Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 850850, 13 pages
doi:10.1155/2011/850850

Research Article

Characterization of Generators for Multiresolution Analyses with Composite Dilations

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Received 20 July 2010; Revised 12 December 2010; Accepted 2 February 2011

Academic Editor: Roman Simon Hilscher

This paper introduces multiresolution analyses with composite dilations (AB-MRAs) and addresses frame multiresolution analyses with composite dilations in the setting of reducing subspaces of $L^2(\mathbb{R}^n)$ (AB-RMRAs). We prove that an AB-MRA can induce an AB-RMRA on a given reducing subspace $L^2(S)^\vee$. For a general expansive matrix, we obtain the characterizations for a scaling function to generate an AB-RMRA, and the main theorems generalize the classical results. Finally, some examples are provided to illustrate the general theory.

1. Introduction

As well known, multiresolution analyses (MRAs) play a significant role in the construction of wavelets for $L^2(\mathbb{R})$ [1, 2]. Up to now, different characterizations of the scaling function for an MRA have been presented. It is shown in [1] that a function $\varphi \in L^2(\mathbb{R})$ is a generator for an MRA if and only if

(1) $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + k)|^2 = 1$, a.e. $\xi \in [-1/2, 1/2]$;

(2) $\lim_{j\to+\infty} |\widehat{\varphi}(2^{-j}\xi)|^2 = 1$, a.e. $\xi \in \mathbb{R}$;

(3) there exists $m_0 \in L^2([0,1])$ such that $\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi)$, a.e. $\xi \in \mathbb{R}$.

If condition (2) is replaced by $\mathbb{R} = \bigcup_{j\in\mathbb{Z}} 2^j \operatorname{supp}(\widehat{\varphi})$ or another condition that the function $F(x,y) = (1/(y-x)) \int_x^y |\widehat{\varphi}(\omega)|^2 d\omega$ is dyadicaly away from zero at the origin, then the two different characterizations of the scaling functions for MRAs are obtained in [3, 4], respectively.

Similarly, under certain conditions, wavelet with composite dilations can be constructed by AB-MRAs which is the generalized definition of MRAs and permits the existence of fast implementation algorithm [5]. Given an $n \times n$ invertible matrix a, $f \in L^2(\mathbb{R}^n)$, and $k \in \mathbb{Z}^n$, we define the dilation operator D and the shift operator T_k on $L^2(\mathbb{R}^n)$ by

$$D_a f(\cdot) := |\det a|^{1/2} f(a\cdot), \qquad T_k f(\cdot) := f(\cdot - k). \tag{1.1}$$

The affine system with composite dilations is defined by $\mathcal{A}_{AB}(\Psi) = \{D_a D_b T_k \Psi : k \in \mathbb{Z}^n, b \in B, a \in A\}$ where $\Psi = \{\psi^1, \psi^2, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. By choosing Ψ, A, and B appropriately, we can make $\mathcal{A}_{AB}(\Psi)$ an orthonormal basis or, more generally, a Parseval frame (PF) for $L^2(\mathbb{R}^n)$ [5–7]. In this case, Ψ is called an AB-multiwavelet or a PF AB-multiwavelet, respectively. Since not all of the AB-multiwavelet come from AB-MRAs, we only focus on the AB-multiwavelet which come from AB-MRAs. For convenience, we denote the operator $D_b T_k$ by $\mathcal{B}$.

Before proceeding, we need some conventions. We denote by $T^n = [-1/2, 1/2]^n$ the n-dimensional torus. For a Lebesgue measurable set E in $\mathbb{R}^n$, we denote by $|E|$ its measure, denote by χ_E the characteristic function of E, and define $E^{\sim} := E + \mathbb{Z}^n$. An $n \times n$ matrix A is called an expansive matrix if it is an integer matrix with all its eigenvalues greater than 1 in the module. G denotes the set of all expansive matrices. We denote by $GL_n(\mathbb{Z})$ the set $\{a : a$ is an $n \times n$ integral matrix and $|\det a| \neq 0\}$, by $\widetilde{SL_n}(\mathbb{Z})$ the set $\{a : a$ is an $n \times n$ integral matrix and $|\det a| = 1\}$, and by B the set of the subgroups of $\widetilde{SL_n}(\mathbb{Z})$, respectively. For a Lebesgue measurable function f, we define its support by

$$\operatorname{supp}(f) := \{x \in \mathbb{R}^n : f(x) \neq 0\}. \tag{1.2}$$

The Fourier transform of $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-i2\pi\langle \xi, x\rangle} \mathrm{d}x \tag{1.3}$$

on $\mathbb{R}^n$, where $\langle \xi, x\rangle$ denotes the inner product in $\mathbb{R}^n$. Let S be a Lebesgue nonzero measurable set in $\mathbb{R}^n$. We denote by $L^2(S)^\vee$ the closed subspace of $L^2(\mathbb{R}^n)$ of the form

$$L^2(S)^\vee := \left\{f \in L^2(\mathbb{R}^n) : \operatorname{supp}\left(\widehat{f}\right) \subseteq S\right\}. \tag{1.4}$$

Definition 1.1 (see [8, 9]). The sequence $\{x_{k,l}\}_{k,l}$ in a separable Hilbert space H is called a semiorthogonal PF for H if $\{x_{k,l}\}_{k,l}$ is a PF for H and satisfies $\langle x_{k_1,l_1}, x_{k_2,l_2}\rangle = 0$ for any $k_1, k_2 \in \Lambda_1$, $l_1, l_2 \in \Lambda_2$, and $k_1 \neq k_2$, where Λ_1, Λ_2 are two countable index sets. In particular, if $\{x_{k,l}\}_{k,l}$ is a semiorthogonal PF for $\overline{\operatorname{span}\{x_{k,l}\}_{k,l}}$, it is called a semiorthogonal sequence.

Definition 1.2 (see [4, 10]). A closed subspace X of $L^2(\mathbb{R}^n)$ is called a reducing subspace if $D_a X = X$ and $T_k X = X$ for any $k \in \mathbb{Z}^n$, $a \in G$.

The following proposition provides a characterization of reducing subspace.

Proposition 1.3 (see [4, 10]). *A closed subspace X of $L^2(\mathbb{R}^n)$ is a reducing subspace if and only if*

$$X = \left\{ f \in L^2(\mathbb{R}^n) : \operatorname{supp}\left(\hat{f}\right) \subseteq S \right\} \tag{1.5}$$

for some measurable set $S \subseteq \mathbb{R}^n$ with $\tilde{a}S = S$. So, to be specific, one denotes a reducing subspace by $L^2(S)^\vee$ instead of X. In particular, $L^2(\mathbb{R}^n)$ is a reducing subspace of $L^2(\mathbb{R}^n)$.

Definition 1.4 (see [5–7]). Let $B \ltimes \mathbb{Z}^n$ be a subgroup of the integral affine group $\widetilde{SL_n}(\mathbb{Z}) \ltimes \mathbb{Z}^n$ (the semidirect product of $\widetilde{SL_n}$ and $\mathbb{Z}^n$). The closed subspace V of $L^2(\mathbb{R}^n)$ is called a $B \ltimes \mathbb{Z}^n$ invariant subspace if $\mathcal{B}V = V$ for any $(b, k) \in B \ltimes \mathbb{Z}^n$.

Definition 1.5 (see [2–4]). Let B be a countable subset of $\widetilde{SL_n}(Z)$ and $A = \{a^i : i \in \mathbb{Z}\}$ where $a \in GL_n(\mathbb{Z})$. We say that a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is an AB-MRA if the following holds:

(1) V_0 is a $B \ltimes \mathbb{Z}^n$ invariant space;

(2) for each $j \in \mathbb{Z}, V_j \subset V_{j+1}$, and $V_j = D_a^j V_0$;

(3) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(4) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(5) there exists $\varphi \in V_0$ such that $\Phi_B = \{D_b T_k \varphi : b \in B,\ k \in \mathbb{Z}^n\}$ is a semiorthogonal PF for V_0.

The space V_0 is called an AB scaling space, and the function φ is an AB scaling function for V_0 or a generator of AB-MRA.

Similarly, we say that a sequence $\{V_j\}_{j\in\mathbb{Z}}$ is an AB-RMRA if it is an AB-MRA on $L^2(S)^\vee$, that is, conditions (1), (2), (4), (5), and (3)′ $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(S)^\vee$ are satisfied.

The fact that an AB-MRA can induce an AB-RMRA will be demonstrated by the obvious following results.

Proposition 1.6. *Let I be a countable index set and P the orthogonal projection operator from a Hilbert space H to its proper subspace K. If $\Psi = \{\psi_i : i \in I\}$ is a Parseval frame on H, then $P(\Psi) = \{P(\psi_i) : i \in I\}$ is a Parseval frame on K.*

Proposition 1.7. *Let P be the orthogonal projection operator from a Hilbert space H to its reducing subspace K. Then P can commutate with the shift and dilation operators T_k and D_a, respectively.*

Theorem 1.8. *Suppose that $\{\varphi; V_j\}$ is an AB-MRA, then $\{\tilde{\varphi}; \tilde{V}_j\}$ is an AB-RMRA for $L^2(S)^\vee$, where $\tilde{\varphi} := P\varphi$, $\tilde{V}_0 := \overline{\operatorname{span}}\{D_b T_k \tilde{\varphi} : b \in B,\ k \in \mathbb{Z}^n\}$, $\tilde{V}_j := \overline{\operatorname{span}}\{D_{a^j} D_b T_k \tilde{\varphi} : b \in B,\ k \in \mathbb{Z}^n\}$, and P is the orthogonal projection operator from $L^2(\mathbb{R}^n)$ to $L^2(S)^\vee$.*

The rest of this paper is organized as follows. Theorem 1.8 and some properties of an AB-RMRA will be proved in Section 2. In Section 3, the characterization of the generator for an AB-RMRA will be established, which is the main purpose of this paper. Finally, some examples are provided to illustrate the general theory.

2. Preliminaries

In this section, we will firstly prove Theorem 1.8 as follows.

We can easily prove that $\{D_bT_k\widetilde{\varphi} : b \in B, k \in \mathbb{Z}^n\}$ is a Parseval frame sequence by Propositions 1.6 and 1.7. Naturally, $\{D_bT_k\widetilde{\varphi} : b \in B, k \in \mathbb{Z}^n\}$ is a semi-Parseval frame for $\widetilde{V}_0$. Let $\varphi \in V_0 \subset L^2(\mathbb{R}^n)$. Thus $P\varphi = \varphi_1$. For any $f \in \widetilde{V}_0$, we have

$$\begin{aligned} f &= \sum_b \sum_k \langle f, D_bT_k\widetilde{\varphi}\rangle D_bT_k\widetilde{\varphi} = \sum_b \sum_k \langle f, D_bT_k\varphi_1\rangle D_bT_k\varphi_1, \\ D_{b'}T_{k'}f &= \sum_b \sum_k \langle f, D_bT_k\varphi_1\rangle \varphi_1(b'bx - bk' - k) = \sum_b \sum_k \langle f, D_bT_k\varphi_1\rangle \varphi_1(bx-k), \end{aligned} \tag{2.1}$$

namely, $D_{b'}T_{k'}f \in \widetilde{V}_0$. So $\widetilde{V}_0$ is a $B \ltimes \mathbb{Z}^n$ invariant space. On the other hand,

$$\begin{aligned} f &= \sum_b \sum_k \langle f, D_bT_kP\varphi\rangle D_bT_kP\varphi = \sum_b \sum_k \langle Pf, D_bT_k\varphi\rangle PD_bT_k\varphi \\ &= \sum_b \sum_k \langle f, D_bT_k\varphi\rangle P\sum_{b'} \sum_{k'} \langle D_bT_k\varphi, D_aD_bT_k\varphi\rangle D_aD_{b'}T_{k'}\varphi \\ &= \sum_b \sum_k c_{b,k}D_aD_bT_k\widetilde{\varphi} \in \widetilde{V}_1. \end{aligned} \tag{2.2}$$

So $\widetilde{V}_0 \subset \widetilde{V}_1$. Notice that $\widetilde{V}_i = D_{a^i}\widetilde{V}_0$. Then $\widetilde{V}_i \subset \widetilde{V}_{i+1}$. Thus, conditions (1), (2), and (5) in Definition 1.5 have been proved. However, condition $(3)'$ is the natural consequence of the later Lemma 3.1 in Section 3. Therefore, we complete the proof of Theorem 1.8.

Some properties of AB-RMRA, which were not discussed in [5–7], will be presented. The first one can be obtained obviously by the definition of AB-RMRA as follows.

Proposition 2.1. *Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is an AB-RMRA. Then*

(1) *for each $j \in \mathbb{Z}$, $\{D_a^jD_bT_k\varphi : b \in B, k \in \mathbb{Z}^n\}$ is a semiorthogonal PF on V_j;*

(2) *V_0 is a $B \ltimes \mathbb{Z}^n$ invariant subspace, while V_j is a $B \ltimes a^{-j}\mathbb{Z}^n$ invariant subspace.*

Condition (5) of AB-RMRA can be characterized by the following proposition.

Proposition 2.2. *Let $\varphi \in L^2(S)^\vee$. Then $\Phi_B = \{D_bT_k\varphi : b \in B, k \in \mathbb{Z}^n\}$ is a semiorthogonal PF sequence if and only if*

(1) *$\sum_{k\in\mathbb{Z}^n} |\widehat{\varphi}(\xi + k)|^2 = \chi_F(\xi)$, a.e., where $F = \{\xi \in T^n \cap \Omega : \widehat{\varphi}(\xi + k) \neq 0, k \in \mathbb{Z}^n\}$;*

(2) *$\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}(\xi + k)\overline{\widehat{\varphi}(\widetilde{b}^{-1}(\xi + k))} = 0$, a.e. $\xi \in \Omega$, for each $b \in B$ and $b \neq I_n$.*

Proof. Necessity. For any $f(x) \in \overline{\operatorname{span}}\{T_k\varphi : k \in \mathbb{Z}^n\}$, we have

$$f(x) = \sum_{b,k} \langle f, \mathcal{B}\varphi\rangle \mathcal{B}\varphi(x) = \sum_k \langle f, T_k\varphi\rangle T_k\varphi(x) + \sum_{b\neq I_n} \sum_k \langle f, \mathcal{B}\varphi\rangle \mathcal{B}\varphi(x) = \sum_k \langle f, T_k\varphi\rangle T_k\varphi(x). \tag{2.3}$$

By Theorem 1.6 in [1] and Theorem 7.2.3 in [8], conclusion (1) holds clearly. Using Parseval theorem, we can deduce

$$\begin{aligned}\langle T_k\varphi, D_bT_{k'}\varphi\rangle &= \int_\Omega \varphi(x-k)\overline{\varphi(bx-k')}\mathrm{d}x \\ &= \int_\Omega \widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\widetilde{b}^{-1}\xi)}e^{-2\pi i(k-\widetilde{b}^{-1}k')\cdot\xi}\mathrm{d}\xi \\ &= \int_\Omega \sum_l \widehat{\varphi}(\xi+l)\overline{\widehat{\varphi}(\widetilde{b}^{-1}(\xi+l))}e^{-2\pi i(k-k_1)\cdot\xi}\mathrm{d}\xi,\end{aligned} \tag{2.4}$$

where $k_1 = \widetilde{b}^{-1}k'$. Note that for any $k, k' \in \mathbb{Z}^n$, $b \neq b' \in B$, $\langle D_bT_k\varphi, D_{b'}T_{k'}\varphi\rangle = 0$ if and only if for any $b \in B$ and $b \neq I_n$,

$$\langle T_k\varphi, D_bT_{k'}\varphi\rangle = 0. \tag{2.5}$$

Then, we have

$$\sum_l \widehat{\varphi}(\xi+l)\overline{\widehat{\varphi}(\widetilde{b}^{-1}(\xi+l))} = 0, \quad \text{a.e. } \xi \in \Omega. \tag{2.6}$$

Sufficiency. By Theorem 7.2.3 in [8] and conclusion (1), $\{T_k\varphi : k \in \mathbb{Z}^n\}$ is a PF sequence. So is $\{D_bT_k\varphi : k \in \mathbb{Z}^n\}$ for any $b \in B$. It follows from (2.4), (2.5), and conclusion (2) that for any $k, k' \in \mathbb{Z}^n$, $b, b' \in B$, and $b \neq b'$, we get $\langle D_bT_k\varphi, D_{b'}T_{k'}\varphi\rangle = 0$. Thus, for any $f(x) \in \overline{\operatorname{span}}\{D_bT_k\varphi : b \in B,\ k \in \mathbb{Z}^n\}$, there exists $b \in B$ and $f_b(x) \in \overline{\operatorname{span}}\{D_bT_k\varphi : k \in \mathbb{Z}^n\}$ such that

$$\|f\|^2 = \sum_{b\in B}\|f_b\|^2 = \sum_{b\in B}\sum_{k\in\mathbb{Z}^n}\left|\langle f_b, D_bT_k\varphi\rangle\right|^2 = \sum_{b\in B}\sum_{k\in\mathbb{Z}^n}\left|\langle f, D_bT_k\varphi\rangle\right|^2. \tag{2.7}$$

By Theorem 1.6 in [1], the proof of Proposition 2.2 is completed. □

Proposition 2.3. *Let $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspace of $L^2(S)^\vee$, where*

$$V_j := \overline{\operatorname{span}}\left\{D_a^jD_bT_k\varphi : b \in B,\ k \in \mathbb{Z}^n\right\}. \tag{2.8}$$

If conditions (1), (2), and (5) of AB-RMRA are satisfied, then one has the following.

(1) *There exists $\{c_{b,k}\} \in l^2(B \times \mathbb{Z}^n)$ such that*

$$\varphi(x) = \sum_b\sum_k c_{b,k}|\det a|^{1/2}\varphi(bax-k). \tag{2.9}$$

(2) *There exists* $\{h_b(\xi)\}_{b\in B} \subseteq L^\infty(T^n)$ *such that*

$$\widehat{\varphi}(\xi) = \sum_b h_b\left[\widetilde{(ba)}^{-1}\xi\right]\widehat{\varphi}\left[\widetilde{(ba)}^{-1}\xi\right], \tag{2.10}$$

for any $b \in B$*, where* $h_b(\xi) = |\det a|^{-1/2}\sum_k c_{b,k}e^{-2\pi ik\cdot\xi}$.

Proof. By conditions (2) and (5) of AB-RMRA and the fact that $\varphi \in V_0 \subset V_1$, we obtain

$$\varphi(x) = \sum_{b,k}\langle\varphi, D_a\mathcal{B}\varphi\rangle D_a\mathcal{B}\varphi(x) = \sum_b\sum_{k\in\mathbb{Z}^n} c_{b,k}|\det a|^{1/2}\varphi(bax - k), \tag{2.11}$$

where $c_{b,k} = \langle\varphi, D_a\mathcal{B}\varphi\rangle$ and $\{c_{b,k}\} \in l^2(B\times\mathbb{Z}^n)$. Therefore (2.9) holds. Taking Fourier transform on both sides of (2.9), we obtain (2.10), where for any $b \in B$, $h_b(\xi) = |\det a|^{-1/2}\sum_k c_{b,k}e^{-2\pi ik\cdot\xi}$. In what follows, we will only prove $\{h_b(\xi)\}_{b\in B} \subseteq L^\infty(T^n)$. Indeed, for $a \in GL_n(\mathbb{Z}), \widetilde{a}\mathbb{Z}^n$ is a subgroup of $\mathbb{Z}^n$ and the quotient group $\mathbb{Z}^n/\widetilde{a}\mathbb{Z}^n$ has order $M = |\det a|$. Thus, we can choose a complete set of representatives of $\mathbb{Z}^n/\widetilde{a}\mathbb{Z}^n$, that is, the set $\{\alpha_0, \alpha_1, \ldots, \alpha_{M-1}\}$ so that each $k \in \mathbb{Z}^n$ can be uniquely expressed in the form $k = \widetilde{a}k' + \alpha_i$ with $k' \in \mathbb{Z}^n$, $0 \le i \le M-1$. For simplicity, we denote $\widetilde{(ba)}^{-1}$ and $\widetilde{(b'a)}^{-1}$ by b^* and b_1^*, respectively. Then we have

$$\begin{aligned}
\sum_k\left|\widehat{\varphi}(\xi+k)\right|^2 &= \sum_{k\in\mathbb{Z}^n}\left|\sum_b h_b[b^*(\xi+k)]\widehat{\varphi}[b^*(\xi+k)]\right|^2 \\
&= \sum_{k\in\mathbb{Z}^n}\sum_{b,b_1} h_b[b^*(\xi+k)]\overline{h_{b_1}[b_1^*(\xi+k)]}\widehat{\varphi}[b^*(\xi+k)]\overline{\widehat{\varphi}[b_1^*(\xi+k)]} \\
&= \sum_{i=0}^{M-1}\sum_{k',b,b_1} h_b[b^*(\xi+\alpha_i)]\overline{h_{b_1}[b_1^*(\xi+\alpha_i)]}\widehat{\varphi}\left[b^*(\xi+\alpha_i)+\widetilde{b}^{-1}k'\right] \\
&\quad\times\overline{\widehat{\varphi}[b_1^*(\xi+\alpha_i)+\widetilde{b_1}^{-1}k']}
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&= \sum_{i=0}^{M-1}\sum_{b,b_1} h_b[b^*(\xi+\alpha_i)]\overline{h_{b_1}[b_1^*(\xi+\alpha_i)]} \\
&\quad\times\sum_{k'\in\mathbb{Z}^n}\widehat{\varphi}\left[b^*(\xi+\alpha_i)+\widetilde{b}^{-1}k'\right]\overline{\widehat{\varphi}[b_1^*(\xi+\alpha_i)+\widetilde{b_1}^{-1}k']} \\
&= \sum_{i=0}^{M-1}\sum_b|h_b[b^*(\xi+\alpha_i)]|^2\sum_{k'\in\mathbb{Z}^n}\left|\widehat{\varphi}\left[b^*(\xi+\alpha_i)+\widetilde{b}^{-1}k'\right]\right|^2,
\end{aligned} \tag{2.13}$$

where (2.12) is obtained by the periodicity of function sequence $\{h_b(\xi)\}_b$ and (2.13) is proved by conclusion (2) in Proposition 2.2. In addition, using Proposition 2.2 again and (2.13) above, for any $\xi \in F$, we get $\sum_{k\in\mathbb{Z}^n}|\widehat{\varphi}(\xi+k)|^2 = \sum_{i=0}^{M-1}\sum_b|h_b[b^*\xi + b^*\alpha_i)]|^2 = 1$. Then, for $\xi \in \mathbb{R}^n$, $|h_b(\xi)|^2 \le 1$, so $h_b(\xi) \in L^\infty(T^n)$. Therefore, the proof of Proposition 2.3 is completed. □

3. Characterization of the Generator for an AB-RMRA

In this section, we will characterize the scaling function of AB-RMRA which will determine a multiresolution structure and AB-wavelets and the obtained results can be easily extended to the whole space $L^2(\mathbb{R}^n)$.

Lemma 3.1. *Let $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces of $L^2(S)^\vee$ and defined by* (2.8). *Assume that conditions (1), (2), and (5) in an AB-RMRA are satisfied. Then the following results are equivalent:*

(1) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(S)^\vee$;

(2) $\lim_{j\to+\infty}\sum_b |\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]|^2 = 1$, *a.e.* $\xi\in\mathbb{R}^n$.

Proof. Theorems 1.7 and 5.2 in [1] imply that for any $f\in L^2(\mathbb{R}^n)$, $\lim_{j\to+\infty}\|P_j f\|^2 = \|f\|^2$ is equivalent to $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$. Thus, for any $f\in L^2(S)^\vee$, $\lim_{j\to+\infty}\|P_j f\|^2 = \|f\|^2$ is equivalent to $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(S)^\vee$. Hence, we have to prove that $\lim_{j\to+\infty}\|P_j f\|^2 = \|f\|^2$ is equivalent to $\lim_{j\to+\infty}\sum_b |\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]|^2 = 1$, a.e. $\xi\in\Omega$. First, we prove (1)⇒(2) For any $f\in L^2(S)^\vee$, $f = P_j f + Q_j f$, where $Q_j : L^2(\mathbb{R}) \to (V_j)^\perp$ is the orthogonal projection operator. Set $\widehat{f}(\xi) = \chi_{T^n\cap S}(\xi)$. Then, when j is large enough, we have

$$
\begin{aligned}
\|P_j f\|^2 &= \sum_b\sum_k \left|\left\langle f, D_a^j D_b T_k \varphi\right\rangle\right|^2 \\
&= \sum_b\sum_k \left|\int_{\mathbb{R}^n} |\det a_j|^{-1/2}\widehat{f}(\xi)\overline{\widehat{\varphi}\left[\widetilde{(ba_j)}^{-1}\xi\right]}e^{2\pi i k\cdot\widetilde{(ba_j)}^{-1}\xi}d\xi\right|^2 \\
&= |\det a_j|\sum_b\sum_k \left|\int_{\widetilde{(ba_j)}^{-1}T^n}\overline{\widehat{\varphi}(\xi)}e^{2\pi i k\cdot\xi}\right|^2 d\xi \\
&= |\det a_j|\sum_b\int_{T^n}\left|\chi_{\widetilde{(ba_j)}^{-1}T^n}(\xi)\overline{\widehat{\varphi}(\xi)}\right|^2 d\xi \\
&= \sum_b\int_{T^n}\left|\widehat{\varphi}\left[\widetilde{(ba_j)}^{-1}(\xi)\right]\right|^2 d\xi \\
&= \int_{T^n}\sum_b\left|\widehat{\varphi}\left[\widetilde{(ba_j)}^{-1}(\xi)\right]\right|^2 d\xi.
\end{aligned}
\tag{3.1}
$$

Before proving the equivalence, we need to prove two assertions as follows:

(i) $\sum_b |\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]|^2 \in L^1(T^n)$;

(ii) $\lim_{j\to+\infty}\sum_b |\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]|^2$ makes sense.

Since

$$\begin{aligned}
\int_{T^n}\sum_b\left|\widehat{\varphi}\left[\widetilde{(ba_j)}^{-1}\xi\right]\right|^2 d\xi &= \sum_b\int_{T^n}\left|\widehat{\varphi}\left[\widetilde{(ba_j)}^{-1}\xi\right]\right|^2 d\xi \\
&= \left|\det a_j\right|\sum_b\int_{\widetilde{(ba_j)}^{-1}T^n}\left|\widehat{\varphi}(\xi)\right|^2 d\xi \\
&= \left|\det a_j\right|\int_{\bigcup_b \widetilde{(ba_j)}^{-1}T^n}\left|\widehat{\varphi}(\xi)\right|^2 d\xi \\
&\le \left|\det a_j\right|\int_{\mathbb{R}^n}\left|\widehat{\varphi}(\xi)\right|^2 d\xi < \infty,
\end{aligned} \tag{3.2}$$

it follows that (i) holds.

For (ii), we will only prove that $\{\sum_b|\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]|^2\}_{j\in\mathbb{Z}}$ is a monotonic bounded sequence when $\xi(\in S)$ is fixed. Indeed, by the orthogonality, for each $b\neq b'\in B$, we have $\operatorname{supp}(\widehat{\varphi}(\widetilde{b}^{-1}\xi))\cap\operatorname{supp}(\widehat{\varphi}(\widetilde{b'}^{-1}\xi))=\emptyset$. In addition, we deduce from (2.10) that, for any $b\in B$,

$$\begin{aligned}
\widehat{\varphi}\left[\widetilde{(ba_j)}^{-1}\xi\right] &= \sum_{b'\in B}h_{b'}\left[\widetilde{(b'a)}^{-1}\widetilde{(ba_j)}^{-1}\xi\right]\widehat{\varphi}\left[\widetilde{(b'a)}^{-1}\widetilde{(ba_j)}^{-1}\xi\right] \\
&= \sum_{b'\in B}h_{b'}\left(\widetilde{b'}^{-1}\widetilde{b}^{-1}\widetilde{a_{j+1}}^{-1}\xi\right)\widehat{\varphi}\left(\widetilde{b'}^{-1}\widetilde{b}^{-1}\widetilde{a_{j+1}}^{-1}\xi\right).
\end{aligned} \tag{3.3}$$

Set $b^*=\widetilde{b}^{-1}\widetilde{a_{j+1}}^{-1}$. Then by the orthogonality and Proposition 2.3, we obtain

$$\begin{aligned}
\sum_{b\in B}\left|\widehat{\varphi}\left[\widetilde{(ba_j)}^{-1}\xi\right]\right|^2 &= \sum_{b\in B}\left|\sum_{b'\in B}h_{b'}\left(\widetilde{b'}^{-1}b^*\xi\right)\widehat{\varphi}\left(\widetilde{b'}^{-1}b^*\xi\right)\right|^2 \\
&= \sum_{b\in B}\sum_{b_1\in B}h_{b_1}\left(\widetilde{b_1}^{-1}b^*\xi\right)\widehat{\varphi}\left(\widetilde{b_1}^{-1}b^*\xi\right)\overline{\sum_{b_2\in B}h_{b_2}\left(\widetilde{b_2}^{-1}b^*\xi\right)\widehat{\varphi}\left(\widetilde{b_2}^{-1}b^*\xi\right)} \\
&= \sum_{b\in B}\left|h_b\left(\widetilde{(ba_{j+1})}^{-1}\xi\right)\widehat{\varphi}\left(\widetilde{(ba_{j+1})}^{-1}\xi\right)\right|^2 \\
&\le \sum_{b\in B}\left|\widehat{\varphi}\left(\widetilde{(ba_{j+1})}^{-1}\xi\right)\right|^2.
\end{aligned} \tag{3.4}$$

Hence, $\{\sum_{b\in B}|\widehat{\varphi}(\widetilde{(ba_j)}^{-1}\xi)|^2\}_{j\in\mathbb{Z}}$ is a monotonic sequence when ξ is fixed. On the other hand, by the property of B, we deduce that $D_bT_k\varphi(x)=\varphi(bx-k)=\varphi(b(x-b^{-1}k))=\varphi(b(x-k'))=T_{k'}D_b\varphi(x)$. Note that $\{D_bT_k\varphi : k\in\mathbb{Z}^n\}$ is a PF for $\overline{\operatorname{span}}\{D_bT_k\varphi : k\in\mathbb{Z}^n\}$ for any $b\in B$. Then $\{T_{k'}D_b\varphi : k'\in\mathbb{Z}^n\}$ is a PF on $\overline{\operatorname{span}}\{D_bT_k\varphi : k\in\mathbb{Z}^n\}$, which implies that there exists a set M_b such that $\sum_k |\widehat{\varphi}(\widetilde{b}^{-1}\xi + k)|^2 = \chi_{M_b}(\xi)$. By the orthogonality, $\sum_b \sum_k |\widehat{\varphi}(\widetilde{b}^{-1}\xi + k)|^2 = \chi_{\bigcup_b M_b}(\xi)$, consequently $\bigcup_b M_b = F$. Hence, $\sum_b |\widehat{\varphi}(\widetilde{b}^{-1}\xi)|^2 \le 1$ holds for $\xi \in S$. So $\lim_{j\to+\infty} \sum_{b\in B} |\widehat{\varphi}(\widetilde{(ba_j)}^{-1}\xi)|^2$ exists. Now we have proved the two assertions. By the Lebesgue dominant convergence theorem, we get $\lim_{j\to+\infty}\|P_j f\|^2 = \int_{T^n} \lim_{j\to+\infty} \sum_b |\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]|^2 \mathrm{d}\xi = \|f\|^2 = |T^n| = 1$. Thus, $\lim_{j\to+\infty} \sum_{b\in B} |\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]|^2 = 1$, a.e. $\xi \in \mathbb{R}^n$.

Next, we prove (2)⇒(1). Let D be the class of all functions $f \in L^2(\mathbb{R}^n)$ such that $\widehat{f} \in L^\infty(\mathbb{R}^n)$ and $\widehat{f}$ is compactly supported in $\mathbb{R}^n \setminus \{0\}$. If we can show that $\lim_{j\to+\infty}\|P_j f\|^2 = \|f\|^2$ for all $f \in D$, then, by Lemma 1.10 in [1], the proof is finished. Indeed, denoting $\widetilde{(ba_j)}^{-1}$ by b^*, we have

$$\begin{aligned}
\|P_j f\|^2 &= \sum_b \sum_k \left|\left\langle f, D_{a_j} D_b T_k \varphi \right\rangle\right|^2 \\
&= \sum_b \sum_k \left| \int_{\mathbb{R}^n} \left|\det a_j\right|^{-1/2} \widehat{f}(\xi)\overline{\widehat{\varphi}(b^*\xi)} e^{2\pi i k\cdot b^*\xi} \mathrm{d}\xi \right|^2 \\
&= \sum_b \sum_k \left| \sum_m \int_{\widetilde{a}_j T^n} \left|\det a_j\right|^{-1/2} \widehat{f}(\xi + \widetilde{a}_j m)\overline{\widehat{\varphi}[b^*(\xi + \widetilde{a}_j m)]} e^{2\pi i k\cdot b^*\xi} \mathrm{d}\xi \right|^2 \\
&= \sum_b \sum_k \left| \int_{\widetilde{a}_j T^n} \left|\det a_j\right|^{-1/2} \sum_m \widehat{f}(\xi + \widetilde{a}_j m)\overline{\widehat{\varphi}[b^*(\xi + \widetilde{a}_j m)]} e^{2\pi i k\cdot b^*\xi} \mathrm{d}\xi \right|^2 \\
&= \sum_b \int_{\widetilde{a}_j T^n} \left| \sum_m \widehat{f}(\xi + \widetilde{a}_j m)\overline{\widehat{\varphi}\left[b^*\left(\xi + a_j^T m\right)\right]} \right|^2 \mathrm{d}\xi \\
&= \sum_b \int_{\widetilde{a}_j T^n} \sum_m \widehat{f}(\xi + \widetilde{a}_j m)\overline{\widehat{\varphi}[b^*(\xi + \widetilde{a}_j m)]} \sum_n \overline{\widehat{f}(\xi + \widetilde{a}_j n)}\widehat{\varphi}[b^*(\xi + \widetilde{a}_j n)] \mathrm{d}\xi \\
&= \sum_b \sum_m \int_{\widetilde{a}_j T^n + m} \sum_p \widehat{f}(\eta)\overline{\widehat{f}(\eta + \widetilde{a}_j p)}\widehat{\varphi}\left[b^*\eta + \widetilde{b}p\right]\overline{\widehat{\varphi}(b^*\eta)} \mathrm{d}\eta \\
&= \sum_b \int_{\mathbb{R}^n} \sum_p \widehat{f}(\eta)\overline{\widehat{f}(\eta + \widetilde{a}_j p)}\widehat{\varphi}\left(b^*\eta + \widetilde{b}p\right)\overline{\widehat{\varphi}(b^*\eta)} \mathrm{d}\eta \\
&= \int_{\mathbb{R}^n} \left|\widehat{f}(\eta)\right|^2 \sum_b \left|\widehat{\varphi}\left(\widetilde{b}^{-1}\widetilde{a}_j\eta\right)\right|^2 \mathrm{d}\eta + R_f,
\end{aligned} \tag{3.5}$$

where $R_f = \sum_b \sum_{p\neq 0} \int_{\mathbb{R}^n} \widehat{f}(\eta)\overline{\widehat{f}(\eta + \widetilde{a}_j p)}\widehat{\varphi}(b^*\eta + \widetilde{b}p)\overline{\widehat{\varphi}(b^*\eta)} d\eta$. Since f has compact support, when j is large enough, $\operatorname{supp}(\widehat{f}(\eta)) \cap \operatorname{supp}(\widehat{f}(\eta + \widetilde{a}_j p)) = \emptyset$, consequently, $R_f = 0$. Thus, taking $j \to +\infty$ in (3.5), we obtain

$$\begin{aligned}
\lim_{j\to+\infty} \left\|P_j f\right\|^2 &= \lim_{j\to+\infty} \sum_b \sum_k \left| \left\langle f, D_{a_j} D_b T_k \varphi \right\rangle \right|^2 \\
&= \lim_{j\to+\infty} \sum_b \int_{\mathbb{R}^n} \left|\widehat{f}(\eta)\right|^2 \sum_b \left|\widehat{\varphi}\left(\widetilde{b}^{-1} \widetilde{a}_j^{-1} \eta\right)\right|^2 \mathrm{d}\eta \\
&= \int_{\mathbb{R}^n} \left|\widehat{f}(\eta)\right|^2 \lim_{j\to+\infty} \sum_b \left|\widehat{\varphi}\left(\widetilde{b}^{-1} \widetilde{a}_j^{-1} \eta\right)\right|^2 \mathrm{d}\eta \\
&= \int_{\mathbb{R}^n} \left|\widehat{f}(\eta)\right|^2 \mathrm{d}\eta = \|f\|^2.
\end{aligned} \tag{3.6}$$

□

Lemma 3.2. *Let $a \in G$, $\varphi \in L^2(\mathbb{R}^n)$ satisfy* (2.10), *and let $\{V_j\}_{j\in\mathbb{Z}}$ be defined by* (2.8). *Then*

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(S)^{\vee}, \tag{3.7}$$

where $S = \bigcup_{j\in\mathbb{Z}} \bigcup_{b\in B} \widetilde{a}_j \widetilde{b} \operatorname{supp}(\widehat{\varphi})$.

Proof. By the definition of $\{V_j\}_{j\in\mathbb{Z}}$, we have $D_a V_j = V_{j+1}$ for any $j \in \mathbb{Z}$. It follows that $D_a(\bigcup_{j\in\mathbb{Z}} V_j) = \bigcup_{j\in\mathbb{Z}} V_j$. Note that D_a is a unitary operator. Hence $D_a(\overline{\bigcup_{j\in\mathbb{Z}} V_j}) = \overline{D_a(\bigcup_{j\in\mathbb{Z}} V_j)} = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$. It is obvious to see that $V_j = \{f : \widehat{f} = \sum_b F_b[\widetilde{(ba_j)}^{-1}\xi]\widehat{\varphi}[\widetilde{(ba_j)}^{-1}\xi]\}$, where $\{F_b\}_b \in L^2(T^n)$. Then, for any $f \in V_0$, we have $\widehat{f}(\xi) = \sum_b F_b(\widetilde{b}^{-1}\xi)\widehat{\varphi}(\widetilde{b}^{-1}\xi)$, $F_b \in L^2(T^n)$. Notice that for any $b \in B$, $\widehat{\varphi}(\widetilde{b}^{-1}\xi) = \sum_b h_b(\widetilde{b}^{-1}\widetilde{a}^{-1}\xi)\widehat{\varphi}(\widetilde{b}^{-1}\widetilde{a}^{-1}\xi)$. Thus

$$\begin{aligned}
\widehat{f}(\xi) &= \sum_{b_1} F_{b_1}\left(b_1^{-T}\xi\right) \sum_{b_2} h_{b_2}\left(\widetilde{b_2}^{-1} \widetilde{a}^{-1}\xi\right) \widehat{\varphi}\left(\widetilde{b_2}^{-1} \widetilde{a}^{-1}\xi\right) \\
&= \sum_{b_1} \sum_{b_2} F_{b_1}\left(\widetilde{b_1}^{-1}\xi\right) h_{b_2}\left(\widetilde{b_2}^{-1} \widetilde{a}^{-1}\xi\right) \widehat{\varphi}\left(\widetilde{b_2}^{-1} \widetilde{a}^{-1}\xi\right).
\end{aligned} \tag{3.8}$$

Put $H_{b_2}(\widetilde{b}^{-1}\widetilde{a}^{-1}\xi) = \sum_{b_1} F_{b_1}(\widetilde{b_1}^{-1}\xi) M_{b_2}(\widetilde{b_2}^{-1}\widetilde{a}^{-1}\xi)$. Then, we obtain $\widehat{f}(\xi) = \sum_{b_2} H_{b_2}(\widetilde{b_2}^{-1}\widetilde{a}^{-1}\xi)\widehat{\varphi}(\widetilde{b_2}^{-1}\widetilde{a}^{-1}\xi)$. Recalling that $F_b(\xi) \in L^2(T^n)$ and $h_b(\xi) \in L^\infty(T^n)$, we have $M_{b_2}(\xi) \in L^2(T^n)$, so $f(x) \in V_1$. Thus, for any $f(x)$, we get $V_0 \subset V_1$, so $V_j \subset V_{j+1}$.

Therefore, for any $f(x) \in \bigcup_{j\in\mathbb{Z}} V_j$, we can choose $j_f > 0$ such that $f(x) \in V_{j_f}$, that is, $f(x) = \sum_b \sum_k c_{b,k}\varphi(ba_{j_f}x - k)$. Hence, for any $m \in \mathbb{Z}^n$, we have

$$\begin{aligned} T_m f(x) &= \sum_b \sum_k c_{b,k}\varphi\Big(ba_{j_f}(x-m) - k\Big) \\ &= \sum_b \sum_k c_{b,k}\varphi\Big(ba_{j_f}x - ba_{j_f}m - k\Big) \\ &= \sum_b \sum_k c_{b,k}\varphi\Big(ba_{j_f}x - k\Big), \end{aligned} \tag{3.9}$$

which implies $T_m(\bigcup_{j\in\mathbb{Z}} V_j) \subseteq \bigcup_{j\in\mathbb{Z}} V_j$. Thus, $T_m(\bigcup_{j\in\mathbb{Z}} V_j) = \bigcup_{j\in\mathbb{Z}} V_j$ holds. Note that T_k is also a unitary operator. Hence, $T_m(\overline{\bigcup_{j\in\mathbb{Z}} V_j}) = \overline{T_m(\bigcup_{j\in\mathbb{Z}} V_j)} = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$. By Proposition 1.3, $\overline{\bigcup_{j\in\mathbb{Z}} V_j}$ is a reducing subspace. We notate $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(S)^\vee$. Next we have to prove $S = \bigcup_{j\in\mathbb{Z}}\bigcup_b \widetilde{a}_j\widetilde{b}\,\mathrm{supp}(\widehat{\varphi})$. By $\varphi(ba_j\cdot) \in V_j$, we have $\mathrm{supp}(\widehat{\varphi}(\widetilde{(ba_j)}^{-1}\xi)) \subset S$. Obviously, we will only prove that $S \setminus \bigcup_{j\in\mathbb{Z}}\bigcup_b \widetilde{a}_j\widetilde{b}\,\mathrm{supp}(\widehat{\varphi})$ is a zero measurable set. If $S \setminus \bigcup_{j\in\mathbb{Z}}\bigcup_b \widetilde{a}_j\widetilde{b}\,\mathrm{supp}(\widehat{\varphi})$ is a set with nonzero measure, then a contradiction is led. In fact, choosing a set $M \subset S \setminus \bigcup_{j\in\mathbb{Z}}\bigcup_b \widetilde{a}_j\widetilde{b}\,\mathrm{supp}(\widehat{\varphi})$ with $0 < |M| < +\infty$ and using Plancherel theorem, we have

$$\|P_j f\|^2 = \langle P_j f, P_j f\rangle = \langle f, P_j f\rangle = \Big\langle \widehat{f}, \widehat{f}_j'\Big\rangle, \tag{3.10}$$

for any $f \in L^2(\mathbb{R}^n)$, where $f_j' := P_j f$. In particular, setting $\widehat{f}(\xi) = \chi_M(\xi)$ and taking $j \to +\infty$ in (3.10), we get $\|P_j f\|^2 = \langle \widehat{f}, \widehat{f}_j'\rangle = 0$, but $\|f\|^2 = |M| > 0$, and this is a contradiction. Therefore, we complete the proof of Lemma 3.2. □

By Lemmas 3.1 and 3.2 and Theorems 1.7 and 5.2 in [1], we characterize the density condition of AB-RMRA as follows.

Theorem 3.3. *Let $a \in G$ and $\{V_j\}_{j\in\mathbb{Z}}$ be defined by* (2.8)*. If conditions (1), (2), and (5) of AB-RMRA are satisfied, then the following results are equivalent:*

(1) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(S)^\vee$;

(2) $\lim_{j\to+\infty}\|P_j f\|^2 = \|f\|^2$, *where $P_j : L^2(\mathbb{R}^n) \to V_j$ denotes an orthogonal projection operator;*

(3) $\lim_{j\to+\infty}\sum_{b\in B}|\widehat{\varphi}(\widetilde{(ba_j)}^{-1}\xi)|^2 = 1$, *a.e.* $\xi \in S$;

(4) $S = \bigcup_{j\in\mathbb{Z}}\bigcup_b \widetilde{a}_j\widetilde{b}\,\mathrm{supp}\,\widehat{\varphi}$.

It is well known that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ can be deduced by the other conditions of MRA. Similarly, the condition $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ of AB-RMRA can be also deduced by the others. Thus, using Proposition 2.2 and Theorem 3.3, we can get the main theorem as follows.

Theorem 3.4. *Let $a \in G$, and let B be a subgroup of $\widetilde{SL_n}(\mathbb{Z})$ with $aBa^{-1} \subseteq B$. Suppose that $\varphi \in L^2(S)^\vee$ and $\{V_j\}_{j\in\mathbb{Z}}$ is defined by* (2.8)*. Then φ is a generator of AB-RMRA if and only if*

(1) *there exists* $\{h_b(\xi)\}_{b\in B} \subset L^\infty(T^n)$ *such that*

$$\widehat{\varphi}(\xi) = \sum_b M_b\left(\widetilde{b}^{-1}\widetilde{a}^{-1}\xi\right)\widehat{\varphi}\left(\widetilde{b}^{-1}\widetilde{a}^{-1}\xi\right); \tag{3.11}$$

(2) $\sum_k |\widehat{\varphi}(\xi+k)|^2 = \chi_F(\xi)$, *a.e., where* $F = \{\xi \in T^n \cap S : \widehat{\varphi}(\xi+k) \neq 0,\ k \in \mathbb{Z}^n\}$;

(3) *for any* $b \in B$ *and* $b \neq I_n$, $\sum_k \widehat{\varphi}(\xi+k)\overline{\widehat{\varphi}(\widetilde{b}^{-1}(\xi+k))} = 0$, *a.e.* $\xi \in S$;

(4) $S = \bigcup_{j\in\mathbb{Z}}\bigcup_b \widetilde{a}_j\widetilde{b}\,\mathrm{supp}(\widehat{\varphi})$.

Remarks 1. (1) Condition (4) in Theorem 3.4 can be replaced by any one of conditions in Theorem 3.3.

(2) If $S = \mathbb{R}^n$ in Theorem 3.4, then we obtain the characterization of generator of AB-MRA.

(3) If $n = 1$, $a = (2)$, and $B = (1)$ in Theorem 3.4, then we obtain the characterization of $\{V_j\}_{j\in\mathbb{Z}}$ as a generator of MRA on $L^2(\mathbb{R})$.

Corollary 3.5. *Let* $a \in G$, *and let* B *be a subgroup of* $\widetilde{SL_n}(\mathbb{Z})$ *with* $aBa^{-1} \subsetneq B$. *Suppose that* E *is a bounded nonzero measurable set satisfying* $\widetilde{a}^{-1}E \subset E$ *with* $S = \bigcup_{j\in\mathbb{Z}}\bigcup_b \widetilde{a}_j\widetilde{b}E$. *Define* $\widehat{\varphi}(\xi) = \chi_E(\xi)$ *and* $F = \{\xi \in T^n \cap \Omega : \widehat{\varphi}(\xi+k) \neq 0,\ k \in \mathbb{Z}^n\}$. *Then* φ *generates an AB-RMRA if and only if*

(1) $F = \bigcup_{k\in\mathbb{Z}^n}(E+k)$ *with* $|E \cap (E+k)| = 0$, $|b^T E \cap b'^T E| = 0$ $(b \neq b')$;

(2) *there exists* $b \in B$ *such that* $\widetilde{(ba)}^{-1}E \subset E$ *and* $[\widetilde{(ba)}^{-1}E]^{\sim} \cap E = \widetilde{(ba)}^{-1}E$;

(3) $T_k\chi_{\widetilde{(ba)}^{-1}E}(\xi) = \chi_{\widetilde{(ba)}^{-1}E}(\xi)$ *on* $\widetilde{(ba)}^{-1}E \cap [\widetilde{(ba)}^{-1}E + k] \neq \emptyset$ *for* $k \in \mathbb{Z}^n \setminus \{0\}$.

4. Some Examples

Three examples are provided to illustrate the general theory in this section.

Example 4.1. Let $\widehat{\varphi}(\xi) = \chi_I(\xi)$, where $I = I^+ \cup I^-$, $I^- = \{\xi \in \mathbb{R}^2 \mid -\xi \in I^+\}$, and I^+ is a triangle region with vertices $(0,0), (\alpha,0), (\alpha,\alpha)$. Let $a = \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$, $B = \{\left(\begin{smallmatrix}1&i\\0&1\end{smallmatrix}\right) : i \in \mathbb{Z}\}$, $S_0 = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : 0 \le \xi_1 \le \alpha,\ \xi_2 \in \mathbb{R}\}$, $S_j = (\widetilde{a^j})S_0$. Define $V_j = L^2(S_j)^\vee$. Then by Corollary 3.5, we get the following:

(1) when $\alpha \le 1$, $\varphi(x)$ is a generator for an AB-RMRA;

(2) when $\alpha > 1$, $\varphi(x)$ is not a generator.

Example 4.2. Given $k \in \mathbb{Z}$, let $E = [0,\beta_0]^2 \cup (\bigcup_{j=-k}^{-1}[2^j\alpha, 2^j\beta]^2)$, where $0 < \beta_0 \le 2^{-k}\alpha$, $\alpha < \beta \le \min\{2\alpha, 2^{k+1}\beta_0, 2\}$. If $\widehat{\varphi} = \chi_E(\cdot)$, then φ generates an AB-RMRA, where $S = \{(x,y) : x \ge 0,\ y \ge 0\}$, $a = \left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$.

Example 4.3. Let $E = [0,1/4]^2 \cup [3/8, 3/8+\varepsilon]^2$, where $0 < \varepsilon \le 1/16$. Assume that $\widehat{\varphi} = \chi_E(\cdot)$. Then φ generates an AB-RMRA, where $S = \{(x,y) : x \ge 0,\ y \ge 0\}$, $a = \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$.

Acknowledgments

The paper was supported by the National Natural Science Foundation of China (no. 61071189) and the Innovation Scientists and Technicians Troop Construction Projects of Henan Province of China (no. 084100510012).

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