ u \geq H. \tag{3.10}$$

Choose $R > \max\{(1 + M/\sigma)H, \ \delta\}$, and let $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$; namely, $u - Au \neq \tau e$, for every $u \in K \cap \partial\Omega_2$ and $\tau \geq 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\tau_1 \geq 0$ such that $u_1 - Au_1 = \tau_1 e$, since $u_1 - \tau_1 e = Au_1$, by definition of A and Lemma 2.1, $u_1(t) \in C^4(I)$ satisfies the differential equation

$$u_1^{(4)}(t) - \beta u_1''(t) + \alpha(u_1(t) - \tau_1) = f(t, u_1(t), u_1''(t)), \quad t \in I, \tag{3.11}$$

and the periodic boundary condition (3.4). Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_1(t) \geq \sigma\|u_1\|_C, \quad |u_1''(t)| \leq Mu_1(t), \quad t \in I. \tag{3.12}$$

By the second inequality of (3.12), we have

$$\|u_1\|_{C^{02}} = \|u_1\|_C + \|u_1''\|_C \leq \|u_1\|_C + M\|u_1\|_C = (1 + M)\|u_1\|_C. \tag{3.13}$$

Consequently,

$$\|u_1\|_C \geq \frac{1}{1 + M}\|u_1\|_{C^{02}}. \tag{3.14}$$

By (3.14) and the first inequality of (3.12), we have

$$u_1(t) \geq \sigma\|u_1\|_C \geq \frac{\sigma}{1 + M}\|u_1\|_{C^{02}} = \frac{\sigma}{1 + M}R > H, \quad t \in I. \tag{3.15}$$

From this, the second inequality of (3.12), and (3.10), it follows that

$$f\left(t, u_1(t), u_1''(t)\right) \geq (\alpha + \varepsilon_1) u_1(t), \quad t \in I. \tag{3.16}$$

By this and (3.11), we have

$$u_1^{(4)}(t) - \beta u_1''(t) + \alpha(u_1(t) - \tau_1) \geq (\alpha + \varepsilon_1) u_1(t), \quad t \in I. \tag{3.17}$$

Integrating this inequality on I and using the periodic boundary condition (3.4), we get that

$$\alpha \int_0^1 u_1(t) dt - \alpha\tau_1 \geq (\alpha + \varepsilon_1) \int_0^1 u_1(t) dt. \tag{3.18}$$

Since $\int_0^1 u_1(t) dt \geq \sigma \|u_1\|_C > 0$, from this inequality it follows that $\alpha \geq \alpha + \varepsilon_1$, which is a contradiction. This means that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$. By Lemma 2.4,

$$i(A, K \cap \Omega_2, K) = 0. \tag{3.19}$$

Now, by the additivity of fixed point index, (3.9), and (3.19), we have

$$i\left(A, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right), K\right) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1. \tag{3.20}$$

Hence, A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is the positive solution of PBVP (1.1).

Case 2. Assume that (F2) holds.

By the assumption of $f_0 > \alpha$ and the definition of f_0, there exist $\varepsilon > 0$ and $\delta > 0$, such that

$$f(t, u, v) \geq (\alpha + \varepsilon) u, \quad t \in I, \ |v| \leq Mu, \ 0 \leq u \leq \delta. \tag{3.21}$$

Let $r \in (0, \delta)$, and let $e(t) \equiv 1$. We prove that A satisfies the hypothesis of Lemma 2.5 in $K \cap \partial\Omega_1$; namely, $u - Au \neq \tau e$ for every $u \in K \cap \partial\Omega_1$ and $\tau \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\tau_0 \geq 0$ such that $u_0 - Au_0 = \tau_0 e$, since $u_0 - \tau_0 e = Au_0$, by the definition of A and Lemma 2.1, $u_0(t) \in C^4(I)$ satisfies the differential equation

$$u_0^{(4)}(t) - \beta u_0''(t) + \alpha(u_0(t) - \tau_0) = f\left(t, u_0(t), u_0''(t)\right), \quad t \in I, \tag{3.22}$$

and the periodic boundary condition (3.4). Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (3.5). From (3.5) and (3.22), it follows that

$$f\left(t, u_0(t), u_0'(t)\right) \geq (\alpha + \varepsilon) u_0(t), \quad t \in I. \tag{3.23}$$

By this inequality and (3.22), we have

$$u_0^{(4)}(t) - \beta u_0''(t) + \alpha(u_0(t) - \tau_0) \geq (\alpha + \varepsilon)\ u_0(t), \quad t \in I. \tag{3.24}$$

Integrating this inequality on I and using the periodic boundary condition (3.4), we have

$$\alpha \int_0^1 u_0(t)dt - \alpha\tau_0 \geq (\alpha + \varepsilon) \int_0^1 u_0(t)dt. \tag{3.25}$$

Since $\int_0^1 u_0(t)dt \geq \sigma\|u_0\|_C > 0$, from this inequality, it follows that $\alpha \geq \alpha + \varepsilon$, which is a contradiction. Hence A satisfies the hypothesis of Lemma 2.5 in $K \cap \partial\Omega_1$. By Lemma 2.5,

$$i(A, K \cap \Omega_1, K) = 0. \tag{3.26}$$

Since $f^\infty < \alpha$, by the definition of f^∞, there exist $\varepsilon_1 \in (0, \alpha)$ and $H > 0$ such that

$$f(t, u, v) \leq (\alpha -\ \varepsilon_1)\ u, \quad t \in I,\ |v| \leq Mu,\ u \geq H. \tag{3.27}$$

Choosing $R > \max\{(1 + M/\sigma)H,\ \delta\}$, we show that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$; namely, $\lambda Au \neq u$, for every $u \in K \cap \partial\Omega_2$ and $0 < \lambda \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \lambda_1 \leq 1$ such that $\lambda_1 Au_1 = u_1$, then by the definition of A and Lemma 2.1, $u_1 \in C^4(I)$ satisfies the differential equation

$$u_1^{(4)}(t) - \beta u_1''(t) + \alpha u_1(t) = \lambda_1 f(t, u_1(t), u_1''(t)), \quad t \in I, \tag{3.28}$$

and the periodic boundary condition (3.4). Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, u_1 satisfies (3.12). From (3.12), we can show that u_1 satisfies (3.14). By (3.14) and the first inequality of (3.12), we have

$$u_1(t) \geq \sigma\|u_1\|_C \geq \frac{\sigma}{1+M}\|u_1\|_{C^{02}} = \frac{\sigma}{1+M}R > H, \quad t \in \mathbb{R}. \tag{3.29}$$

From this, the second inequality of (3.12), and (3.27), it follows that

$$f(t, u_1(t), u_1''(t)) \leq (\alpha - \varepsilon_1)u_1(t), \quad t \in I. \tag{3.30}$$

By this inequality and (3.28), we have

$$u_1^{(4)}(t) - \beta u_1''(t) + \alpha u_1(t) \leq \lambda_1(\alpha - \varepsilon_1)u_1(t) \leq (\alpha - \varepsilon_1)u_1(t), \quad t \in I. \tag{3.31}$$

Integrating this inequality on I and using the periodic boundary condition (3.4), we obtain that

$$\alpha \int_0^1 u_1(t)dt \leq (\alpha - \varepsilon_1) \int_0^1 u_1(t)dt. \tag{3.32}$$

Since $\int_0^1 u_1(t)dt \geq \sigma\|u_1\|_C > 0$, form this inequality it follows that $\alpha \leq \alpha - \varepsilon_1$, which is a contradiction. This means that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$. By Lemma 2.4,

$$i(A, K \cap \Omega_2, K) = 1. \tag{3.33}$$

From (3.26) and (3.33), it follows that

$$i\left(A, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right), K\right) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = 1. \tag{3.34}$$

Hence, A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is the positive solution of PBVP (1.1).

The proof of Theorem 1.1 is completed. □

Example 3.1. Consider the superlinear fourth-order periodic boundary problem

$$\begin{gathered} u^{(4)} - u'' + u = a_1(t)u^2 + a_2(t)\left(u''\right)^2, \quad 0 \leq t \leq 1, \\ u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, 2, 3, \end{gathered} \tag{3.35}$$

where $a_1,\ a_2 \in C(I)$ and $a_1(t),\ a_2(t) > 0$ for $t \in I$. It is easy to verify that $\alpha = 1$ and $\beta = 1$ satisfy the assumption (P). $f(t, u, v) = a_1(t)u^2 + a_2(t)v^2$ satisfies the condition (F1), in which $f^0 = 0$ and $f_\infty = +\infty$. Hence, by Theorem 1.1, (3.35) has at least one positive solution.

Example 3.2. Consider the sublinear fourth-order periodic boundary problem

$$\begin{gathered} u^{(4)} + u'' + u = b_1(t)\sqrt{u} + b_2\sqrt{|u''|}, \quad 0 \leq t \leq 1, \\ u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, 2, 3, \end{gathered} \tag{3.36}$$

where $b_1,\ \ b_2 \in C(I)$ and $b_1(t),\ \ b_2(t) > 0$ for $t \in I$. For PBVP (3.36), it is easy to verify that $\alpha = 1$ and $\beta = -1$ satisfy the assumption (1.3), and $f(t, u, v) = b_1(t)\sqrt{u} + b_2(t)\sqrt{|v|}$ satisfies the condition (F2) with $f_0 = +\infty$ and $f^\infty = 0$. By Theorem 1.1, (3.36) has a positive solution.

Since (3.35) and (3.36) have nonlinear terms of u'', which are not in the range considered by [1–6], the existence results in Example 3.1, and Example 3.2 cannot be obtained from [1–6].

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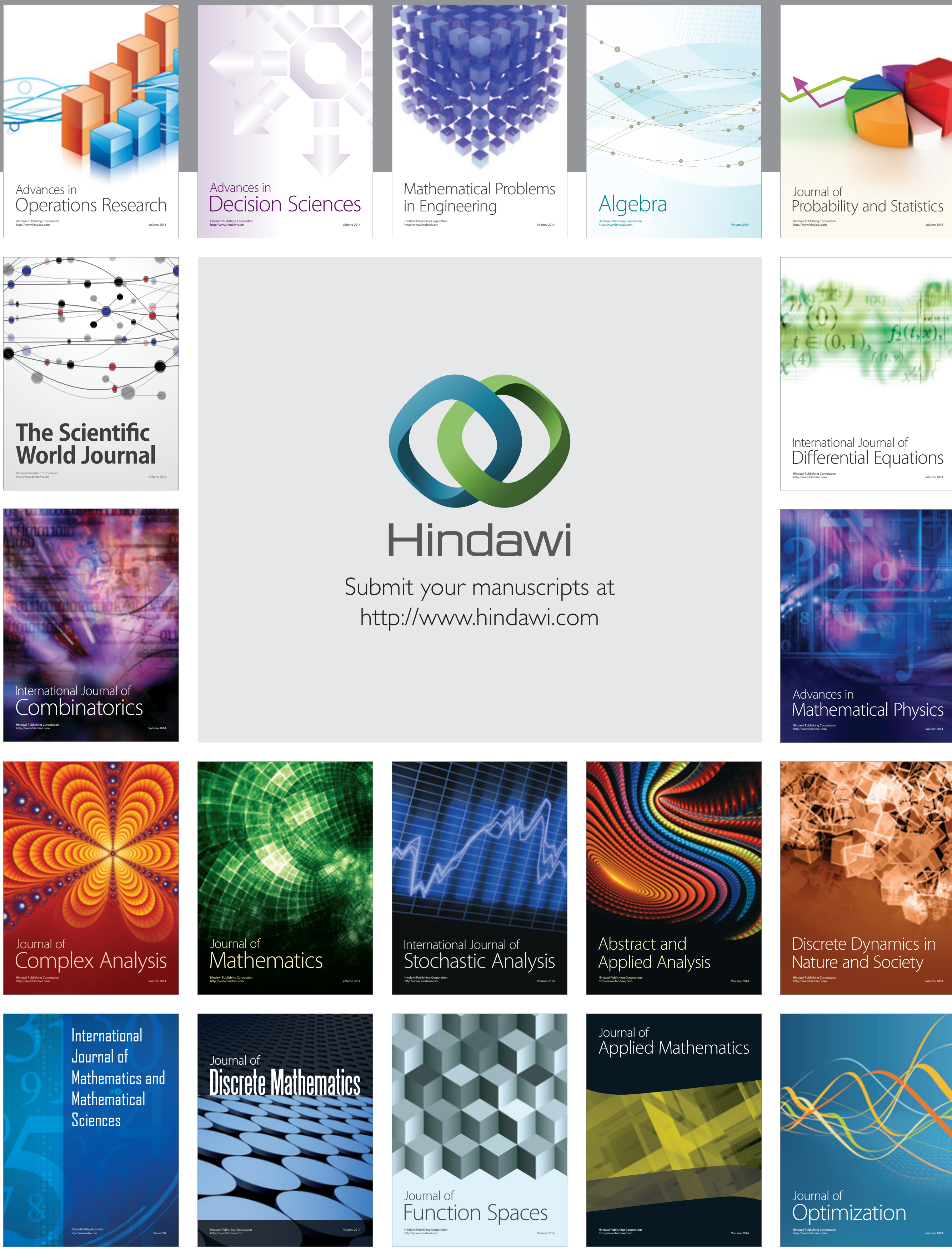
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