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Research Article

Asymptotic Behavior of Solutions to Half-Linear q-Difference Equations

Pavel Řehák

Institute of Mathematics, Academy of Sciences of the Czech Republic, Žižkova 22, 61662 Brno, Czech Republic

Correspondence should be addressed to Pavel Řehák, rehak@math.cas.cz

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We derive necessary and sufficient conditions for (some or all) positive solutions of the half-linear q-difference equation $D_q(\Phi(D_q y(t))) + p(t)\Phi(y(qt)) = 0$, $t \in \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$ with $\alpha > 1$, to behave like q-regularly varying or q-rapidly varying or q-regularly bounded functions (that is, the functions y, for which a special limit behavior of $y(qt)/y(t)$ as $t \to \infty$ is prescribed). A thorough discussion on such an asymptotic behavior of solutions is provided. Related Kneser type criteria are presented.

1. Introduction

In this paper we recall and survey the theory of q-Karamata functions, that is, of the functions $y : q^{\mathbb{N}_0} \to (0,\infty)$, where $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, and for which some special limit behavior of $y(qt)/y(t)$ as $t \to \infty$ is prescribed, see [1–3]. This theory corresponds with the classical "continuous" theory of regular variation, see, for example, [4], but shows some special features (see Section 2), not known in the continuous case, which are due to the special structure of $q^{\mathbb{N}_0}$. The theory of q-Karamata functions provides a powerful tool, which we use in this paper to establish sufficient and necessary conditions for some or all positive solutions of the half-linear q-difference equation

$$D_q\big(\Phi\big(D_q y(t)\big)\big) + p(t)\Phi\big(y(qt)\big) = 0, \tag{1.1}$$

where $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$ with $\alpha > 1$, to behave like q-regularly varying or q-rapidly varying or q-regularly bounded functions. We stress that there is no sign condition on p. We also present Kneser type (non)oscillation criteria for (1.1), existing as well as new ones, which are somehow related to our asymptotic results.

The main results of this paper can be understood as a q-version of the continuous results for

$$\left(\Phi\left(y'(t)\right)\right)' + p(t)\Phi\left(y(t)\right) = 0 \tag{1.2}$$

from [5] (with noting that some substantial differences between the parallel results are revealed), or as a half-linear extension of the results for $D_q^2 y(t) + p(t)y(qt) = 0$ from [1]. In addition, we provide a thorough description of asymptotic behavior of solutions to (1.1) with respect to the limit behavior of $t^\alpha p(t)$ in the framework of q-Karamata theory. For an explanation why the q-Karamata theory and its applications are not included in a general theory of regular variation on measure chains see [6]. For more information on (1.2) see, for example, [7]. Many applications of the theory of regular variation in differential equations can be found, for example, in [8]. Linear q-difference equations were studied, for example, in [1, 9–11]; for related topics see, for example, [12, 13]. Finally note that the theory of q-calculus is very extensive with many aspects; some people speak bout different tongues of q-calculus. In our paper we follow essentially its "time-scale dialect".

2. Preliminaries

We start with recalling some basic facts about q-calculus. For material on this topic see [9, 12, 13]. See also [14] for the calculus on time-scales which somehow contains q-calculus. First note that some of the below concepts may appear to be described in a "nonclassical q-way", see, for example, our definition of q-integral versus original Jackson's definition [9, 12, 13], or the q-exponential function. But, working on the lattice $q^{\mathbb{N}_0}$ (which is a time-scale), we can introduce these concepts in an alternative and "easier" way (and, basically, we avoid some classical q-symbols). Our definitions, of course, naturally correspond with the original definitions. The q-derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $D_q f(t) = [f(qt) - f(t)]/[(q-1)t]$. The q-integral $\int_a^b f(t)\mathrm{d}_q t$, $a, b \in q^{\mathbb{N}_0}$, of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $\int_a^b f(t)\mathrm{d}_q t = (q-1)\sum_{t\in[a,b)\cap q^{\mathbb{N}_0}} t f(t)$ if $a < b$; $\int_a^b f(t)\mathrm{d}_q t = 0$ if $a = b$; $\int_a^b f(t)\mathrm{d}_q t = (1-q)\sum_{t\in[b,a)\cap q^{\mathbb{N}_0}} t f(t)$ if $a > b$. The improper q-integral is defined by $\int_a^\infty f(t)\mathrm{d}_q t = \lim_{b\to\infty}\int_a^b f(t)\mathrm{d}_q t$. We use the notation $[a]_q = (q^a - 1)/(q-1)$ for $a \in \mathbb{R}$. Note that $\lim_{q\to 1^+}[a]_q = a$. It holds that $D_q t^\vartheta = [\vartheta]_q t^{\vartheta-1}$. In view of the definition of $[a]_q$, it is natural to introduce the notation $[\infty]_q = \infty$, $[-\infty]_q = 1/(1-q)$. For $p \in \mathcal{R}$ (i.e., for $p : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfying $1 + (q-1)tp(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$) we denote $e_p(t,s) = \prod_{u\in[s,t)\cap q^{\mathbb{N}_0}}[(q-1)up(u) + 1]$ for $s < t$, $e_p(t,s) = 1/e_p(s,t)$ for $s > t$, and $e_p(t,t) = 1$, where $s, t \in q^{\mathbb{N}_0}$. For $p \in \mathcal{R}$, $e(\cdot, a)$ is a solution of the IVP $D_q y = p(t)y$, $y(a) = 1$, $t \in q^{\mathbb{N}_0}$. If $s \in q^{\mathbb{N}_0}$ and $p \in \mathcal{R}^+$, where $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + (q-1)tp(t) > 0 \text{ for all } t \in q^{\mathbb{N}_0}\}$. then $e_p(t,s) > 0$ for all $t \in q^{\mathbb{N}_0}$. If $p, r \in \mathcal{R}$, then $e_p(t,s)e_p(s,u) = e_p(t,u)$ and $e_p(t,s)e_r(t,s) = e_{p+r+t(q-1)pr}(t,s)$. Intervals having the subscript q denote the intervals in $q^{\mathbb{N}_0}$, for example, $[a,\infty)_q = \{a, aq, aq^2, \ldots\}$ with $a \in q^{\mathbb{N}_0}$.

Next we present auxiliary statements which play important roles in proving the main results. Define $F : (0,\infty) \to \mathbb{R}$ by $F(x) = \Phi(x/q-1/q)-\Phi(1-1/x)$ and $h : (\Phi([-\infty]_q),\infty) \to \mathbb{R}$ by

$$h(x) = \frac{x}{1-q^{1-\alpha}}\left[1-\left(1+(q-1)\Phi^{-1}(x)\right)^{1-\alpha}\right]. \tag{2.1}$$

For $y : q^{\mathbb{N}_0} \to \mathbb{R}\setminus\{0\}$ define the operator $\mathcal{L}$ by

$$\mathcal{L}[y](t) = \Phi\left(\frac{y(q^2t)}{qy(qt)}-\frac{1}{q}\right)-\Phi\left(1-\frac{y(t)}{y(qt)}\right). \tag{2.2}$$

We denote $\omega_q = ([(\alpha-1)/\alpha]_q)^{\alpha}$. Let β mean the conjugate number of α, that is, $1/\alpha+1/\beta = 1$.

The following lemma lists some important properties of F, h, $\mathcal{L}$ and relations among them.

Lemma 2.1. (i) *The function F has the global minimum on $(0,\infty)$ at $q^{(\alpha-1)/\alpha}$ with*

$$F\left(q^{(\alpha-1)/\alpha}\right) = -\frac{\omega_q(q-1)^{\alpha}}{q^{\alpha-1}} \tag{2.3}$$

and $F(1) = 0 = F(q)$. Further, F is strictly decreasing on $(0,q^{(\alpha-1)/\alpha})$ and strictly increasing on $(q^{(\alpha-1)/\alpha},\infty)$ with $\lim_{x\to 0+}F(x) = \infty$, $\lim_{t\to\infty}F(x) = \infty$.

(ii) *The graph of $x \mapsto h(x)$ is a parabola-like curve with the minimum at the origin. The graph of $x \mapsto h(x)+\gamma_\alpha$ touches the line $x \mapsto x$ at $x = \lambda_0 := ([(\alpha-1)/\alpha]_q)^{\alpha-1}$. The equation $h(\lambda)+\gamma = \lambda$ has*

(a) *no real roots if $\gamma > \omega_q/[\alpha-1]_q$,*

(b) *the only root λ_0 if $\gamma = \omega_q/[\alpha-1]_q$,*

(c) *two real roots λ_1, λ_2 with $0 < \lambda_1 < \lambda_0 < \lambda_2 < 1$ if $\gamma \in (0,\omega_q/[\alpha-1]_q)$,*

(d) *two real roots 0 and 1 if $\gamma = 0$,*

(e) *two real roots λ_1, λ_2 with $\lambda_1 < 0 < 1 < \lambda_2$ if $\gamma < 0$.*

(iii) *It holds that $F(q^{\vartheta_1}) = F(q^{\vartheta_2})$, where $\vartheta_i = \log_q[(q-1)\Phi^{-1}(\lambda_i)+1]$, $i = 1,2$, with $\lambda_1 < \lambda_2$ being the real roots of the equation $\lambda = h(\lambda)+A$ with $A \in (-\infty,\omega_q/[\alpha-1]_q)$.*

(iv) *If $q \to 1+$, then $h(x) \to |x|^{\beta}$.*

(v) *For $\vartheta \in \mathbb{R}$ it hold that $\Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}} = \Phi([\vartheta]_q)-h(\Phi([\vartheta]_q))$.*

(vi) *For $\vartheta \in \mathbb{R}$ it hold that $F(q^{\vartheta}) = (q-1)^{\alpha}[1-\alpha]_q\Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}}$.*

(vii) *For $y \neq 0$, (1.1) can be written as $\mathcal{L}[y](t) = -(q-1)^{\alpha}t^{\alpha}p(t)$.*

(viii) *If the $\lim_{t\to\infty}y(qt)/y(t)$ exists as a positive real number, then $\lim_{t\to\infty}\mathcal{L}[y](t) = \lim_{t\to\infty}F(y(qt)/y(t))$.*

Proof. We prove only (iii). The proofs of other statements are either easy or can be found in [3].

(iii) Let λ_1, λ_2 be the real roots of $\lambda = h(\lambda) + A$. We have $\lambda_i = \Phi([\vartheta_i]_q)$, $i = 1, 2$, and so, by virtue of identities (v) and (vi), we get $F(q^{\vartheta_1}) = (q-1)^\alpha [1-\alpha]_q (\lambda_1 - h(\lambda_1)) = (q-1)^\alpha [1-\alpha]_q A = (q-1)^\alpha [1-\alpha]_q (\lambda_2 - h(\lambda_2)) = F(q^{\vartheta_2})$. □

Next we define the basic concepts of q-Karamata theory. Note that the original definitions (see [1–3]) was more complicated; they were motivated by the classical continuous and the discrete (on the uniform lattices) theories. But soon it has turned out that simpler (and equivalent)definitions can be established. Also, there is no need to introduce the concept of normality, since every q-regularly varying or q-rapidly varying or q-regularly bounded function is automatically normalized. Such (and some other) simplifications are not possible in the original continuous theory or in the classical discrete theory; in q-calculus, they are practicable thanks to the special structure of $q^{\mathbb{N}_0}$, which is somehow natural for examining regularly varying behavior.

For $f : q^{\mathbb{N}_0} \to (0, \infty)$ denote

$$K_* = \liminf_{t\to\infty} \frac{f(qt)}{f(t)}, \qquad K^* = \limsup_{t\to\infty} \frac{f(qt)}{f(t)}, \qquad K = \lim_{t\to\infty} \frac{f(qt)}{f(t)}. \tag{2.4}$$

Definition 2.2. A function $f : q^{\mathbb{N}_0} \to (0, \infty)$ is said to be

(i) *q-regularly varying of index ϑ*, $\vartheta \in \mathbb{R}$, if $K = q^\vartheta$; we write $f \in \mathcal{RV}_q(\vartheta)$,

(ii) *q-slowly varying* if $K = 1$; we write $f \in \mathcal{SV}_q$,

(iii) *q-rapidly varying of index ∞* if $K = \infty$; we write $f \in \mathcal{RPV}_q(\infty)$,

(iv) *q-rapidly varying of index $-\infty$* if $K = 0$; we write $f \in \mathcal{RPV}_q(-\infty)$,

(v) *q-regularly bounded* if $0 < K_* \le K^* < \infty$; we write $f \in \mathcal{RB}_q$.

Clearly, $\mathcal{SV}_q = \mathcal{RV}_q(0)$. We have defined q-regular variation, q-rapid variation, and q-regular boundedness at infinity. If we consider a function $f : q^{\mathbb{Z}} \to (0, \infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(t)$ is said to be q-regularly varying/q-rapidly varying/q-regularly bounded at zero if $f(1/t)$ is q-regularly varying/q-rapidly varying/q-regularly bounded at infinity. But it is apparent that it is sufficient to examine just the behavior at ∞.

Next we list some selected important properties of the above-defined functions. We define $\tau : [1, \infty) \to q^{\mathbb{N}_0}$ as $\tau(x) = \max\{s \in q^{\mathbb{N}_0} : s \le x\}$.

Proposition 2.3. (i) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow \lim_{t\to\infty} t D_q f(t)/f(t) = [\vartheta]_q$.

(ii) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow f(t) = \varphi(t) e_\psi(t, 1)$, *where a positive φ satisfies* $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$, $\lim_{t\to\infty} t\psi(t) = [\vartheta]_q$, $\psi \in \mathcal{R}^+$ *(w.l.o.g., φ can be replaced by C).*

(iii) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow f(t) = t^\vartheta L(t)$, *where* $L \in \mathcal{SV}_q$.

(iv) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow f(t)/t^\gamma$ *is eventually increasing for each* $\gamma < \vartheta$ *and* $f(t)/t^\eta$ *is eventually decreasing for each* $\eta > \vartheta$.

(v) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow \lim_{t\to\infty} f(\tau(\lambda t))/f(t) = (\tau(\lambda))^\vartheta$ *for every* $\lambda \ge 1$.

(vi) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow R : [1, \infty) \to (0, \infty)$ *defined by* $R(x) = f(\tau(x))(x/\tau(x))^\vartheta$ *for* $x \in [1, \infty)$ *is regularly varying of index* ϑ.

(vii) $f \in \mathcal{RV}_q(\vartheta) \Rightarrow \lim_{t\to\infty} \log f(t)/\log t = \vartheta$.

Proof. See [2]. □

Proposition 2.4. (i) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow \lim_{t\to\infty} tD_q f(t)/f(t) = [\pm\infty]_q$.

(ii) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow f(t) = \varphi(t)e_\psi(t,1)$, *where a positive* φ *satisfies* $\liminf_{t\to\infty}\varphi(qt)/\varphi(t) > 0$ *for index* ∞, $\limsup_{t\to\infty}\varphi(qt)/\varphi(t) < \infty$ *for index* $-\infty$, *and* $\lim_{t\to\infty} t\psi(t) = [\pm\infty]_q$, $\psi \in \mathcal{R}^+$ *(w.l.o.g.,* φ *can be replaced by* $C \in (0,\infty)$*).*

(iii) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow$ *for each* $\vartheta \in [0,\infty)$, $f(t)/t^\vartheta$ *is eventually increasing (towards* ∞*) for index* ∞ *and* $f(t)t^\vartheta$ *is eventually decreasing (towards 0) for index* $-\infty$.

(iv) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow$ *for every* $\lambda \in [q,\infty)$ *it holds,* $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = \infty$ *for index* ∞ *and* $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = 0$ *for index* $-\infty$.

(v) *Let* $R : [1,\infty) \to (0,\infty)$ *be defined by* $R(x) = f(\tau(x))$ *for* $x \in [1,\infty)$. *If* R *is rapidly varying of index* $\pm\infty$, *then* $f \in \mathcal{RPV}_q(\pm\infty)$. *Conversely, if* $f \in \mathcal{RPV}_q(\pm\infty)$, *then* $\lim_{x\to\infty} R(\lambda x)/R(x) = \infty$, *resp.,* $\lim_{x\to\infty} R(\lambda x)/R(x) = 0$ *for* $\lambda \in [q,\infty)$.

(vi) $f \in \mathcal{RPV}_q(\pm\infty) \Rightarrow \lim_{t\to\infty} \log f(t)/\log t = \pm\infty$.

Proof. We prove only the "if" part of (iii). The proofs of (iv), (v), and (vi) can be found in [1]. The proofs of other statements can be found in [3].

Assume that $f(t)/t^\vartheta$ is eventually increasing (towards ∞) for each $\vartheta \in [0,\infty)$. Because of monotonicity, we have $f(t)/t^\vartheta \le f(qt)/(q^\vartheta t^\vartheta)$, and so $f(qt)/f(t) \ge q^\vartheta$ for large t. Since ϑ is arbitrary, we have $f(qt)/f(t) \to \infty$ as $t \to \infty$, thus $f \in \mathcal{RPV}_q(\infty)$. The case of the index $-\infty$ can be treated in a similar way. □

Proposition 2.5. (i) $f \in \mathcal{RB}_q \Leftrightarrow [-\infty]_q < \liminf_{t\to\infty} tD_q f(t)/f(t) \le \limsup_{t\to\infty} tD_q f(t)/f(t) < [\infty]_q$.

(ii) $f \in \mathcal{RB}_q \Leftrightarrow f(t) = t^\vartheta\varphi(t)e_\psi(t,1)$, *where* $0 < C_1 \le \varphi(t) \le C_2 < \infty$, $[-\infty]_q < D_1 \le t\psi(t) \le D_2 < [\infty]_q$ *(w.l.o.g.,* φ *can be replaced by* $C \in (0,\infty)$*).*

(iii) $f \in \mathcal{RB}_q \Leftrightarrow f(t)/t^{\gamma_1}$ *is eventually increasing and* $f(t)/t^{\gamma_2}$ *is eventually decreasing for some* $\gamma_1 < \gamma_2$ *(w.l.o.g., monotonicity can be replaced by almost monotonicity; a function* $f : q^{\mathbb{N}_0} \to (0,\infty)$ *is said to be almost increasing (almost decreasing) if there exists an increasing (decreasing) function* $g : q^{\mathbb{N}_0} \to (0,\infty)$ *and* $C, D \in (0,\infty)$ *such that* $Cg(t) \le f(t) \le Dg(t)$*).*

(iv) $f \in \mathcal{RB}_q \Leftrightarrow 0 < \liminf_{t\to\infty} f(\tau(\lambda t))/f(t) \le \limsup_{t\to\infty} f(\tau(\lambda t))/f(t) < \infty$ *for every* $\lambda \in [q,\infty)$ *or for every* $\lambda \in (0,1)$.

(v) $f \in \mathcal{RB}_q \Leftrightarrow R : [1,\infty) \to (0,\infty)$ *defined by* $R(x) = f(\tau(x))$ *for* $x \in [1,\infty)$ *is regularly bounded.*

(vi) $f \in \mathcal{RB}_q \Rightarrow -\infty < \liminf_{t\to\infty} \log f(t)/\log t \le \limsup_{t\to\infty} \log f(t)/\log t < \infty$.

Proof. See [1]. □

For more information on q-Karamata theory see [1–3].

3. Asymptotic Behavior of Solutions to (1.1) in the Framework of q-Karamata Theory

First we establish necessary and sufficient conditions for positive solutions of (1.1) to be q-regularly varying or q-rapidly varying or q-regularly bounded. Then we use this result to provide a thorough discussion on Karamata-like behavior of solutions to (1.1).

Theorem 3.1. (i) *Equation* (1.1) *has eventually positive solutions* u, v *such that* $u \in \mathcal{RV}_q(\vartheta_1)$ *and* $v \in \mathcal{RV}_q(\vartheta_2)$ *if and only if*

$$\lim_{t\to\infty} t^\alpha p(t) = P \in \left(-\infty, \frac{\omega_q}{q^{\alpha-1}}\right), \tag{3.1}$$

where $\vartheta_i = \log_q[(q-1)\Phi^{-1}(\lambda_i)+1]$, $i = 1,2$, *with* $\lambda_1 < \lambda_2$ *being the real roots of the equation* $\lambda = h(\lambda) - P/[1-\alpha]_q$. *For the indices* ϑ_i, $i = 1,2$, *it holds that* $\vartheta_1 < 0 < 1 < \vartheta_2$ *provided* $P < 0$; $\vartheta_1 = 0$, $\vartheta_2 = 1$ *provided* $P = 0$; $0 < \vartheta_1 < (\alpha-1)/\alpha < \vartheta_2 < 1$ *provided* $P > 0$. *Any of two conditions* $u \in \mathcal{RV}_q(\vartheta_1)$ *and* $v \in \mathcal{RV}_q(\vartheta_2)$ *implies* (3.1).

(ii) *Let* (1.1) *be nonoscillatory (which can be guaranteed, for example, by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large t; with the note that it allows* (3.2)*). Equation* (1.1) *has an eventually positive solution* u *such that* $u \in \mathcal{RV}_q((\alpha-1)/\alpha)$ *if and only if*

$$\lim_{t\to\infty} t^\alpha p(t) = \frac{\omega_q}{q^{\alpha-1}}. \tag{3.2}$$

All eventually positive solutions of (1.1) *are q-regularly varying of index* $(\alpha-1)/\alpha$ *provided* (3.2) *holds.*

(iii) *Equation* (1.1) *has eventually positive solutions* u, v *such that* $u \in \mathcal{RPV}_q(-\infty)$ *and* $u \in \mathcal{RPV}_q(\infty)$ *if and only if*

$$\lim_{t\to\infty} t^\alpha p(t) = -\infty. \tag{3.3}$$

All eventually positive solutions of (1.1) *are q-rapidly varying provided* (3.3) *holds.*

(iv) *If* (1.1) *is nonoscillatory (which can be guaranteed, e.g., by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large t) and*

$$\liminf_{t\to\infty} t^\alpha p(t) > -\infty, \tag{3.4}$$

then all eventually positive solutions of (1.1) *are q-regularly bounded.*

Conversely, if there exists an eventually positive solution u *of* (1.1) *such that* $u \in \mathcal{RB}_q$, *then*

$$-\infty < \liminf_{t\to\infty} t^\alpha p(t) \le \limsup_{t\to\infty} t^\alpha p(t) < \frac{1+q^{1-\alpha}}{(q-1)^\alpha}. \tag{3.5}$$

If, in addition, p *is eventually positive or* u *is eventually increasing, then the constant on the right-hand side of* (3.5) *can be improved to* $1/(q-1)^\alpha$.

Proof. (i) *Necessity*. Assume that u is a solution of (1.1) such that $u \in \mathcal{RV}_q(\vartheta_1)$. Then, by Lemma 2.1,

$$\begin{aligned}\lim_{t\to\infty} t^\alpha p(t) &= -(q-1)^{-\alpha}\lim_{t\to\infty}\mathcal{L}[u](t) = -(q-1)^{-\alpha}\lim_{t\to\infty} F\left(\frac{u(qt)}{u(t)}\right)\\ &= -(q-1)^{-\alpha}F\left(q^{\vartheta_1}\right) = -[1-\alpha]_q\left[\Phi\left([\vartheta_1]_q\right) - h\left(\Phi[\vartheta_1]_q\right)\right]\\ &= \frac{[1-\alpha]_q P}{[1-\alpha]_q} = P.\end{aligned} \tag{3.6}$$

The same arguments work when dealing with $v \in \mathcal{RV}_q(\vartheta_2)$ instead of u.

Sufficiency. Assume that (3.1) holds. Then there exist $N \in [0,\infty)$, $t_0 \in q^{\mathbb{N}_0}$, and $P_\eta \in (0,\omega_q/q^{\alpha-1})$ such that $-N \le t^\alpha p(t) \le P_\eta$ for $t \in [t_0,\infty)_q$. Let $\mathcal{X}$ be the Banach space of all bounded functions $[t_0,\infty)_q \to \mathbb{R}$ endowed with the supremum norm. Denote $\Omega = \{w \in \mathcal{X} : \Phi(q^{-\eta}-1) \le w(t) \le \widetilde{N}$ for $t \in [t_0,\infty)_q\}$, where $\widetilde{N} = N(q-1)^\alpha + q^{1-\alpha}$, $\eta = \log_q[(q-1)\Phi^{-1}(\lambda_\eta)+1]$, λ_η being the smaller root of $\lambda = h(\lambda) - P_\eta/[1-\alpha]_q$. In view of Lemma 2.1, it holds that $\eta < (\alpha-1)/\alpha$. Moreover, if $P_\eta \ge P$ (which must be valid in our case), then $\vartheta_1 \le \eta$. Further, by Lemma 2.1, $-(q-1)^P_\eta = \Phi(q^{-\eta}-1)(1-q^{(\alpha-1)(\eta-1)})$. Let $\mathcal{T} : \Omega \to \mathcal{X}$ be the operator defined by

$$(\mathcal{T}w)(t) = -(q-1)^\alpha t^\alpha p(t) - \Phi\left(\frac{1}{q\Phi^{-1}(w(qt)) + q} - \frac{1}{q}\right). \tag{3.7}$$

By means of the contraction mapping theorem we will prove that $\mathcal{T}$ has a fixed-point in Ω. First we show that $\mathcal{T}\Omega \subseteq \Omega$. Let $w \in \Omega$. Then, using identities (v) and (vi) from Lemma 2.1,

$$\begin{aligned}
(\mathcal{T}w)(t) &\ge -(q-1)^\alpha P_\eta - \Phi\left(\frac{1}{qq^{-\eta}} - \frac{1}{q}\right)\\
&= (\lambda_\eta - h(\lambda_\eta))(q-1)^\alpha [1-\alpha]_q - q^{(\eta-1)(\alpha-1)}\Phi(1-q^{-\eta})\\
&= F(q^\eta) - q^{(\eta-1)(\alpha-1)}\Phi(1-q^{-\eta})\\
&= \Phi(q^{-\eta}-1)\left(1-q^{(\alpha-1)(\eta-1)}\right) - q^{(\eta-1)(\alpha-1)}\Phi(1-q^{-\eta})\\
&= \Phi(q^{-\eta}-1)
\end{aligned} \tag{3.8}$$

and $(\mathcal{T}w)(t) \le -(q-1)^\alpha t^\alpha p(t) + q^{1-\alpha} \le \widetilde{N}$ for $t \in [t_0,\infty)_q$. Now we prove that $\mathcal{T}$ is a contraction mapping on Ω. Consider the function $g : (-1,\infty) \to \mathbb{R}$ defined by $g(x) = -\Phi(1/(q\Phi^{-1}(x) + q) - 1/q)$. It is easy to see that $|g'(x)| = q^{1-\alpha}(\Phi^{-1}(x)+1)^{-\alpha}$. Let $w, z \in \Omega$. The Lagrange mean value theorem yields $|g(w(t)) - g(z(t))| = |w(t) - z(t)||g'(\xi(t))|$, where $\xi : q^{\mathbb{N}_0} \to \mathbb{R}$ is such that $\min\{w(t), z(t)\} \le \xi(t) \le \max\{w(t), z(t)\}$ for $t \in [t_0,\infty)_q$. Hence,

$$\begin{aligned}
|(\mathcal{T}w)(t) - (\mathcal{T}z)(t)| &= |g(w(qt)) - g(z(qt))|\\
&= |w(qt) - z(qt)||g'(\xi(t))|\\
&\le |w(qt) - z(qt)||g'(\Phi(q^{-\eta}-1))|\\
&= q^{\eta\alpha+1-\alpha}|w(qt) - z(qt)|\\
&\le q^{\eta\alpha+1-\alpha}\|w - z\|
\end{aligned} \tag{3.9}$$

for $t \in [t_0,\infty)_q$. Thus $\|\mathcal{T}w - \mathcal{T}z\| \le q^{\eta\alpha+1-\alpha}\|w-z\|$, where $q^{\eta\alpha+1-\alpha} \in (0,1)$ by virtue of $q > 1$ and $\eta < (\alpha-1)/\alpha$. The Banach fixed-point theorem now guarantees the existence of $w \in \Omega$ such that $w = \mathcal{T}w$. Define u by $u(t) = \prod_{s\in[t_0,t)_q}(\Phi^{-1}(w(s)) + 1)^{-1}$. Then u is a positive solution of $\mathcal{L}[u](t) = -(q-1)^\alpha t^\alpha p(t)$ on $[t_0,\infty)_q$, and, consequently, of (1.1) (this implies nonoscillation of (1.1)). Moreover, $q^{-\eta} \le \Phi^{-1}(w(t)) + 1 \le 1/\overline{N}$, where $\overline{N} = 1/(\Phi^{-1}(\widetilde{N}) + 1)$, and thus

$\overline{N} \le u(qt)/u(t) \le q^{\eta}$. Denote $M_* = \liminf_{t\to\infty} u(qt)/u(t)$ and $M^* = \limsup_{t\to\infty} u(qt)/u(t)$. Rewrite $\mathcal{L}[u](t) = -(q-1)^{\alpha} t^{\alpha} p(t)$ as

$$\Phi\left(\frac{u(q^2t)}{qu(qt)} - \frac{1}{q}\right) = \Phi\left(1 - \frac{u(t)}{u(qt)}\right) - (q-1)^{\alpha} t^{\alpha} p(t). \tag{3.10}$$

Taking lim inf and lim sup as $t \to \infty$ in (3.10), we get $\Phi(M_*/q - 1/q) = \Phi(1 - 1/M_*) - (q-1)^{\alpha} P$ and $\Phi(M^*/q - 1/q) = \Phi(1 - 1/M^*) - (q-1)^{\alpha} P$, respectively. Hence, $F(M_*) = F(M^*)$. Since $M_*, M^* \in [\overline{N}, q^{\eta}]$ and F is strictly decreasing on $(0, q^{(\alpha-1)/\alpha})$ (by Lemma 2.1), we have $M := M_* = M^*$. Moreover,

$$F(M) = -(q-1)^{\alpha} P = (q-1)^{\alpha} [1-\alpha]_q \left(\Phi\left([\vartheta_i]_q\right) - h\left(\Phi[\vartheta_i]_q\right)\right) = F\left(q^{\vartheta_i}\right), \tag{3.11}$$

$i = 1, 2$, which implies $M = q^{\vartheta_1}$, in view of the facts that $M, q^{\vartheta_1} \in (0, q^{(\alpha-1)/\alpha})$, $q^{\vartheta_2} > q^{(\alpha-1)/\alpha}$, and F is monotone on $(0, q^{(\alpha-1)/\alpha})$. Thus $u \in \mathcal{RV}_q(\vartheta_1)$. Now we show that there exists a solution v of (1.1) with $v \in \mathcal{RV}_q(\vartheta_2)$. We can assume that N, t_0, and P_η are the same as in the previous part. Consider the set $\Gamma = \{w \in \mathcal{X} : \Phi(q^{\zeta-1} - 1/q) \le w(t) \le \widetilde{M}$ for $t \in [t_0, \infty)_q\}$, where $\widetilde{M} = 1 + (q-1)^{\alpha} N$, $\zeta = \log_q[(q-1)\Phi^{-1}(\lambda_\zeta) + 1]$, λ_ζ being the larger root of $\lambda = h(\lambda) - P_\eta/[1-\alpha]_q$. It is clear that N can be chosen in such a way that $\Phi(q^{\vartheta_2 - 1} - 1/q) < \widetilde{M}$. It holds $(\alpha-1)/\alpha < \zeta \le \vartheta_2$ and $-(q-1)^{\alpha} P_\eta = \Phi(q^{-\zeta} - 1)(1 - q^{(\alpha-1)(\zeta-1)})$. Define $\mathcal{S} : \Gamma \to \mathcal{X}$ by $(\mathcal{S}w)(t) = \Phi(1 - 1/(q\Phi^{-1}(w(t/q)) + 1)) - (q-1)^{\alpha} t^{\alpha} p(t)$ for $t \in [qt_0, \infty)_q$, and $(\mathcal{S}w)(t_0) = \Phi(q^{\vartheta_2 - 1} - 1/q)$. Using similar arguments as above it is not difficult to see that $\mathcal{S}\Gamma \subseteq \Gamma$ and $\|\mathcal{S}w - \mathcal{S}z\| < q^{\alpha-1-\alpha\zeta}\|w - z\|$ for $w, z \in \Gamma$. So there exists $w \in \Gamma$ such that $w = \mathcal{S}w$. If we define $v(t) = \prod_{s \in [qt_0, t)_q} (q\Phi^{-1}(w(s/q)) + 1)$, then v is a positive solution of (1.1) on $[qt_0, \infty)_q$, which satisfies $q^{\zeta} \le v(qt)/v(t) \le q\Phi^{-1}(\widetilde{M}) + 1$. Arguing as above we show that $v \in \mathcal{RV}_q(\vartheta_2)$.

(ii) *Necessity*. The proof is similar to that of (i).

Sufficiency. The condition $t^{\alpha} p(t) \le \omega_q / q^{\alpha-1}$ for large t implies nonoscillation of (1.1). Indeed, it is easy to see that $y(t) = t^{(\alpha-1)/\alpha}$ is a nonoscillatory solution of the Euler type equation $D_q(\Phi(D_q y(t))) + \omega_q q^{1-\alpha} t^{-\alpha} \Phi(y(qt)) = 0$. Nonoscillation of (1.1) then follows by using the Sturm type comparison theorem, see also Section 4(i). Let us write P as $P = [1-\alpha]_q (h(\Phi([(\alpha-1)/\alpha]_q)) - \Phi([(\alpha-1)/\alpha]_q))$, with noting that $\lambda = \Phi([(\alpha-1)/\alpha]_q)$ is the double root of $\lambda = h(\lambda) - \omega_q q^{1-\alpha}/[1-\alpha]_q$, see Lemma 2.1. Then, in view of Lemma 2.1, we obtain

$$\begin{aligned} F\left(q^{\vartheta}\right) &= (q-1)^{\alpha} [1-\alpha]_q \left[\Phi\left([\vartheta]_q\right) - h\left(\Phi[\vartheta]_q\right)\right] = -\frac{(q-1)^{\alpha} \omega_q}{q^{\alpha-1}} \\ &= -(q-1)^{\alpha} \lim_{t\to\infty} t^{\alpha} p(t) = \lim_{t\to\infty} \mathcal{L}[u](t). \end{aligned} \tag{3.12}$$

Let us denote $U_* = \liminf_{t\to\infty} u(qt)/u(t)$ and $U^* = \limsup_{t\to\infty} u(qt)/u(t)$. It is impossible to have $U_* = 0$ or $U^* = \infty$, otherwise $\lim_{t\to\infty} \mathcal{L}[u](t) = \infty$, which contradicts to (3.12). Thus $0 < U_* \le U^* < \infty$. Consider (1.1) in the form (3.10). Taking lim sup, respectively, lim inf as $t \to \infty$ in (3.10), into which our u is plugged, we obtain $F(U_*) = F(q^{(\alpha-1)/\alpha}) = F(U^*)$. Thanks to the properties of F, see Lemma 2.1, we get $U_* = U^* = q^{(\alpha-1)/\alpha}$. Hence, $u \in \mathcal{RV}_q((\alpha-1)/\alpha)$.

Since we worked with an arbitrary positive solution, it implies that all positive solutions must be q-regularly varying of index $(\alpha-1)/\alpha$.

(iii) The proof repeats the same arguments as that of [3, Theorem 1] (in spite of no sign condition on p). Note just that condition (3.3) compels p to be eventually negative and the proof of necessity does not depend on the sign of p.

(iv) *Sufficiency*. Let u be an eventually positive solution of (1.1). Assume by a contradiction that $\limsup_{t\to\infty} y(qt)/y(t)=\infty$. Then, in view of Lemma 2.1(vii),

$$\infty=\limsup_{t\to\infty}\left(\Phi\left(\frac{y(q^2t)}{qy(qt)}-\frac{1}{q}\right)-1\right)\leq\limsup_{t\to\infty}\mathcal{L}[y](t)=-(q-1)^{\alpha}\liminf_{t\to\infty}t^{\alpha}p(t)<\infty \tag{3.13}$$

by (3.4), a contradiction. If $\liminf_{t\to\infty} y(qt)/y(t)=0$, then $\limsup_{t\to\infty} y(t)/y(qt)=\infty$ and we proceed similarly as in the previous case. Since we worked with an arbitrary positive solution, it implies that all positive solutions must be q-regularly bounded.

Necessity. Let $y\in\mathcal{RB}_q$ be a solution of (1.1). Taking lim sup as $t\to\infty$ in $-(q-1)^{\alpha}t^{\alpha}p(t)=\mathcal{L}[y](t)$, we get

$$\begin{aligned}&-(q-1)^{\alpha}\liminf_{t\to\infty}t^{\alpha}p(t)\\&\quad=\limsup_{t\to\infty}\mathcal{L}[y](t)\leq\limsup_{t\to\infty}\Phi\left(\frac{y(q^2t)}{qy(qt)}-\frac{1}{q}\right)+\limsup_{t\to\infty}\Phi\left(\frac{y(t)}{y(qt)}-1\right)<\infty,\end{aligned} \tag{3.14}$$

which implies the first inequality in (3.5). Similarly, the lim inf as $t\to\infty$ yields $-(q-1)^{\alpha}\limsup_{t\to\infty}t^{\alpha}p(t)>-1/q^{\alpha-1}-1$, which implies the last inequality in (3.5). If p is eventually positive, then every eventually positive solution of (1.1) is eventually increasing, which can be easily seen from its concavity. Hence, $y(qt)/y(t)\geq 1$ for large t. Thus the last inequality becomes $-(q-1)^{\alpha}\limsup_{t\to\infty}t^{\alpha}p(t)>-1$. □

We are ready to provide a summarizing thorough discussion on asymptotic behavior of solutions to (1.1) with respect to the limit behavior of $t^{\alpha}p(t)$ in the framework of q-Karamata theory. Denote

$$P=\lim_{t\to\infty}t^{\alpha}p(t),\qquad P_*=\liminf_{t\to\infty}t^{\alpha}p(t),\qquad P^*=\limsup_{t\to\infty}t^{\alpha}p(t). \tag{3.15}$$

The set of all q-regularly varying and q-rapidly varying functions is said to be *q-Karamata functions*. With the use of the previous results we obtain the following statement.

Corollary 3.2. (i) *Assume that there exists $P\in\mathbb{R}\cup\{-\infty,\infty\}$. In this case,* (1.1) *possesses solutions that are q-Karamata functions provided* (1.1) *is nonoscillatory. Moreover, we distinguish the following subcases:*

(a) $P=-\infty$*:* (1.1) *is nonoscillatory and all its positive solutions are q-rapidly varying (of index $-\infty$ or ∞).*

(b) $P\in(-\infty,\omega_q/q^{\alpha-1})$*:* (1.1) *is nonoscillatory and there exist a positive solution which is q-regularly varying of index ϑ_1 and a positive solution which is q-regularly varying of index ϑ_2.*

(c) $P = \gamma_q$: (1.1) *either oscillatory or nonoscillatory (the latter one can be guaranteed, e.g., by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large* t*). In case of nonoscillation of* (1.1) *all its positive solutions are* q*-regularly varying of index* $(\alpha-1)/\alpha$.

(d) $P \in (\omega_q/q^{\alpha-1}, \infty) \cup \{\infty\}$: (1.1) *is oscillatory.*

(ii) *Assume that* $\mathbb{R} \cup \{-\infty\} \ni P_* < P^* \in \mathbb{R} \cup \{\infty\}$. *In this case, there are no* q*-Karamata functions among positive solutions of* (1.1). *Moreover, we distinguish the following subcases:*

(a) $P_* \in (\omega_q/q^{\alpha-1}, \infty) \cup \{\infty\}$: (1.1) *is oscillatory.*

(b) $P_* \in \{-\infty\} \cup (-\infty, \omega_q/q^{\alpha-1}]$: (1.1) *is either oscillatory (this can be guaranteed, e.g., by* $P^* > (1+q^{1-\alpha})/(q-1)^\alpha$ *or by* $p > 0$ *and* $P^* \ge 1/(q-1)^\alpha$*) or nonoscillatory (this can be guaranteed, e.g., by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large* t*). If, in addition to nonoscillation of* (1.1), *it holds* $P_* > -\infty$, *then all its positive solutions are* q*-regularly bounded, but there is no* q*-regularly varying solution. If* $P_* = -\infty$, *then there is no* q*-regularly bounded or* q*-rapidly varying solution.*

4. Concluding Remarks

(i) We start with some remarks to Kneser type criteria. As a by product of Theorem 3.1(i) we get the following nonoscillation Kneser type criterion: if $\lim_{t\to\infty} t^\alpha p(t) < \omega_q/q^{\alpha-1}$, then (1.1) is nonoscillatory. However, its better variant is known (it follows from a more general time-scale case involving Hille-Nehari type criterion [15]), where the sufficient condition is relaxed to $\limsup_{t\to\infty} t^\alpha p(t) < \omega_q/q^{\alpha-1}$. The constant $\omega_q/q^{\alpha-1}$ is sharp, since $\liminf_{t\to\infty} t^\alpha p(t) > \omega_q/q^{\alpha-1}$ implies oscillation of (1.1), see [15]. But no conclusion can be generally drawn if the equality occurs in these conditions. The above lim sup nonoscillation criterion can be alternatively obtained also from the observation presented at the beginning of the proof of Theorem 3.1(ii) involving the Euler type q-difference equation. And it is worthy of note that the conclusion of that observation can be reached also when modifying the proof of Hille-Nehari type criterion in [15]. A closer examination of the proof of Theorem 3.1(iv) shows that a necessary condition for nonoscillation of (1.1) is $-(q-1)^\alpha \limsup_{t\to\infty} t^\alpha p(t) \ge -q^{1-\alpha} - 1$. Thus we have obtained quite new Kneser type oscillation criterion: if $\limsup_{t\to\infty} t^\alpha p(t) > (1+q^{1-\alpha})/(q-1)^\alpha$, then (1.1) is oscillatory. If p is eventually positive, then the constant on the right-hand side can be improved to $1/(q-1)^\alpha$ and the strict inequality can be replaced by the nonstrict one (this is because of q-regular boundedness of possible positive solutions). A continuous analog of this criterion is not known, which is quite natural since $1/(q-1)^\alpha \to \infty$ as $q \to 1$. Compare these results with the Hille-Nehari type criterion, which was proved in general setting for dynamic equations and time-scales, and is valid no matter what the graininess is (see [15]); in q-calculus it reads as follows: if $p \ge 0$ and $\limsup_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\mathrm{d}_q s > 1$, then (1.1) is oscillatory. This criterion holds literally also in the continuous case. Finally note that, in general, $\limsup_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\mathrm{d}_q s \le \limsup_{t\to\infty} -[1-\alpha]_q t^\alpha p(t)$.

(ii) The results contained in Theorem 3.1 can understood at least in the three following ways:

(a) As a q-version of the continuous results for (1.2) from [5]. However, there are several substantial differences: The conditions in the continuous case are (and somehow must be) in the integral form (see also the item (iii) of this section); there is a different approach in the proof (see also the item (iv) of this section); the rapid variation has not been treated in such detail in the continuous case; in the case of the existence of the double root, we show that all (and not just some) positive solutions are q-regularly varying under quite mild assumptions; for positive solutions to be q-regularly bounded we obtain quite simple and natural sufficient and also necessary conditions.

(b) As a half-linear extension of the results for $D_q^2 y(t) + p(t)y(qt) = 0$ from [1]. In contrast to the linear case, in the half-linear case a reduction of order formula is not at disposal. Thus to prove that there are two q-regularly varying solutions of two different indices we need immediately to construct both of them. Lack of a fundamental like system for half-linear equations causes that, for the time being, we are not able to show that all positive solutions are q-regularly varying. This is however much easier task when $p(t) < 0$, see [3].

(c) As a generalization of the results from [3] in the sense of no sign condition on the coefficient p.

(iii) From the continuous theory we know that the sufficient and necessary conditions for regularly or rapidly varying behavior of solutions to (1.2) are in terms of limit behavior of integral expressions, typically $t^{\alpha-1}\int_t^\infty p(s)\mathrm{d}s$ or $t^{\alpha-1}\int_t^{\lambda t} p(s)\mathrm{d}s$. In contrast to that, in q-calculus case the conditions have nonintegral form. This is the consequence of specific properties of q-calculus: one thing is that we use a different approach which does not apply in the continuous case. Another thing is that the limit $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\mathrm{d}_q s$ can be expressed in terms of $\lim_{t\to\infty} t^\alpha p(t)$ (and vice versa), provided it exists. Such a relation does not work in the continuous case.

(iv) As already said, our approach in the proof of Theorem 3.1 is different from what is known in the continuous theory. Our method is designed just for q-difference equations and roughly speaking, it is based on rewriting a q-difference equation in terms of the fractions which appear in Definition 2.2. Such a technique cannot work in the continuous case. Since this method uses quite natural and simple relations (which are possible thanks to the special structure of $q^{\mathbb{N}_0}$), we believe that it will enable us to prove also another results which are q-versions of existing or nonexisting continuous results; in the latter case, such results may serve to predict a possible form of the continuous counterpart, which may be difficult to handle directly. We just take, formally, the limit as $q \to 1+$.

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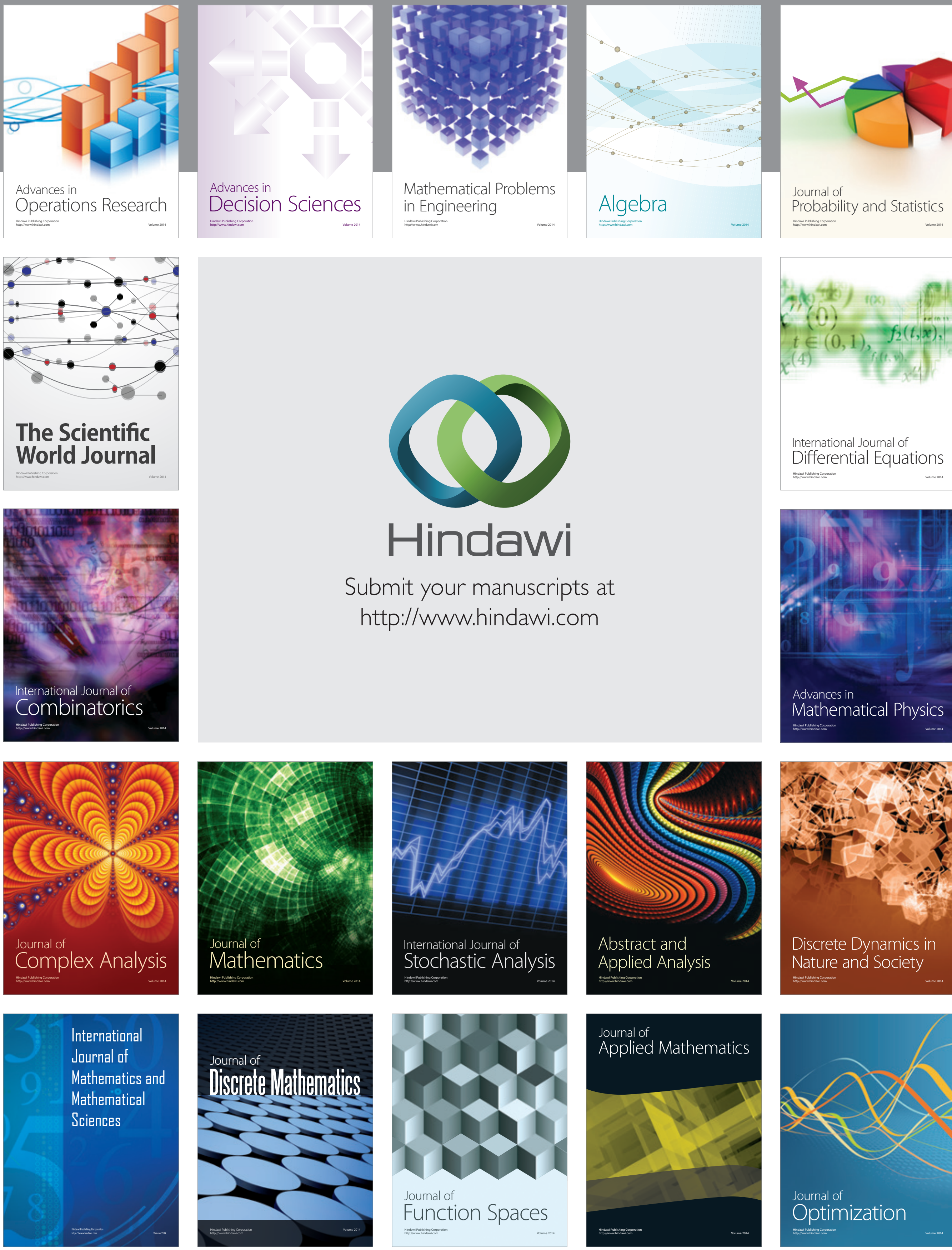
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