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Research Article

Algebraic Properties of Toeplitz Operators on the Polydisk

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We discuss some algebraic properties of Toeplitz operators on the Bergman space of the polydisk $\mathbb{D}^n$. Firstly, we introduce Toeplitz operators with quasihomogeneous symbols and property (P). Secondly, we study commutativity of certain quasihomogeneous Toeplitz operators and commutators of diagonal Toeplitz operators. Thirdly, we discuss finite rank semicommutators and commutators of Toeplitz operators with quasihomogeneous symbols. Finally, we solve the finite rank product problem for Toeplitz operators on the polydisk.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and its boundary the unit circle $\mathbb{T}$. For a fixed positive integer n, the unit polydisk $\mathbb{D}^n$ and the torus $\mathbb{T}^n$ are the subsets of $\mathbb{C}^n$ which are Cartesian products of n copies $\mathbb{D}$ and $\mathbb{T}$, respectively. Let $dV(z) = dV_n(z)$ denote the Lebesgue volume measure on the polydisk $\mathbb{D}^n$, normalized so that the measure of $\mathbb{D}^n$ equals 1. Let $L^p = L^p(\mathbb{D}^n)$ denote the usual Lebesgue space. The Bergman space $A^2 = A^2(\mathbb{D}^n)$ is the Hilbert space consisting of holomorphic functions on $\mathbb{D}^n$ that are also in $L^2(\mathbb{D}^n, dV(z))$. Since every point evaluation is a bounded linear functional on A^2, there corresponds to every $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$ a unique function $K_z \in A^2$ which has the following reproducing property:

$$f(z) = \langle f, K_z \rangle, \quad f \in A^2, \tag{1.1}$$

where the notation $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2. The function K_z is the well-known Bergman kernel and its explicit formula is given by

$$K_z(w) = \prod_{j=1}^{n} \frac{1}{\left(1 - w_j \overline{z_j}\right)^2}, \quad w \in \mathbb{D}^n. \tag{1.2}$$

Here and elsewhere z_j denotes the jth component of z. The Bergman projection P is defined for the Hilbert space orthogonal projection from L^2 onto A^2. Given a function $\varphi \in L^\infty(\mathbb{D}^n, dV)$, the Toeplitz operator $T_\varphi : A^2 \to A^2$ is defined by the formula

$$T_\varphi(f)(z) = P(\varphi f)(z) = \int_{\mathbb{D}^n} f(w)\varphi(w)\overline{K_z(w)}dV(w) \tag{1.3}$$

for all $f \in A^2$. Since the Bergman projection P has norm 1, it is clear that Toeplitz operators defined in this way are bounded linear operators on A^2 and $\|T_\varphi\| \leq \|\varphi\|_\infty$.

We now consider a more general class of Toeplitz operators. For $F \in L^1(\mathbb{D}^n, dV)$, in analogy to (1.3) we define an operator T_F by

$$T_F f(z) = \int_{\mathbb{D}^n} F(w) f(w)\overline{K_z(w)}dV(w). \tag{1.4}$$

Since the Bergman projection P can be extended to $L^1(\mathbb{D}^n, dV)$, the operator T_F is well defined on H^∞, where H^∞ is the space of bounded holomorphic functions on $\mathbb{D}^n$. Hence, T_F is always densely defined on $A^2(\mathbb{D}^n)$. Since P is not bounded on $L^1(\mathbb{D}^n, dV)$, it is well known that T_F can be unbounded in general. This motivates the following definition, which is based on the definitions on unit ball in [1].

Definition 1.1. Let $F \in L^1(\mathbb{D}^n, dV)$.

(a) F is called a T-function if (1.4) defines a bounded operator on A^2.

(b) If F is a T-function, one writes T_F for the continuous extension of the operator (it is defined on the dense subset H^∞ of $L^2(\mathbb{D}^n)$) defined by (1.4). T_F is called a Toeplitz operator on A^2.

(c) If there exist $r_j \in (0,1)$, $1 \leq j \leq n$, such that F is (essentially) bounded on $\{z = (z_1, z_2, \ldots, z_n) : r_j < |z_j| < 1, 1 \leq j \leq n\}$, then one says F is "nearly bounded."

Notice that the T-functions form a proper subset of $L^1(\mathbb{D}^n, dV)$ which contains all bounded and "nearly bounded" functions. In this paper, the functions which we considered are all T-functions without special introduction. We denote the semicommutator and commutator of two Toeplitz operators T_f and T_g by

$$(T_f, T_g] = T_{fg} - T_f T_g, \qquad [T_f, T_g] = T_f T_g - T_g T_f. \tag{1.5}$$

The commuting problem and the finite-rank product problem for Toeplitz operators on the Hardy and Bergman spaces over various domains are some of the most interesting problems in operator theory.

For commuting problem, in 1963, Brown and Halmos [2] showed that two bounded Toeplitz operators T_φ and T_ψ on the classical Hardy space commute if and only if (i) both φ and ψ are analytic, (ii) both $\overline{\varphi}$ and $\overline{\psi}$ are analytic, or (iii) one is a linear function of the other. On the Bergman space of the unit disk, some similar results were obtained for Toeplitz operators with bounded harmonic symbols or analytic symbols (see [2–4]). The problem of characterizing commuting Toeplitz operators with arbitrary bounded symbols seems quite challenging and is not fully understood until now. In recent years, by Mellin transform, some results with quasihomogeneous symbols (it is of the form $e^{ik\theta}\phi$, where ϕ is a radial function) or monomial symbols were obtained (see [5–7]). On the Hardy and Bergman spaces of several complex variables, the situation is much more complicated. On the unit ball, Toeplitz operators with pluriharmonic or quasihomogeneous symbols were studied in [1, 8–11]. On the polydisk, some results about Toeplitz operators with pluriharmonic symbols were obtained in [10, 12–14].

For finite-rank product problem, Luecking recently proved that a Toeplitz operator with measure symbol on the Bergman space of unit disk has finite rank if and only if its symbols are a linear combination of point masses (see [15]). In [16], Choe extended Luecking's theorem to higher-dimensional cases. Using those results, Le studied finite-rank products of Toeplitz operators on the Bergman space of the unit disk and unit ball in [17, 18].

Motivated by recent work in [1, 5, 7, 17, 18], we define quasihomogeneous functions on the polydisk and study Toeplitz operators with quasihomogeneous symbols on the Bergman space of the polydisk. The present paper is assembled as follows. In Section 2, we introduce Mellin transform, Toeplitz operators with quasihomogeneous symbols and property (P). In Section 3, we study commutativity of certain quasihomogeneous Toeplitz operators and commutators of diagonal Toeplitz operators. In Sections 4 and 5, we prove that finite rank semicommutators and commutators of Toeplitz operators with quasihomogeneous symbols must be zero operator and we also solve the finite-rank product problem for Toeplitz operators on the Bergman space of the polydisk.

2. Mellin Transform, Toeplitz Operators with Quasihomogeneous Symbols and Property (P)

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ (here $\mathbb{N}$ denotes the set of all nonnegative integers), we write $a_\alpha = \alpha_1 \cdots \alpha_n$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$. The standard orthonormal basis for A^2 is $\{e_\alpha : \alpha \in \mathbb{N}^n\}$, where

$$e_\alpha(z) = \sqrt{(\alpha_1 + 1) \cdots (\alpha_n + 1)}\, z^\alpha, \quad \alpha \in \mathbb{N}^n,\ z \in \mathbb{D}^n. \tag{2.1}$$

For two n-tuples of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, we define $\alpha \succ \beta$ if $\alpha_j > \beta_j$ for all $0 \le j \le n$. Similarly, we write $\alpha \succeq \beta$ if $\alpha_j \ge \beta_j$ for all $1 \le j \le n$ and $\alpha \not\succeq \beta$ if otherwise. We also define $\alpha \perp \beta$ if $\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 0$ and $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$.

For any $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, particularly we write $\vec{k}_1 = (k_1, \ldots, k_1)$ and put $k^* = (|k_1|, \ldots, |k_n|)$, $k^+ = (1/2)(k^* + k)$ and $k^- = (1/2)(k^* - k)$. Then, $k^+, k^- \succeq 0$, $k = k^+ - k^-$, and $k^+ \perp k^-$.

Recall that a function φ on $\mathbb{D}^n$ is radial if and only if $\varphi(z)$ depends only on $(|z_1|, |z_2|, \ldots, |z_n|)$, that is, $\varphi(e^{i\theta_1}z_1, e^{i\theta_2}z_2, \ldots, e^{i\theta_n}z_n) = \varphi(z_1, z_2, \ldots, z_n)$ for any $\theta_1, \theta_2, \ldots, \theta_n \in R$. For any function $f \in L^1(\mathbb{D}^n, dV)$, we define the radicalization of f by

$$\operatorname{rad}(f)(z_1, z_2, \ldots, z_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f\left(e^{it_1}z_1, e^{it_2}z_2, \ldots, e^{it_n}z_n\right) dt_1 \cdots dt_n. \tag{2.2}$$

Then, f is radial if and only if $\operatorname{rad}(f) = f$. For $\alpha \in \mathbb{N}^n$, we have

$$\begin{aligned} \left\langle T_{\operatorname{rad}(f)} z^\alpha, z^\alpha \right\rangle &= \left\langle \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f\left(e^{it_1}z_1, e^{it_2}z_2, \ldots, e^{it_n}z_n\right) dt_1 \cdots dt_n z^\alpha, z^\alpha \right\rangle \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{\mathbb{D}^n} f\left(e^{it_1}z_1, e^{it_2}z_2, \ldots, e^{it_n}z_n\right) z^\alpha \overline{z}^\alpha dV(z) dt_1 \cdots dt_n \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} dt_1 \cdots dt_n \int_{\mathbb{D}^n} f(w_1, w_2, \ldots, \mathrm{w}_n) w^\alpha \overline{w}^\alpha dV(w) \\ &= \left\langle T_f z^\alpha, z^\alpha \right\rangle. \end{aligned} \tag{2.3}$$

The main tool in this paper will be the Mellin transform. which is defined by the equation

$$\widehat{\varphi}(z_1, \ldots, z_n) = \int_0^\infty \cdots \int_0^\infty \varphi(s_1, \ldots, s_n) s_1^{z_1 - 1} \cdots s_n^{z_n - 1} ds_1 \cdots ds_n. \tag{2.4}$$

We apply the Mellin transform to functions in $L^1([0,1]^n, r_1 \cdots r_n dr_1 \cdots dr_n)$; then,

$$\widehat{\varphi}(z_1, \ldots, z_n) = \int_0^1 \cdots \int_0^1 \varphi(s_1, \ldots, s_n) s_1^{z_1 - 1} \cdots s_n^{z_n - 1} ds_1 \cdots ds_n. \tag{2.5}$$

For convenience, we denote $\widehat{\varphi}(z_1, \ldots, z_n)$ by $\varphi^\wedge(z_1, \ldots, z_n)$ when the form of φ is complicated. It is clear that $\widehat{\varphi}$ is well defined on $I_n = \{z = (z_1, z_2, \ldots, z_n) : \operatorname{Re} z_j > 2,\ j = 1, 2, \ldots, n\}$. Using the Hartogs theorem, for any function $\varphi \in L^1([0,1]^n, r_1 \cdots r_n dr_1 \cdots dr_n)$, the Mellin transform of φ is a bounded holomorphic function on I_n.

By calculation, we can get

$$\widehat{\varphi}(z + p) = \widehat{\varphi}(z_1 + p_1, z_2 + p_2, \ldots, z_n + p_n) = \widehat{r^p \varphi}(z_1, z_2, \ldots, z_n) = \widehat{r^p \varphi}(z), \tag{2.6}$$

where $p = (p_1, p_2, \ldots, p_n) \geq 0$, $z = (z_1, z_2, \ldots, z_n) \in I_n$, and $r^p = r_1^{p_1} r_2^{p_2} \cdots r_n^{p_n}$.

The quasihomogeneous functions have been defined in many spaces (see [5, 7]). In the following, we give a similar definition on the polydisk $\mathbb{D}^n$.

Definition 2.1. Let $k \in \mathbb{Z}^n$. A function $f \in L^1(\mathbb{D}^n, dV)$ is called a quasihomogeneous function of degree k if f is of the form $\xi^k\varphi$ where φ is a radial function, that is,

$$f(r\xi) = \xi^k \varphi(r) \tag{2.7}$$

for any ξ in the torus $\mathbb{T}^n$ and $r \in [0,1)^n$.

As in [19], for any n-tuple $k \in \mathbb{Z}^n$, let $H_k = \{f \in L^2 : f$ is a quasihomogeneous function of degree $k\}$. It is clear that H_k is a closed subspace of L^2. By Lemma 3.2 in [19], $L^2 = \bigoplus_{s\in\mathbb{Z}^n} H_s$. In particular, for all $z = (r_1\xi_1, \ldots, r_n\xi_n) \in \mathbb{D}$, if $f \in H_k$, that is, $f(r_1\xi_1, \ldots, r_n\xi_n) = \xi^k f_k(r_1, \ldots, r_n)$, then we conclude that $L^2(\mathbb{D}^n, dV) = \bigoplus_{k\in\mathbb{Z}^n} \xi^k f_k(r_1, \ldots, r_n)$, $f_k \in \mathfrak{R}$, where $\mathfrak{R} = \{\varphi : \mathbb{D}^n \to \mathbb{C} \text{ radial} \mid \int_{[0,1]^n} |\varphi(r_1, \ldots, r_n)|^2 \prod_{i=1}^n r_i dr_i < +\infty\}$.

Lemma 2.2. *Let $k, l \in \mathbb{Z}^n$, and let φ, ψ be radial functions on $\mathbb{D}^n$, such that $\xi^k\varphi$, $\xi^l\psi$, and $\xi^{k+l}\varphi\psi$ are all T-functions. Then, the following equation holds for every $\alpha \in \mathbb{N}^n$:*

$$T_{\xi^k\varphi}(z^\alpha) = \begin{cases} 0 & \text{if } \alpha \not\geq k^-, \\ 2^n a_{\alpha+k+\vec{1}}\widehat{\varphi}\left(2\alpha + k + \vec{2}\right) z^{\alpha+k}, & \text{if } \alpha \geq k^-, \end{cases} \tag{2.8}$$

where $a_{\alpha+k+\vec{1}} = (\alpha_1 + k_1 + 1)\cdots(\alpha_n + k_n + 1)$ and $\widehat{\varphi}(2\alpha + k + \vec{2}) = \widehat{\varphi}(2\alpha_1 + k_1 + 2, 2\alpha_2 + k_2 + 2, \ldots, 2\alpha_n + k_n + 2)$.

Using Lemma 2.2, we can get the following two results:

$$\left(T_{\xi^k\varphi}, T_{\xi^l\psi}\right](z^\alpha) = \begin{cases} 0 & \text{if } \alpha \in E_1, \\ 2^n a_{\alpha+m+\vec{1}}\widehat{\varphi\psi}\left(2\alpha + m + \vec{2}\right) z^{\alpha+m} & \text{if } \alpha \in E_1^c \cap E_2^c, \\ \left(2^n a_{\alpha+m+\vec{1}}\widehat{\varphi\psi}\left(2\alpha + m + \vec{2}\right) - \lambda\right) z^{\alpha+m} & \text{if } \alpha \in E_2, \end{cases} \tag{2.9}$$

where $m = k + l$, $\lambda = 4^n a_{\alpha+l+\vec{1}} a_{\alpha+m+\vec{1}} \widehat{\psi}(2\alpha + l + \vec{2})\widehat{\varphi}(2(\alpha + l) + k + \vec{2})$, $E_1 = \{\alpha : \alpha \not\geq m^-\}$, $E_2 = \{\alpha : \alpha \geq l^-\} \cap \{\alpha : \alpha + l \geq k^-\}$, $E_1^c = \mathbb{N}^n \setminus E_1$, and $E_2^c = \mathbb{N}^n \setminus E_2$. It is easy to check that $E_1 \cap E_2 = \phi$

$$\left[T_{\xi^k\varphi}, T_{\xi^l\psi}\right](z^\alpha) = \begin{cases} 0 & \text{if } \alpha \in F_1^c \cap F_2^c, \\ \lambda_1 z^{\alpha+k+l} & \text{if } \alpha \in F_1 \cap F_2^c, \\ -\lambda_2 z^{\alpha+k+l} & \text{if } \alpha \in F_1^c \cap F_2, \\ (\lambda_1 - \lambda_2) z^{\alpha+k+l} & \text{if } \alpha \in F_1 \cap F_2, \end{cases} \tag{2.10}$$

where $\lambda_1 = 4^n a_{\alpha+l+\vec{1}} a_{\alpha+k+l+\vec{1}} \widehat{\psi}(2\alpha + l + \vec{2})\widehat{\varphi}(2(\alpha + l) + k + \vec{2})$, $\lambda_2 = 4^n a_{\alpha+k+\vec{1}} a_{\alpha+k+l+\vec{1}} \widehat{\varphi}(2\alpha + k + \vec{2})\widehat{\psi}(2(\alpha + k) + l + \vec{2})$, $F_1 = \{\alpha : \alpha \geq l^-\} \cap \{\alpha : \alpha + l \geq k^-\}$, $F_2 = \{\alpha : \alpha \geq k^-\} \cap \{\alpha : \alpha + k \geq l^-\}$, $F_1^c = \mathbb{N}^n \setminus F_1$ and $F_2^c = \mathbb{N}^n \setminus F_2$.

Let G be a region in complex plane $\mathbb{C}$ and f holomorphic on G. If $\{z_k\}_{k=1}^{\infty}$ has a limit point in G, such that $f(z_k) = 0$, then $f \equiv 0$. For functions of several complex variables, the above conclusion does not hold. For example, $f(z_1, z_2) = z_1 z_2$ is analytic on bidisk $\{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}$, point sequence $\{(0, 1/k)\}$, $k = 2, 3, \ldots$, has a limit point $(0, 0)$, and $f(0, 1/k) = 0$, but f is not a zero function on the bidisk. So we need the following definition, which is given in [9, 17].

For any $1 \leq j \leq n$, let $\sigma_j : \mathbb{N} \times \mathbb{N}^{n-1} \to \mathbb{N}^n$ be the map defined by the formula $\sigma_j(s, (\alpha_1, \ldots, \alpha_{n-1})) = (\alpha_1, \ldots, \alpha_{j-1}, s, \alpha_j, \ldots, \alpha_{n-1})$ for all $s \in \mathbb{N}$ and $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}^{n-1}$. If M is a subset of $\mathbb{N}^n$ and $1 \leq j \leq n$, we define

$$\widetilde{M_j} = \left\{ \tilde{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}^{n-1} : \sum_{s \in \mathbb{N}, \sigma_j(s, \tilde{\alpha}) \in M} \frac{1}{s+1} = \infty \right\}. \tag{2.11}$$

As in [9, 17], we say that M has property (P) if one of the following statements holds:

(1) $M = \emptyset$,

(2) $M \neq \emptyset, n = 1$, and $\sum_{s \in M} 1/s < \infty$, or

(3) $M \neq \emptyset, n \geq 2$, and, for any $1 \leq j \leq n$, the set $\widetilde{M_j}$ has property (P) as a subset of $\mathbb{N}^{n-1}$.

Let M and N be two sets that have property (P). It is not difficult to check that the following statements hold:

(1) $M \bigcap N$ and $M \bigcup N$ have property (P);

(2) $\mathbb{N}^n \setminus M$ do not have property (P).

Lemma 2.3. *If $\varphi \in L^1([0,1]^n, r_1 \cdots r_n dr_1 \cdots dr_n)$ and $Z(\widehat{\varphi}) = \{\alpha \in \mathbb{N}^n : \widehat{\varphi}(\alpha) = 0\}$ does not have property (P), then φ is identically zero.*

Proof. By the Müntz theorem, we can prove that it is true when $n = 1$ (see [7] for more details). Suppose that the conclusion of the lemma holds whenever $n \leq N$, where N is a positive integer. Consider the case $n = N + 1$. Since $Z(\widehat{\varphi})$ does not have property (P), there must be a $1 \leq j \leq N + 1$, such that $\widetilde{Z(\widehat{\varphi})}_j$ does not have property (P). Without loss of generality, taking $j = N + 1$, then, $\widetilde{Z(\widehat{\varphi})}_{N+1} \neq \emptyset$. For each $\tilde{r} \in \widetilde{Z(\widehat{\varphi})}_{N+1}$, $\sum_{s \in \mathbb{N},\ \widehat{\varphi}(\tilde{r}, s) = 0} 1/(s+1) = \infty$. So $\widehat{\varphi}(\tilde{r}, z_{N+1}) = 0$, for all $z_{N+1} \in I_1$. For every $\lambda \in I_1$, let $\widehat{\varphi}_\lambda(z') = \widehat{\varphi}(z', \lambda)$; then, $\widehat{\varphi}_\lambda$ is an analytic function on I_N and $Z(\widehat{\varphi}_\lambda) = \widetilde{Z(\widehat{\varphi})}_{N+1}$, which does not have property (P). By the induction hypothesis, we have $\widehat{\varphi}(z', \lambda) = 0, z' \in I_N$. Thus, $\widehat{\varphi}(z) = 0$ on I_{N+1}. Therefore, φ is identically zero. □

Theorem 2.4. *Let $p = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n$, and let f be a T-function. Then, the following statements hold.*

(i) *If $E_p = \{\alpha; \langle T_f z^{\alpha+p}, z^\alpha \rangle = 0$ for all $\alpha \geq p^-\}$ does not have property (P), then $\langle T_f z^{\alpha+p}, z^\alpha \rangle = 0$ for all $\alpha \geq p^-$.*

(ii) *Let E, $E' \subseteq \mathbb{N}^n$ be the sets that have property (P). If $\langle T_f z^\alpha, z^\beta \rangle = 0$ for all $\alpha \in \mathbb{N}^n \setminus E$, $\beta \in \mathbb{N}^n \setminus E'$, then $f(z) \equiv 0$ for almost all $z \in \mathbb{D}^n$.*

(iii) *If $\langle T_f z^\alpha, z^\beta \rangle = 0$ for all $\alpha + p \neq \beta$, then f is a quasihomogeneous function of degree p.*

Proof. (i) By direct computation, we have

$$\begin{aligned}
&\left\langle T_f z^{\alpha+p}, z^{\alpha}\right\rangle \\
&\quad = \int_{\mathbb{D}^n} f(z) z^{\alpha+p} \overline{z}^{\alpha} dV(z) \\
&\quad = \frac{1}{\pi^n} \int_{[0,1]^n} \int_{[0,2\pi]^n} f\left(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}\right) \prod_{j=1}^{n} r_j^{2\alpha_j+|p_j|+1} \prod_{j=1}^{n} \left(e^{i\theta_j}\right)^{p_j} d\theta_1 \cdots d\theta_n dr_1 \cdots dr_n.
\end{aligned} \tag{2.12}$$

Let

$$F(r_1, \ldots, r_n) = \frac{1}{\pi^n} \int_{[0,2\pi]^n} f\left(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}\right) \prod_{j=1}^{n} \left(e^{i\theta_j}\right)^{p_j} d\theta_1 \cdots d\theta_n. \tag{2.13}$$

Then, $F \in L^1([0,1]^n, r_1 \cdots r_n dr_1 \cdots dr_n)$. In fact, $\|F(r_1, \ldots, r_n)\|_{L^1([0,1]^n, r_1 \cdots r_n dr_1 \cdots dr_n)} \leq \|f\|_{L^1(\mathbb{D}^n, dV)}$. Therefore, equality (2.12) shows that $\widehat{F}(2\alpha_1 + |p_1| + 2, \ldots, 2\alpha_n + |p_n| + 2) = 0$ for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. That is, $Z(\widehat{F})$ does not have property (P). Thus, Lemma 2.3 implies that $F \equiv 0$ and $\langle T_f z^{\alpha+p}, z^{\alpha}\rangle = 0$ for all $\alpha \geq p^-$.

(ii) For each $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$, $\langle T_f z^{\alpha}, z^{\alpha+l}\rangle = 0$ for $\alpha \in (\mathbb{N}^n \setminus E) \cap (\mathbb{N}^n \setminus (E' - l))$. Since E and E' have property (P), the subset $(\mathbb{N}^n \setminus E) \cap (\mathbb{N}^n \setminus (E' - l)) = (\mathbb{N}^n \setminus (E \cup (E' - l))$ does not have property (P). By (i), we have $\langle T_f z^{\alpha}, z^{\alpha+l}\rangle = 0$ for $\alpha \in \mathbb{N}^n$. It is easy to prove that $\langle T_f z^{\alpha+l}, z^{\alpha}\rangle = 0$ for $\alpha \in \mathbb{N}^n$. So $\langle T_f z^{\alpha}, z^{\beta}\rangle = 0$ for all $\alpha, \beta \in \mathbb{N}^n$, that is, $T_f = 0$ and $f(z) \equiv 0$ for almost all $z \in \mathbb{D}^n$.

(iii) Since

$$\begin{aligned}
\operatorname{rad}\left(\overline{\xi}^{p} f\right)(z_1, \ldots, z_n) &= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \left(\overline{\xi}^{p} f\right)\left(e^{it_1} z_1, e^{it_2} z_2, \ldots, e^{it_n} z_n\right) dt_1 \cdots dt_n \\
&= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} e^{-i(t_1+\theta_1)p_1} \cdots e^{-i(t_n+\theta_n)p_n} f\left(r_1 e^{i(t_1+\theta_1)}, \ldots, r_n e^{i(t_n+\theta_n)}\right) dt_1 \cdots dt_n \\
&= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} e^{-i(\sum_{j=1}^{n} t_j p_j)} f\left(r_1 e^{it_1}, \ldots, r_n e^{it_n}\right) dt_1 \cdots dt_n,
\end{aligned} \tag{2.14}$$

we have

$$\begin{aligned}
&\int_{\mathbb{D}^n} \left[\xi^{p} \operatorname{rad}\left(\overline{\xi}^{p} f\right)\right](z) z^{\alpha} \overline{z}^{\beta} dV(z) \\
&\quad = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \int_{[0,1]^n} \int_{[0,2\pi]^n} e^{-i(\sum_{j=1}^{n} \theta_j (p_j+\alpha_j-\beta_j))} f\left(r_1 e^{it_1}, \ldots, r_n e^{it_n}\right) d\theta_1 \cdots d\theta_n \\
&\qquad \times \left(\prod_{j=1}^{n} r_j^{\alpha_j+\beta_j+1}\right) e^{-i(\sum_{j=1}^{n} t_j p_j)} dr_1 \cdots dr_n dt_1 \cdots dt_n.
\end{aligned} \tag{2.15}$$

If $\alpha + p \neq \beta$, then $\int_{\mathbb{D}^n} [\xi^p \mathrm{rad}(\overline{\xi}^p f)](z) z^\alpha \overline{z}^\beta dV(z) = 0$. Otherwise, if $\alpha + p = \beta$, then

$$\begin{aligned}
&\int_{\mathbb{D}^n} \left[\xi^p \mathrm{rad}\left(\overline{\xi}^p f\right)\right](z) z^\alpha \overline{z}^\beta dV(z) \\
&= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \int_{[0,1]^n} f\left(r_1 e^{it_1}, \dots, r_n e^{it_n}\right) \left(\prod_{j=1}^n r_j^{\alpha_j + \beta_j + 1}\right) e^{-i(\sum_{j=1}^n t_j p_j)} dr_1 \cdots dr_n dt_1 \cdots dt_n \\
&= \int_{\mathbb{D}^n} f(z) z^\alpha \overline{z}^\beta dV(z).
\end{aligned} \tag{2.16}$$

Thus, we get $\int_{\mathbb{D}^n} [\xi^p \mathrm{rad}(\overline{\xi}^p f)](z) z^\alpha \overline{z}^\beta dV(z) = \int_{\mathbb{D}^n} f(z) z^\alpha \overline{z}^\beta dV(z)$ for any $\alpha, \beta \in \mathbb{Z}^n$. So $\xi^p \mathrm{rad}(\overline{\xi}^p f) = f$, this means that there exists $\varphi(r)$ such that $\overline{\xi}^p f(z) = \varphi(r)$, that is, $f(z) = \xi^p \varphi(r)$ is a quasihomogeneous function of degree p. □

Remark 2.5. Let f be as in Theorem 2.4. Then, $T_f = \sum_{\alpha \in \mathbb{N}^n} w(f, \alpha, \alpha + p) e_{\alpha+p} \otimes e_\alpha$, where

$$\begin{aligned}
w(f, \alpha, \alpha+p) = \langle T_f e_\alpha, e_{\alpha+p} \rangle &= \sqrt{a_{\alpha+\vec{1}}} \sqrt{a_{\alpha+p+\vec{1}}} \int_{\mathbb{D}^n} \xi^p \varphi(r) z^\alpha \overline{z}^{\alpha+p} dV(z) \\
&= \sqrt{a_{\alpha+\vec{1}}} \sqrt{a_{\alpha+p+\vec{1}}} \widehat{\varphi}\left(2\alpha + p + \vec{2}\right).
\end{aligned} \tag{2.17}$$

Recall that a densely defined operator on $A^2(\mathbb{D}^n)$ is said to be diagonal if it is diagonal with respect to the standard orthonormal basis. In particular, for $f \in L^\infty(\mathbb{D}^n)$, T_f is diagonal if and only if $\mathrm{rad}(f) = f$. In this case, $T_f = \sum_{\alpha \in \mathbb{N}^n} w(f, \alpha) e_\alpha \otimes e_\alpha$, where $w(f, \alpha) = \langle T_f e_\alpha, e_\alpha \rangle = a_{\alpha+\vec{1}} \widehat{f}(2\alpha + \vec{2})$.

3. Commutativity of Toeplitz Operators

In this section, we study the commutativity of the Toeplitz operators with some special quasihomogeneous symbols and give the characterizations, respectively.

Theorem 3.1. *Let $g = \xi^p \varphi(r) \in L^2(\mathbb{D}^n)$ be a quasihomogeneous function of degree p and $f = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r_1, \dots, r_n) \in L^2(\mathbb{D}^n)$. Then, $T_f T_g = T_g T_f$ if and only if $T_{\xi^k f_k} T_g = T_g T_{\xi^k f_k}$ for any $k \in \mathbb{Z}^n$. Moreover, the following statements hold.*

(i) *If $Q_1 = \{\alpha : \alpha + p \ngeq 0\} \cap \{\alpha : \alpha + k \geq 0\} \cap \{\alpha : \alpha + k + p \geq 0\} \neq \emptyset$, then, for each $\alpha \in Q_1$, $\widehat{\varphi}(2\alpha + 2k + p + \vec{2}) \widehat{f}_k(2\alpha + k + \vec{2}) = 0$.*

(ii) *If $Q_2 = \{\alpha : \alpha + k \ngeq 0\} \cap \{\alpha : \alpha + p \geq 0\} \cap \{\alpha : \alpha + k + p \geq 0\} \neq \emptyset$, then, for each $\alpha \in Q_2$, $\widehat{\varphi}(2\alpha + p + \vec{2}) \widehat{f}_k(2\alpha + 2p + k + \vec{2}) = 0$.*

Proof. Note that, for $\alpha \in \mathbb{N}^n$,

$$\begin{aligned} T_g T_f(e_\alpha) &= \sum_{k\in\mathbb{Z}^n} T_g T_{\xi^k f_k} e_\alpha = \sum_{k\in\mathbb{Z}^n}\sum_{\beta\in\mathbb{N}^n} \left\langle T_g T_{\xi^k f_k} e_\alpha, e_\beta \right\rangle e_\beta \\ &= \sum_{k\in\mathbb{Z}^n} \left\langle T_g T_{\xi^k f_k} e_\alpha, e_{\alpha+k+p} \right\rangle e_{\alpha+k+p}. \end{aligned} \tag{3.1}$$

The second equality follows that $\langle T_g T_{\xi^k f_k} e_\alpha, e_\beta\rangle = 0$, when $\beta \neq \alpha + k + p$.

Similarly,

$$\begin{aligned} T_f T_g(e_\alpha) &= \sum_{k\in\mathbb{Z}^n} T_{\xi^k f_k} T_g e_\alpha = \sum_{k\in\mathbb{Z}^n}\sum_{\beta\in\mathbb{N}^n} \left\langle T_{\xi^k f_k} T_g e_\alpha, e_\beta \right\rangle e_\beta \\ &= \sum_{k\in\mathbb{Z}^n} \left\langle T_{\xi^k f_k} T_g e_\alpha, e_{\alpha+k+p} \right\rangle e_{\alpha+k+p}. \end{aligned} \tag{3.2}$$

Since $\{e_\alpha\}$ are the standard orthogonal basis and $\langle T_g T_{\xi^k f_k} e_\alpha, e_\beta\rangle = \langle T_{\xi^k f_k} T_g e_\alpha, e_\beta\rangle = 0$ for $\beta \neq \alpha + k + p$, it is easy to check that the following statements are equal:

(I) $T_f T_g = T_g T_f$;

(II) $\langle T_g T_{\xi^k f_k} e_\alpha, e_{\alpha+k+p}\rangle = \langle T_{\xi^k f_k} T_g e_\alpha, e_{\alpha+k+p}\rangle$, $\alpha \in \mathbb{N}^n$;

(III) $T_g T_{\xi^k f_k} e_\alpha = T_{\xi^k f_k} T_g e_\alpha$, $\alpha \in \mathbb{N}^n$;

(IV) $T_g T_{\xi^k f_k} = T_{\xi^k f_k} T_g$.

Furthermore,

$$\begin{aligned} &\left\langle T_g T_{\xi^k f_k} e_\alpha, e_{\alpha+k+p} \right\rangle = \left\langle T_{\xi^k f_k} e_\alpha, e_{\alpha+k} \right\rangle \langle T_g e_{\alpha+k}, e_{\alpha+k+p}\rangle \\ &\quad = \begin{cases} 0, & \alpha + k \not\succeq 0 \ \text{ or } \ \alpha + k + p \not\succeq 0, \\ \sqrt{a_{\alpha+\vec{1}}}\sqrt{a_{\alpha+k+p+\vec{1}}}\, a_{\alpha+k+\vec{1}}\, \widehat{\varphi}\left(2\alpha + 2k + p + \vec{2}\right) \widehat{f}_k\left(2\alpha + k + \vec{2}\right), & \alpha + k \succeq 0,\ \alpha + k + p \succeq 0, \end{cases} \\ &\left\langle T_{\xi^k f_k} T_g e_\alpha, e_{\alpha+k+p} \right\rangle = \langle T_g e_\alpha, e_{\alpha+p}\rangle \left\langle T_{\xi^k f_k} e_{\alpha+p}, e_{\alpha+k+p} \right\rangle \\ &\quad = \begin{cases} 0, & \alpha + p \not\succeq 0 \ \text{ or } \ \alpha + k + p \not\succeq 0, \\ \sqrt{a_{\alpha+\vec{1}}}\sqrt{a_{\alpha+k+p+\vec{1}}}\, a_{\alpha+p+\vec{1}}\, \widehat{f}_k\left(2\alpha + 2p + k + \vec{2}\right) \widehat{\varphi}\left(2\alpha + p + \vec{2}\right), & \alpha + p \succeq 0,\ \alpha + k + p \succeq 0. \end{cases} \end{aligned} \tag{3.3}$$

Thus, the statements (i) and (ii) hold. □

Theorem 3.2. *Let f_1, f_2 be quasihomogeneous functions of degree p and $-s$, where $p \succeq s \succ 0$. If $[T_{f_1}, T_{f_2}] = 0$, then $f_1 = 0$ or $f_2 = 0$.*

Proof. If f_1, f_2 are quasihomogeneous functions of degree p and $-s$, then there exist radial functions φ_1 and φ_2, such that $f_1 = \xi^p \varphi_1$ and $f_2 = \overline{\xi}^s \varphi_2$. If $T_{f_1}T_{f_2} = T_{f_2}T_{f_1}$, (2.10) implies that, for all $\alpha \in \mathbb{N}^n$,

$$\begin{gathered} a_{\alpha-s+\vec{1}}\widehat{\varphi_2}\left(2\alpha - s + \vec{2}\right)\widehat{\varphi_1}\left(2\alpha - 2s + p + \vec{2}\right) = a_{\alpha+p+\vec{1}}\widehat{\varphi_1}\left(2\alpha + p + \vec{2}\right)\widehat{\varphi_2}\left(2\alpha + 2p - s + \vec{2}\right), \quad \text{if } \alpha \geq s, \\ \widehat{\varphi_1}\left(2\alpha + p + \vec{2}\right)\widehat{\varphi_2}\left(2\alpha + 2p - s + \vec{2}\right) = 0, \quad \text{if } \alpha \not\geq s. \end{gathered} \tag{3.4}$$

We claim that there exist $\{\lambda_k\}_{k=1}^{\infty}$ with $\sum 1/\lambda_k = \infty$ such that

$$\widehat{\varphi_1}(2\lambda_k + p_1 + 2, z')\widehat{\varphi_2}(2\lambda_k + 2p_1 - s_1 + 2, z') = 0, \quad \text{for any } z' \in I_{n-1}. \tag{3.5}$$

It follows that $Z(\widehat{\varphi_1}\widehat{\varphi_2})$ do not have property (P). So we can get $f_1 = 0$ or $f_2 = 0$ by Lemma 2.3.

We only need to prove the claim. Since $s \succ 0$, there exists $1 \leq j \leq n$, such that $s_j \geq 1$. Without losing generality, suppose that $j = 1$. Let $\lambda_0 = s_1 - 1 \geq 0$; then, $\alpha^0 = (\lambda_0, \alpha') \not\geq s$ and $\widehat{\varphi_1}(2\lambda_0 + p_1 + 2, \alpha' + p' + \vec{2})\widehat{\varphi_2}(2\lambda_0 + p_1 + 2, \alpha' + 2p' - s' + \vec{2}) = 0$ where $\alpha' \in \mathbb{N}^{n-1}$, $p' = (p_2, \dots, p_n)$ and $s' = (s_2, \dots, s_n)$. Denote $E_0 = \{z' \in I_{n-1} : \widehat{\varphi_1}(2\lambda_0 + p_1 + 2, z') = 0\}$ and $F_0 = \{z' \in I_{n-1} : \widehat{\varphi_1}(2\lambda_0 + p_1 + 2, z') = 0\}$. Note that at least one of the sets E_0 and F_0 does not have property (P). Since $\widehat{\varphi_1}(2\lambda_0 + p_1 + 2, z') = 0$ and $\widehat{\varphi_2}(2\lambda_0 + p_1 + 2, z') = 0$ are analytic on I_{n-1}, Lemma 2.3 shows that $E_0 = I_{n-1}$ or $F_0 = I_{n-1}$.

Case 1. If $E_0 = I_{n-1}$, then

$$\widehat{\varphi_1}\left(2\left(\alpha^0 + s\right) + p + \vec{2}\right)\widehat{\varphi_2}\left(2\left(\alpha^0 + s\right) + 2p - s + \vec{2}\right) = \frac{a_{\alpha-s+\vec{1}}}{a_{\alpha+p+\vec{1}}}\widehat{\varphi_1}\left(2\varphi^0 + p + \vec{2}\right)\widehat{\varphi_2}\left(2\alpha^0 + s + \vec{2}\right) = 0, \tag{3.6}$$

where $\alpha^0 = (\lambda_0 + s, \alpha')$ with $\alpha' \in N^{n-1}$. Let $\lambda_1 = \lambda_0 + s$. Denote by $E_1 = \{z' \in I_{n-1} : \widehat{\varphi_1}(2\lambda_1 + p_1 + 2, z') = 0\}$ and $F_1 = \{z' \in I_{n-1} : \widehat{\varphi_1}(2\lambda_1 + p_1 + 2, z') = 0\}$. Then, at least one of the sets E_1 and F_1 does not have property (P). By Lemma 2.3 again, we have $E_1 = I_{n-1}$ or $F_1 = I_{n-1}$. Thus $\widehat{\varphi_1}(\lambda_1, z')\widehat{\varphi_2}(\lambda_1, z') = 0$, for any $z' \in I_{n-1}$.

Case 2. If $F_0 = I_{n-1}$, then

$$\begin{aligned} &\widehat{\varphi_1}\left(2\left(\alpha^0 + p\right) + p + \vec{2}\right)\widehat{\varphi_2}\left(2\left(\alpha^0 + p\right) + 2p - s + \vec{2}\right) \\ &\quad = \frac{a_{\alpha-s+\vec{1}}}{a_{\alpha+p+\vec{1}}}\widehat{\varphi_1}\left(2\alpha^0 + 3p - 2s + \vec{2}\right)\widehat{\varphi_2}\left(2\alpha^0 + 2p - s + \vec{2}\right) = 0. \end{aligned} \tag{3.7}$$

By the same technique, we can get that (3.5) holds when $\lambda_1 = \lambda_0 + p_1$.

Similarly, we can find a sequence $\lambda_k = \lambda_0 + u(k)p_1 + v(k)s_1$, where the functions $u(k) = 1$ or 0, $v(k) = 1$ or 0, and $u(k) + v(k) = 1$ for $k \in \mathbb{Z}^+$. Then, (i) $\lambda_1 \geq \min\{p_1, s_1\}$, (ii) $\sum_{k\in\mathbb{N}} 1/\lambda_k = +\infty$, and (iii) for every $k \geq 1$, λ_k satisfies (3.5). So we complete the proof. □

For $n = 2$, we have the following results.

Theorem 3.3. *Let $g(r) = r_1^{m_1} r_2^{m_2}$, where $m_1 \geq 0$ and $m_2 \geq 0$. Let $f(r_1, r_2) \in L^\infty([0,1]^2)$, $p = (p_1, p_2) \in \mathbb{N}^2$ and $p_1 \cdot p_2 \neq 0$. Then, $T_{\xi^p g} T_{\xi^p f} = T_{\xi^p f} T_{\xi^p g}$ if and only if there exists an analytic function on $\mathbb{C}$, such that the function $\varphi((z_2 p_1 - z_1 p_2)/(p_1^2 + p_2^2))/((z_1 + m_1)(z_2 + m_2))$ is bounded on I_2 and*

$$\widehat{f}(z_1, z_2) = \frac{\varphi\left((z_2 p_1 - z_1 p_2)/(p_1^2 + p_2^2)\right)}{(z_1 + m_1)(z_2 + m_2)}, \quad \forall z = (z_1, z_2) \in I_2. \tag{3.8}$$

Proof. As in the proof of Theorem 3.2, it is easy to check that $T_{\xi^p g} T_{\xi^p f} = T_{\xi^p f} T_{\xi^p g}$ if and only if

$$\left\langle T_{\xi^p g} T_{\xi^p f} e_\alpha, e_{\alpha + 2p} \right\rangle = \left\langle T_{\xi^p f} T_{\xi^p g} e_\alpha, e_{\alpha + 2p} \right\rangle, \tag{3.9}$$

which is equal to

$$\widehat{g}\left(2\alpha + p + \vec{2}\right) \widehat{f}\left(2\alpha + 3p + \vec{2}\right) = \widehat{f}\left(2\alpha + p + \vec{2}\right) \widehat{g}\left(2\alpha + 3p + \vec{2}\right), \quad \forall \alpha \in \mathbb{N}^2. \tag{3.10}$$

Suppose that there is a function φ as in this theorem.
Note that

$$\widehat{g}(z) = \int_{[0,1]^2} r^{m + z - \vec{1}} dr_1 dr_2 = \frac{1}{(z_1 + m_1)(z_2 + m_2)} \neq 0, \quad \forall z \in I_2, \tag{3.11}$$

and $(2\alpha_2 + 3p_2 + 2)p_1 - (2\alpha_1 + 3p_1 + 2)p_2 = (2\alpha_2 + p_2 + 2)p_1 - (2\alpha_1 + p_1 + 2)p_2$, for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$. Then, it is easy to check that equality (3.10) holds, that is, $T_{\xi^p g} T_{\xi^p f} = T_{\xi^p f} T_{\xi^p g}$.

Conversely, if $T_{\xi^p g}$ and $T_{\xi^p f}$ commute, we will structure an analytic φ which satisfies the conditions in this theorem.

Since $\widehat{g}(z) \neq 0$ for all $z \in I_2$, the function $\widehat{f}(z)/\widehat{g}(z)$ is analytic on I_2. Note that $|r^{z-1}| < 1$ for $0 < r_j < 1$, $j = 1, 2$ and $z \in I_2$. Thus,

$$\begin{gathered} \left|\widehat{f}(z)\right| \leq \int_{\mathbb{D}^2} |f| dV(z) = \|f\|_{L^1([0,1]^2)}, \\ \left|\widehat{g}(z)\right| \leq \|g\|_{L^1([0,1]^2)}. \end{gathered} \tag{3.12}$$

Fix $\alpha_0 \in \mathbb{N}^2$, and let $z_0 = 2\alpha_0 + p + \vec{2}$, then,

$$\frac{\widehat{f}(z_0)}{\widehat{g}(z_0)} = \frac{\widehat{f}(z_0 + 2p)}{\widehat{g}(z_0 + 2p)} = \cdots = \frac{\widehat{f}(z_0 + 2kp)}{\widehat{g}(z_0 + 2kp)}, \quad k = 0, 1, 2, \ldots. \tag{3.13}$$

Combining this with Lemma 2.3, we can get that the above equality holds for any $z_0 \in I_2$. Let $p^{\perp} = (-p_2, p_1)$; then, $p \perp p^{\perp}$. For each $z_0 \in I_2$, there exist $\mu_1, \mu_2 \in \mathbb{C}$ such that $z_0 = \mu_1 p + \mu_2 p^{\perp}$. So

$$\frac{\widehat{f}(\mu_1 p + \mu_2 p^{\perp})}{\widehat{g}(\mu_1 p + \mu_2 p^{\perp})} = \frac{\widehat{f}((\mu_1 + 2)p + \mu_2 p^{\perp})}{\widehat{g}((\mu_1 + 2)p + \mu_2 p^{\perp})} = \cdots = \frac{\widehat{f}((\mu_1 + 2k)p + \mu_2 p^{\perp})}{\widehat{g}((\mu_1 + 2k)p + \mu_2 p^{\perp})}, \quad k = 0, 1, 2, \ldots. \tag{3.14}$$

Put

$$F(\lambda) = \frac{\widehat{f}(\lambda p + \mu_2 p^{\perp})}{\widehat{g}(\lambda p + \mu_2 p^{\perp})} - \frac{\widehat{f}(\mu_1 p + \mu_2 p^{\perp})}{\widehat{g}(\mu_1 p + \mu_2 p^{\perp})}, \tag{3.15}$$

then, $F(\lambda)$ is analytic on $\{z \in \mathbb{C} : \operatorname{Re}(zp_1 - \mu_2 p_2) > 2 \text{ and } \operatorname{Re}(zp_2 + \mu_2 p_1) > 2\}$ and

$$|F(\lambda)| \leq \|f\|_{L^1} \left(|m_1 + \lambda p_1 - \mu_2 p_2| \cdot |m_2 + \lambda p_2 + \mu_2 p_1| + C_1 \right) \leq \|f\|_{L^1} \left(D_1 |\lambda|^2 + D_2 |\lambda| + D_3 \right), \tag{3.16}$$

where C_1, D_1, D_2, D_3 are all constants. Since $F(\mu_1 + 2k) = 0$ and $\sum_{k=0}^{+\infty} 1/2k = +\infty$, the set $\{\mu_1 + 2k : k = 0, 1, 2, \ldots\} \subseteq Z(F)$. Thus $F(\lambda) \equiv 0$. That is

$$\widehat{f}\left(\lambda p + \mu_2 p^{\perp}\right) = \frac{\widehat{f}(\mu_1 p + \mu_2 p^{\perp})}{\widehat{g}(\mu_1 p + \mu_2 p^{\perp})} \widehat{g}\left(\lambda p + \mu_2 p^{\perp}\right). \tag{3.17}$$

For each $\mu \in \mathbb{C}$, there exists $\lambda_0 \in \mathbb{C}$ such that $\operatorname{Re}(\lambda_0 p_1 - \mu p_2) > 2$ and $\operatorname{Re}(\lambda_0 p_2 + \mu p_1) > 2$; then, let $\psi(\mu) = \widehat{f}(\lambda_0 p + \mu p^{\perp}) / \widehat{g}(\lambda_0 p + \mu p^{\perp})$. By equality (3.17), we conclude that the function ψ is well defined. Since the function $\widehat{f}/\widehat{g}$ is analytic on I_2, we can prove that ψ is an analytic function on $\mathbb{C}$. Let

$$\begin{aligned} z_1 &= \lambda p_1 - \mu_2 p_2, \\ z_2 &= \lambda p_2 + \mu_2 p_1; \end{aligned} \tag{3.18}$$

Then,

$$\begin{aligned} \lambda &= \frac{z_1 p_1 - z_2 p_2}{p_1^2 + p_2^2}, \\ \mu_2 &= \frac{z_2 p_1 - z_1 p_2}{p_1^2 + p_2^2}. \end{aligned} \tag{3.19}$$

So (3.17) is equal to

$$\widehat{f}(z_1, z_2) = \frac{\psi\left((z_2 p_1 - z_1 p_2) / (p_1^2 + p_2^2)\right)}{(z_1 + m_1)(z_2 + m_2)}, \tag{3.20}$$

where $|\varphi((z_2p_1 - z_1p_2)/(p_1^2 + p_2^2))/((z_1 + m_1)(z_2 + m_2))| \le \|f\|_{L^1([0,1]^2)}$ and $(z_1, z_2) \in I_2$. This completes the proof. □

Corollary 3.4. *Let f, g be as in Theorem 3.3 and $p \in \mathbb{N}^+$; then, the following statements hold:*

(i) $T_{e^{ip\theta_1}g}T_{e^{ip\theta_1}f} = T_{e^{ip\theta_1}f}T_{e^{ip\theta_1}g}$ *if and only if* $f = r_1^{m_1}\varphi(r_2)$, *where* $\varphi \in L^\infty([0,1])$;

(ii) $T_{e^{ip\theta_2}g}T_{e^{ip\theta_2}f} = T_{e^{ip\theta_2}f}T_{e^{ip\theta_2}g}$ *if and only if* $f = r_2^{m_2}\varphi(r_1)$, *where* $\varphi \in L^\infty([0,1])$.

Proof. (i) By (3.17) we have

$$\int_{[0,1]^2} f(r_1, r_2) r_1^{z_1-1} r_2^{z_2-1} dr_1 dr_2 = \varphi\left(\frac{z_2}{p}\right) \int_{[0,1]^2} r_1^{m_1+z_1-1} r_2^{m_2+z_2-1} dr_1 dr_2, \tag{3.21}$$

that is,

$$\int_{[0,1]} \left[\int_{[0,1]} f(r_1, r_2) r_2^{z_2-1} dr_2 - r_1^{m_1}\varphi\left(\frac{z_2}{p}\right)\int_{[0,1]} r_2^{m_2+z_2-1} dr_2\right] r_1^{z_1-1} dr_1 = 0. \tag{3.22}$$

Then,

$$\int_{[0,1]} f(r_1, r_2) r_2^{z_2-1} dr_2 = r_1^{m_1}\varphi\left(\frac{z_2}{p}\right)\frac{1}{m_2 + z_2}. \tag{3.23}$$

It follows that there exists $\varphi = \varphi(r_2) \in L^\infty([0,1])$ such that $f = r_1^{m_1}\varphi(r_2)$.

On the other hand, if $f = r_1^{m_1}\varphi(r_2)$, then

$$\begin{aligned} \widehat{f}(z)\widehat{g}(z + 2(p,0)) &= \frac{1}{m_1 + z_1} \cdot \widehat{\varphi}(z_2) \cdot \frac{1}{m_1 + z_1 + 2p} \cdot \frac{1}{m_2 + z_2}, \\ \widehat{f}(z + 2(p,0))\widehat{g}(z) &= \frac{1}{m_1 + z_1 + 2p} \cdot \widehat{\varphi}(z_2) \cdot \frac{1}{m_1 + z_1} \cdot \frac{1}{m_2 + z_2}. \end{aligned} \tag{3.24}$$

Thus, we have $T_{\xi^p g}T_{\xi^p f} = T_{\xi^p f}T_{\xi^p g}$.

(ii) Can also be proved in the same way. □

In [6], Čučković and Rao showed that if $f, g \in L^\infty(\mathbb{D})$ and g is a nonconstant radial function, then $T_fT_g = T_gT_f$ implies that f is a radial function. However, this is not true if $f, g \in L^\infty(\mathbb{D}^n)$, where $n \ge 2$. For example, $g(z) = g(z_1, \dots, z_{j-1}, |z_j|, z_{j+1}, \dots, z_n)$ and $f(z) = f(|z_1|, \dots, |z_n|)$ only for $|z_j|$, and it is clear that $T_gT_f = T_fT_g$, but $g(z)$ may not be a radial function. Let $G = \{g \in L^\infty(\mathbb{D}^n) : g$ is radial and for $f \in L^\infty(\mathbb{D}^n), T_fT_g = T_gT_f$ implies that f is radial$\}$(if $n = 1$, this set is exactly the set of all non-constant bounded radial functions). In the following, we can give a complete description of G.

Theorem 3.5. $G = \{g(z)$ *is a bounded radial function: for each* $k = (k_1, \dots, k_n) \ne 0$, $a_{(2z)}\widehat{g}(2z) \ne a_{(2z+2k)}\widehat{g}(2z + 2k)$, *where* $z \in I_n$ *and* $z + k \in I_n\}$.

Proof. Suppose that $f(z) = \sum_{k\in\mathbb{Z}^n} \xi^k f_k(r_1,\dots,r_n) \in L^\infty(\mathbb{D}^n)$ and g is a radial function. Lemma 2.2 shows that

$$\begin{aligned} T_f T_g(z^\alpha) &= \sum_{k\in\mathbb{Z}^n} 2^n a_{\alpha+\vec{1}} \widehat{g}\left(2\alpha+\vec{2}\right) C_k \widehat{f}_k\left(2\alpha+k+\vec{2}\right) z^{\alpha+k}, \\ T_g T_f(z^\alpha) &= \sum_{k\in\mathbb{Z}^n} C_k \widehat{f}_k\left(2\alpha+k+\vec{2}\right) 2^n a_{\alpha+k+\vec{1}} \widehat{g}\left(2\alpha+2k+\vec{2}\right) z^{\alpha+k}, \end{aligned} \tag{3.25}$$

where $\alpha \in \mathbb{N}^n$ and

$$C_k = \begin{cases} 0 & \text{if } \alpha \not\geq k^-, \\ 2^n a_{\alpha+k+\vec{1}} & \text{if } \alpha \geq k^-. \end{cases} \tag{3.26}$$

It follows that $T_f T_g = T_g T_f$ if and only if

$$a_{\alpha+\vec{1}} \widehat{g}\left(2\alpha+\vec{2}\right) C_k \widehat{f}_k\left(2\alpha+k+\vec{2}\right) = C_k \widehat{f}_k\left(2\alpha+k+\vec{2}\right) a_{\alpha+k+\vec{1}} \widehat{g}\left(2\alpha+2k+\vec{2}\right), \tag{3.27}$$

for any $k \in \mathbb{Z}^n$. Let $E_1 = \{\alpha : (a_{\alpha+\vec{1}} g - a_{\alpha+k+\vec{1}} r^k g)^\wedge(2\alpha+\vec{2}) = 0\}$ and $E_2 = \{\alpha : \widehat{r^k f}_k(2\alpha+\vec{2}) = 0\}$. Then $E_1 \cup E_2 = \{\alpha; \alpha \geq k^-\}$. The commutativity of T_f and T_g is equivalent to that at least one of E_1 and E_2 does not have property (P); then, Lemma 2.3 shows that $f_k \equiv 0$ or $a_{(2z)} \widehat{g}(2z) \neq a_{(2z+2k)} \widehat{g}(2z+2k)$, where $z \in I_n$ and $z+k \in I_n$. The rest of the proof is obvious. □

Remark 3.6. In Theorem 3.5, particularly if g is a radial function such that $g(z) = \prod_{j=1}^n g_j(z_j)$ or $g(z) = \sum_{j=1}^n g_j(z_j)$, where each $g_j(z_j)(1 \leq j \leq n)$ is a non-constant radial function, then for each $k = (k_1,\cdots,k_n) \neq 0$, $a_{(2z)} \widehat{g}(2z) \neq a_{(2z+2k)} \widehat{g}(2z+2k)$, where $z \in I_n$ and $z+k \in I_n$, so $g \in G$. It follows that G is nonempty.

4. Finite Rank Semicommutators and Commutators

Recall that Čučković and Louhichi (see [5]) have found some nonzero finite rank semicommutators of quasihomogeneous symbol Toeplitz operators on the Bergman space of unit disk. In this section, we will show that the finite rank semicommutators and commutators of Toeplitz operators with quasihomogeneous symbols must be zero on $A^2(\mathbb{D}^n)$ with $n \geq 2$. Our idea is mainly from [17].

Theorem 4.1. *Let $k, l \in \mathbb{Z}^n$ with $n \geq 2$, $k+l = m$, and let φ, ψ be radial functions such that $f_1 = \xi^k \varphi$, $f_2 = \xi^l \psi$, and $\xi^m \varphi\psi$ are all T-functions. If the semicommutator $(T_{f_1}, T_{f_2}]$ has finite rank, then it must be zero.*

Proof. Let S denote the semicommutator $(T_{f_1}, T_{f_2}]$. For $\alpha \in \mathbb{N}^n$, if S is finite rank, by equality (2.9), we have that there exists $\alpha^0 \geq k^- + l^-$ such that

$$\begin{aligned} S(z^\alpha) = 2^n a_{\alpha+m+\vec{1}} \Big(& 2^n a_{\alpha+l+\vec{1}} \widehat{\psi}\left(2\alpha+l+\vec{2}\right) \widehat{\varphi}\left(2(\alpha+l)+k+\vec{2}\right) \\ & - \widehat{\varphi\psi}\left(2\alpha+m+\vec{2}\right) \Big) = 0 \quad \text{for } \alpha \geq \alpha^0, \end{aligned} \tag{4.1}$$

which is equivalent to

$$\begin{aligned}&\left(r^{k^++l^+}\varphi\right)^{\wedge}\left(2\alpha+l-k^--l^-+\vec{2}\right)\left(r^{k^-+l^-}\psi\right)^{\wedge}\left(2\alpha+l-k^--l^-+\vec{2}\right)\\&\quad=\left(r^{l^++l^-}\right)^{\wedge}\left(2\alpha+l-k^--l^-+\vec{2}\right)\left(r^{k}\varphi\psi\right)^{\wedge}\left(2\alpha+l-k^--l^-+\vec{2}\right),\end{aligned}\tag{4.2}$$

for $\alpha \succeq \alpha^0$. Combining this with Lemma 2.3, we get

$$\left(r^{k^++l^+}\varphi\right)^{\wedge}(z)\left(r^{k^-+l^-}\psi\right)^{\wedge}(z)=\left(r^{l^++l^-}\right)^{\wedge}(z)\left(r^{k}\varphi\psi\right)^{\wedge}(z),\quad \text{for}\quad z\in I_n.\tag{4.3}$$

Hence,

$$S(z^\alpha)=0\quad \forall\alpha\in E_2.\tag{4.4}$$

In the following, we only need to prove that $\widehat{\varphi\psi}(\alpha)=0$ for all $\alpha\in E_1^c\cap E_2^c$.

If $E_1^c\cap E_2^c\neq\emptyset$, there is a j $(0\le j\le n)$ such that $m_j^-<l_j^-$. Without loss of generality, assume that $j=1$. Then, $\{(\alpha_1,m_2+\alpha_2,\dots,m_n+\alpha_n):m_1^-\le\alpha_1\le l_1^-,\ \alpha_j\ge 0,\ j=2,\dots,n\}\subseteq E_2$. For each $m_1^-\le\alpha_1$, let $F_{a_1}(r_2,\dots,r_n)=\int_0^1(\varphi\psi)(r_1,r_1,\dots,r_n)r_1^{a_1-1}dr_1$. Since $Z(\widehat{F_{a_1}})\supseteq\{(m_2+\alpha_2,\dots,m_n+\alpha_n):(\alpha_2,\dots,\alpha_n)\in N^{n-1}\}$ does not have property (P), we have $\widehat{F_{a_1}}\equiv 0$. Therefore, $\widehat{\varphi\psi}(a_1,a_2,\dots,a_n)=0$ for $m_1^-\le a_1\le l_1^-$ and $a_j\ge m_j^-$, $j=2,\dots,n$. So $S(z^\alpha)=0$ for $\alpha\in E_1^c\cap E_2^c$.

This completes the proof. □

We now pass to the commutator of two quasihomogeneous Toeplitz operators. Here the situation is the same as for the semicommutator.

Theorem 4.2. *Let $k,l\in\mathbb{Z}^n$ with $n\ge 2$, and let φ, ψ be radial functions such that $f_1=\xi^k\varphi$ and $f_2=\xi^l\psi$ are both T-functions. The commutator $[T_{f_1},T_{f_2}]$ has finite rank if and only if it is a zero operator.*

Proof. Let S denote the commutator $[T_{f_1},T_{f_2}]$. For $\alpha\in\mathbb{N}^n$, if S has finite rank N, by equality (2.10), we have that there exists $\alpha^0\succeq k^-+l^-$ such that

$$\begin{aligned}S(z^\alpha)=4^n a_{\alpha+k+l+\vec{1}}&\Big(a_{\alpha+l+\vec{1}}\widehat{\psi}\left(2\alpha+l+\vec{2}\right)\widehat{\varphi}\left(2(\alpha+l)+k+\vec{2}\right)\\&-a_{\alpha+k+\vec{1}}\widehat{\varphi}\left(2\alpha+k+\vec{2}\right)\widehat{\psi}\left(2(\alpha+k)+l+\vec{2}\right)\Big)(z^\alpha)=0\end{aligned}\tag{4.5}$$

for $\alpha\succeq\alpha^0$. As in the proof of Theorem 4.1, the above equation implies that

$$\begin{aligned}&\left(\chi_{[0,1]^n}\right)^{\wedge}\left(2\alpha+2k+\vec{2}\right)\widehat{\psi}\left(2\alpha+l+\vec{2}\right)\widehat{\varphi}\left(2(\alpha+l)+k+\vec{2}\right)\\&\quad=\left(\chi_{[0,1]^n}\right)^{\wedge}\left(2\alpha+2l+\vec{2}\right)\widehat{\varphi}\left(2\alpha+k+\vec{2}\right)\widehat{\psi}\left(2(\alpha+k)+l+\vec{2}\right)\end{aligned}\tag{4.6}$$

for $\alpha \in F_1 \cap F_2$. Hence,

$$S(z^\alpha) = 0 \quad \forall \alpha \in F_4. \tag{4.7}$$

For $\alpha \in F_1 \cap F_2^c$ or $\alpha \in F_1^c \cap F_2$, following the same way as above, we can also prove that $S(z^\alpha) = 0$.

This completes the proof. □

5. Finite Rank Products of Toeplitz Operators

In [17], the author showed that under certain conditions on the bounded operators S_1 and S_2 on $A^2(\mathbb{B}_n)$, if $f \in L^2(\mathbb{B}_n)$, such that $S_2 T_f S_1$ is a finite-rank operator, then f must be zero almost everywhere on $\mathbb{B}_n$. On the Bergman space of the polydisk, using the same method as in [17], we can prove Theorem 5.1. Using Theorem 5.1, we get two useful theorems for Toeplitz operators with quasihomogeneous symbols.

Theorem 5.1. *Let S_1, S_2 be two bounded operators on $A^2(\mathbb{D}^n)$. Suppose that there is a set $\mathbb{S} \subseteq \mathbb{N}^n$ which has property (P), such that $\ker(S_2) \subseteq \overline{\mathcal{M}}$ and $\mathcal{N} \subseteq \operatorname{ran}(S_1)$. Here, $\mathcal{M}$ (resp., $\mathcal{N}$) is the linear subspace of $A^2(\mathbb{D}^n)$ spanned by $\{z^m, m \in \mathbb{S}\}$ (resp., $\{z^m, m \in \mathbb{N}^n \setminus \mathbb{S}\}$). Suppose that $f \in L^2(\mathbb{D}^n)$ such that the operator $S_2 T_f S_1$ has finite rank; then f is the zero function.*

Theorem 5.2. *Let M and W be two positive integers. Let $f_1, \ldots, f_M$ and $g_1, \ldots, g_W$ be quasihomogeneous functions, none of which is the zero function. If $f \in L^2$ such that the operator $T_{f_M} \cdots T_{f_1} T_f T_{g_W} \cdots T_{g_1}$ has finite rank, then f is the zero function.*

Proof. Let $S_2 = T_{f_M} \cdots T_{f_1}$ and $S_1 = T_{g_W} \cdots T_{g_1}$. Suppose that $f_j = \xi^{p_j} \varphi_j(r)$, $1 \le j \le M$, and $g_l = \xi^{q_l} \psi_l(r)$, $1 \le l \le W$, where $p_j, q_l \in \mathbb{Z}^n$. By Lemma 2.2, for $\alpha \ge \sum_{l=1}^{w} q_l^-$, we have

$$S_1(z^\alpha) = \left(2^{2nW} \prod_{j=1}^{W} a_{\alpha + \sum_{l=1}^{j} q_l + \vec{1}} \widehat{\varphi_j}\left(2\left(\alpha + \sum_{l=1}^{j} q_l \right) - q_j + \vec{2} \right) \right) z^{\alpha + \sum_{j=1}^{W} q_j}. \tag{5.1}$$

Define $\mathcal{Q} = \{\alpha \in \mathbb{N}^n : \alpha \not\ge \sum_{l=1}^{W} q_l^-\} \bigcup (\bigcup_{j=1}^{W} \{\alpha \in \mathbb{N}^n : \widehat{\varphi_j}(2(\alpha + \sum_{l=1}^{j} q_l) - q_j + \vec{2}) = 0\})$. Since none of the functions $\psi_1, \ldots, \psi_W$ is the zero function, the set $\mathcal{Q}$ has property (P).

For $\alpha \in \mathbb{N}^n \setminus \mathcal{Q}$, we see that $S_1(z^\alpha) \ne 0$. Suppose that $\varphi \in A^2_\alpha$ such that $S_1(\varphi) = 0$; then,

$$0 = S_1(\varphi) = S_1\left(\sum_{\alpha \in \mathbb{N}^n} \langle \varphi, z^\alpha \rangle z^\alpha \right) = \sum_{\alpha \in \mathbb{N}^n} \langle \varphi, z^\alpha \rangle S_1 z^\alpha. \tag{5.2}$$

So (5.1) implies that for any $\alpha \in \mathbb{N}^n \setminus \mathcal{Q}$, $\langle \varphi, z^\alpha \rangle = 0$. Therefore $\ker(S_1)$ is contained in the closure of the linear span of $\{z^\alpha : \alpha \in \mathcal{Q}\}$ in A^2_α. Now suppose that

$$\mathcal{O} = \left\{ \alpha : \alpha \not\ge \sum_{l=1}^{W} q_l^- \right\} \bigcup \left(\mathbb{N}^n \bigcap \left(\mathcal{Q} + \sum_{j=1}^{W} q_j \right) \right). \tag{5.3}$$

Then the set $\mathcal{O}$ has property (P) and, for any $\alpha \in \mathbb{N}^n \setminus \mathcal{O}$, $\beta = \alpha - \sum_{j=1}^{W} q_j$ belongs to $\mathbb{N}^n \setminus \mathcal{Q}$. Equality (5.1) implies that $z^\alpha = z^{\beta+\sum_{j=1}^{W} q_j}$ is a multiple of $S_1 z^\beta$. So the linear span of $\{z^\alpha : \alpha \in \mathbb{N}^n \setminus \mathcal{O}\}$ is contained in the range of S_1. So there exist subsets $\mathcal{Q}$ and $\mathcal{O}$ of $\mathbb{N}^n$ that have property (P) such that $\ker(S_2)$ is contained in the closure in A_α^2 of $\mathrm{Span}(\{z^\alpha : \alpha \in \mathcal{Q}\})$ and $\mathrm{Span}(\{z^\alpha : \alpha \in \mathbb{N}^n \setminus \mathcal{O}\}$ is a subspace of $S_1(A_\alpha^2)$. Let $\mathbb{S} = \mathcal{O} \bigcup \mathcal{Q}$; then, Theorem 5.1 implies that f is the zero function. □

Theorem 5.3. *Suppose that the function $f(z) \in L^2(\mathbb{D}^n)$ has the expansion*

$$f(z) = \sum_{k \le M} \xi^k f_k(r_1, \ldots, r_n), \tag{5.4}$$

and $\widehat{f}_M(l) \neq 0$ for all $l \ge l_0$, where $l_0 \in \mathbb{N}^n$, if there is a function $g(z) \in L^2(\mathbb{D}^n)$, such that $T_g T_f$ has finite rank; then $g = 0$.

Proof. For $\alpha \in \mathbb{N}^n$,

$$\begin{aligned} T_f(z^\alpha) &= \sum_{M \ge k \ge -\alpha} 2^n a_{\alpha+k+\vec{1}} \widehat{f_k}\left(2\alpha + k + \vec{2}\right) z^{\alpha+k} \\ &= 2^n a_{\alpha+M+\vec{1}} \widehat{f_M}\left(2\alpha + M + \vec{2}\right) z^{\alpha+M} + \sum_{M \ge k \ge -\alpha,\ k \ne M} 2^n a_{\alpha+k+\vec{1}} \widehat{f_k}\left(2\alpha + k + \vec{2}\right) z^{\alpha+k}. \end{aligned} \tag{5.5}$$

By hypothesis, there exists $\alpha_0 \in \mathbb{N}^n$, such that, for any $\alpha \ge \alpha_0$, $\widehat{f}_M(2\alpha + M + \vec{2}) \neq 0$; then, $\widehat{f}_M(2\alpha_0 + M + \vec{2}) \neq 0$. Thus, we have

$$z^{\alpha_0+M} \in \mathrm{Span}\{T_f(z^{\alpha_0}), z^\alpha : 0 \le \alpha \le \alpha_0 + M, \alpha \neq \alpha_0 + M\}. \tag{5.6}$$

Considering the same argument, we get, for all $l \ge 0$,

$$z^{\alpha_0+M+l} \in \mathrm{Span}\left\{T_f\left(z^\beta\right), z^\alpha : \alpha_0 \le \beta \le \alpha_0 + l, 0 \le \alpha \le \alpha_0 + M, \alpha \neq \alpha_0 + M\right\}. \tag{5.7}$$

Now suppose that $T_g T_f$ has finite rank, and let $\{\varphi_1, \ldots, \varphi_N\}$ be the set that spans $T_g T_f(\mathcal{P})$, where $\mathcal{P}$ is the space of all holomorphic polynomials in the variable z. Then, for any $l \in \mathbb{N}^n$, we see that $T_g z^{\alpha_0+M+l}$ is a linear combination of $\{\varphi_1, \ldots, \varphi_N\} \bigcup \{T_g(z^\alpha)$, where $0 \le \alpha \le \alpha_0 + M$ and $\alpha \neq \alpha_0 + M\}$, and it follows that T_g is a finite-rank operator. By Theorem 2.4, we conclude that g is the zero function. □

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