Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 125987, 16 pages
doi:10.1155/2012/125987

Research Article

Multiplicative Isometries on F-Algebras of Holomorphic Functions

Osamu Hatori,[1] Yasuo Iida,[2] Stevo Stević,[3] and Sei-Ichiro Ueki[4]

[1] *Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan*
[2] *Department of Mathematics, Iwate Medical University, Yahaba, Iwate 028-3694, Japan*
[3] *Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia*
[4] *Faculty of Engineering, Ibaraki University, Hitachi 316-8511, Japan*

Correspondence should be addressed to Stevo Stević, sstevic@ptt.rs

Received 12 July 2011; Accepted 11 October 2011

Academic Editor: Norio Yoshida

We study multiplicative isometries on the following F-algebras of holomorphic functions: Smirnov class $N_*(X)$, Privalov class $N^p(X)$, Bergman-Privalov class $AN^p_\alpha(X)$, and Zygmund F-algebra $N\log^\beta N(X)$, where X is the open unit ball $\mathbb{B}_n$ or the open unit polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$.

1. Introduction

Complex-linear isometries on function spaces of holomorphic functions have been studied for almost five decades by many mathematicians. In this paper we study multiplicative isometries on certain F-algebras of holomorphic functions. Recall that an F-algebra is a topological algebra in which the topology arises from a complete metric. For a positive integer n let $\mathbb{B}_n$ denote the open unit ball in the n-dimensional complex vector space $\mathbb{C}^n$ and $\mathbb{D}^n$ the unit polydisk in $\mathbb{C}^n$. We characterize multiplicative isometries on the Smirnov class, the Privalov class, the Bergman-Privalov class and the Zygmund F-algebras on $\mathbb{B}_n$ or $\mathbb{D}^n$. Surjective multiplicative maps on the Smirnov class, and the Bergman-Privalov class have already been correspondingly characterized in [1, 2].

2. Preliminaries

In studying surjective isometries in [1, 2] we applied the Mazur-Ulam theorem for surjective maps on certain subspaces, which themselves are Banach spaces, of the given F-algebras.

Generally we do not assume surjectivity of the isometries in this paper, so instead of the Mazur-Ulam theorem we use Lemma 2.1. Recall that a normed real-linear space L is *uniformly convex* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $\|a+b\| \le 2-\delta$ holds for every pair of $a, b \in L$ with $\|a\| \le 1$, $\|b\| \le 1$, and $\|a-b\| \ge \varepsilon$. It is well known that Hilbert spaces and L^p-spaces for $1 < p < \infty$ are uniformly convex.

Lemma 2.1. *Let L_1 and L_2 be normed real-linear spaces with L_2 uniformly convex. Let S be an isometry from L_1 into L_2 such that $S(0) = 0$. Then S is real-linear.*

The lemma might be well known, but we give a sketch of the proof for the completeness and the benefit of the reader.

Proof of Lemma 2.1. Let a, b be arbitrary elements of L_1. Put $2r = \|a-b\|$. Then since S is an isometry, $\|S(a)-S(b)\| = 2r$ and $\|S(a)-S((a+b)/2)\| = \|S(b)-S((a+b)/2)\| = r$. We also have $\|S(a)-(S(a)+S(b))/2\| = \|S(b)-(S(a)+S(b))/2\| = r$.

Suppose that $S((a+b)/2) \neq (S(a)+S(b))/2$. Set

$$\varepsilon = \left\| S\left(\frac{a+b}{2}\right) - \frac{S(a)+S(b)}{2} \right\|. \tag{2.1}$$

Since L_2 is uniformly convex and ε is positive there exists a $\delta > 0$ such that

$$\begin{aligned} \left\| \left(S(a) - S\left(\frac{a+b}{2}\right)\right) + \left(S(a) - \frac{S(a)+S(b)}{2}\right) \right\| &\le 2r-\delta, \\ \left\| \left(S(b) - S\left(\frac{a+b}{2}\right)\right) + \left(S(b) - \frac{S(a)+S(b)}{2}\right) \right\| &\le 2r-\delta. \end{aligned} \tag{2.2}$$

Then by the triangle inequality

$$\|2S(a) - 2S(b)\| \le 4r - 2\delta \tag{2.3}$$

holds, which contradicts to $\|S(a)-S(b)\| = 2r$. Thus we get $S((a+b)/2) = (S(a)+S(b))/2$, from which for $b = 0$ we obtain $S(a/2) = S(a)/2$. Substituting a by $a+b$ in the last equality we get

$$\frac{S(a+b)}{2} = S\left(\frac{a+b}{2}\right) = \frac{S(a)+S(b)}{2}, \tag{2.4}$$

so that $S(a+b) = S(a)+S(b)$. A routine argument yields $S(ta) = tS(a)$, $t \in \mathbb{R}$. □

For $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$, we denote by ∂X its distinguished boundary. For $X = \mathbb{B}_n$, this is the topological boundary $\partial\mathbb{B}_n$, and for the polydisk $\mathbb{D}^n$, it is the torus $\mathbb{T}^n$. Denote the normalized Lebesgue measure on ∂X by σ. A holomorphic map ψ is inner if $\lim_{r\to 1-0}\psi(rz)$ exists and lies in ∂X for almost all $z \in \partial X$ with respect to σ. We say that $\lim_{r\to 1-0}\psi(rz)$ is the boundary map of ψ and denote it by ψ^*. We say that ψ^* is measure preserving if $\sigma((\psi^*)^{-1}(E)) = \sigma(E)$ for every Borel set $E \subset \partial X$.

Now we recall definitions and some properties of the Smirnov class, the Privalov class, the Bergman-Privalov class, and the Zygmund F-algebra on $\mathbb{B}_n$ or $\mathbb{D}^n$. The space of all holomorphic functions on $X = \mathbb{B}_n$ or $\mathbb{D}^n$ is denoted by $H(X)$. For each $0 < p \le \infty$, the Hardy space is denoted by $H^p(X)$ with the norm $\|\cdot\|_p$.

2.1. Smirnov Class $N_*(X)$

Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. The Nevanlinna class $N(X)$ on X is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \le r < 1} \int_{\partial X} \ln\left(1 + \left|f(r\zeta)\right|\right) d\sigma(\zeta) < \infty \tag{2.5}$$

holds. It is known that every $f \in N(X)$ has a finite nontangential limit, denoted by f^*, almost everywhere on ∂X.

The Smirnov class $N_*(X)$ is defined as

$$N_*(X) = \left\{ f \in N(X) : \sup_{0 \le r < 1} \int_{\partial X} \ln\left(1 + \left|f(r\zeta)\right|\right) d\sigma(\zeta) = \int_{\partial X} \ln\left(1 + \left|f^*(\zeta)\right|\right) d\sigma(\zeta) \right\}. \tag{2.6}$$

Define a metric

$$d_{N_*(X)}(f, g) = \int_{\partial X} \ln\left(1 + \left|f^*(\zeta) - g^*(\zeta)\right|\right) d\sigma(\zeta) \tag{2.7}$$

for $f, g \in N_*(X)$. With the metric $d_{N_*(X)}(\cdot, \cdot)$ the Smirnov class $N_*(X)$ becomes an F-algebra and

$$\bigcup_{q > 0} H^q(X) \subset N_*(X), \tag{2.8}$$

in particular, $H^\infty(X)$ is a dense subalgebra of $N_*(X)$. The convergence in the metric is stronger than uniform convergence on compact subsets of X.

Complex-linear isometries on the Smirnov class were characterized by Stephenson in [3].

2.2. Privalov Class $N^p(X)$

Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as (for the original source see [4, 5])

$$N^p(X) = \left\{ f \in H(X) : \sup_{0 \le r < 1} \int_{\partial X} \left(\ln\left(1 + \left|f(r\zeta)\right|\right)\right)^p d\sigma(\zeta) < \infty \right\}. \tag{2.9}$$

It is well known that $N^p(X)$ is a subalgebra of $N_*(X)$, hence every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on ∂X. Define a metric

$$d_p(f,g) = \left(\int_{\partial X} \left(\ln\left(1 + \left| f^*(\zeta) - g^*(\zeta) \right| \right) \right)^p d\sigma(\zeta) \right)^{1/p} \tag{2.10}$$

for $f, g \in N^p(X)$. With this metric $N^p(X)$ is an F-algebra (cf. [6, 7]) and

$$\bigcup_{q>0} H^q(X) \subset N^p(X) \subset N_*(X). \tag{2.11}$$

The Hardy algebra $H^\infty(X)$ is dense in $N^p(X)$. The convergence on the metric is stronger than uniform convergence on compacts of X.

Complex-linear isometries on $N^p(X)$ are investigated by Iida and Mochizuki [8] for one-dimensional case, and by Subbotin [7] for a general case.

2.3. Bergman-Privalov Class $AN^p_\alpha(X)$

Let $1 \le p < \infty$ and $\alpha > -1$. The Bergman-Privalov class on the unit ball $\mathbb{B}_n$ and the polydisk $\mathbb{D}^n$ are defined, respectively, as

$$\begin{aligned} AN^p_\alpha(\mathbb{B}_n) &= \left\{ f \in H(\mathbb{B}_n) : \|f\|^p_{AN^p_\alpha(\mathbb{B}_n)} = \int_{\mathbb{B}_n} \left(\ln\left(1 + |f(z)|\right) \right)^p dV_{\alpha,n}(z) < \infty \right\}, \\ AN^p_\alpha(\mathbb{D}^n) &= \left\{ f \in H(\mathbb{D}^n) : \|f\|^p_{AN^p_\alpha(\mathbb{D}^n)} = \int_{\mathbb{D}^n} \left(\ln\left(1 + |f(z)|\right) \right)^p \prod_{j=1}^n dV_{\alpha,1}(z_j) < \infty \right\}, \end{aligned} \tag{2.12}$$

where $dV_{\alpha,n}(z) = c_{\alpha,n}(1 - |z|^2)^\alpha dV(z)$ for the normalized Lebesgue volume measure dV on $\mathbb{B}_n$ and $c_{\alpha,n}$ is a normalization constant, that is $V_{\alpha,n}(\mathbb{B}_n) = 1$. Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. In what follows $dV_\alpha(z)$ denotes $dV_{\alpha,n}(z)$ for $X = \mathbb{B}_n$ and $\prod_{j=1}^n dV_{\alpha,1}(z_j)$ for $X = \mathbb{D}^n$, respectively. The Bergman-Privalov class $AN^p_\alpha(X)$ is an F-algebra with respect to the metric

$$d_{AN^p_\alpha(X)}(f,g) = \|f - g\|_{AN^p_\alpha(X)} \tag{2.13}$$

for $f, g \in AN^p_\alpha(X)$. For some results in the case $p = 1$ see [9].

The weighted Bergman space for $q > 0$ and $\alpha > -1$ on the unit ball $\mathbb{B}_n$ and the polydisk $\mathbb{D}_n$ are defined, respectively, as

$$\begin{aligned} A^q_\alpha(\mathbb{B}_n) &= \left\{ f \in H(\mathbb{B}_n) : \|f\|^q_{A^q_\alpha(\mathbb{B}_n)} = \int_{\mathbb{B}_n} |f(z)|^q dV_{\alpha,n}(z) < \infty \right\}, \\ A^q_\alpha(\mathbb{D}^n) &= \left\{ f \in H(\mathbb{D}^n) : \|f\|^q_{A^q_\alpha(\mathbb{D}^n)} = \int_{\mathbb{D}^n} |f(z)|^q \prod_{j=1}^n dV_{\alpha,1}(z_j) < \infty \right\}. \end{aligned} \tag{2.14}$$

It is known that

$$\bigcup_{q>0} A^q_\alpha(X) \subset AN^p_\alpha(X). \tag{2.15}$$

Complex-linear isometries on the Bergman-Privalov class on the unit ball were characterized by Matsugu and Ueki in [10] and on the polydisk by Stević in [2].

2.4. Zygmund F-Algebra $N\log^\beta N(X)$

Let $\beta > 0$ and $\varphi_\beta(t) = t(\ln(\gamma_\beta + t))^\beta$, where $\gamma_\beta = \max\{e,\ e^\beta\}$. Let $X \in \{\mathbb{B}_n,\ \mathbb{D}^n\}$. The Zygmund F-algebra $N\log^\beta N(X)$ on X is defined as

$$N\log^\beta N(X) = \left\{ f \in H(X) : \sup_{0\le r<1} \int_{\partial X} \varphi_\beta(\ln(1 + |f(r\zeta)|)) d\sigma(\zeta) < \infty \right\}. \tag{2.16}$$

It is known that

$$N\log^\beta N(X) = \left\{ f \in H(X) : \sup_{0\le r<1} \int_{\partial X} \varphi_\beta(\ln^+|f(r\zeta)|) d\sigma(\zeta) < \infty \right\}, \tag{2.17}$$

$$\bigcup_{p>0} H^p(X) \subset N\log^\beta N(X) \subset N_*(X). \tag{2.18}$$

This implies that the finite nontangential limit f^* exists almost everywhere on ∂X, for any $f \in N\log^\beta N$. For $f, g \in N\log^\beta N$

$$d_{N\log^\beta N(X)}(f, g) = \int_{\partial X} \varphi_\beta(\ln(1 + |f^*(\zeta) - g^*(\zeta)|)) d\sigma(\zeta) \tag{2.19}$$

defines a complete metric on $N\log^\beta N(X)$ and $N\log^\beta N(X)$ is an F-algebra with this metric (cf. [11]).

Ueki [12] characterized the complex-linear isometries on the Zygmund F-algebra on the balls.

3. Main Results

In this section we formulate and prove the main results in this paper.

3.1. Multiplicative Isometries on $N_*(X)$

Our first result concerns the Smirnov class.

Theorem 3.1. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N_*(X) \to N_*(X)$ is a (not necessarily linear) multiplicative isometry. Then there is an inner map ψ on X whose boundary map ψ^* is measure preserving and such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N_*(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N_*(X). \end{aligned} \tag{3.1}$$

Proof. First we claim that $T(1) = 1$. Since $T(1) = T(1)^2$ and $T(1)$ is a holomorphic function on the connected open set X we get $T(1) = 0$ or $T(1) = 1$. But $T(1) = 0$ is impossible because if it were $T(1) = 0$, then $0 = T(f)T(1) = T(f)$, for each $f \in N_*(X)$, which contradicts with the assumption that T is an isometry. As $T(0) = T(0)^2$ and T is injective, we obtain $T(0) = 0$. Similarly $T(-1) = -1$ is also observed by making use of $T(-1)^2 = T(1) = 1$. Then $T(i)^2 = T(i^2) = -1$ assert that $T(i) = i$ or $T(i) = -i$. If $T(i) = i$, then the first formula of the conclusion will follow and the second one will follow from $T(i) = -i$.

Next we show $T(1/2) = 1/2$. Put $r = 1/2$. Suppose that $|T(r)^*| > r$ on a set of positive measure on ∂X. Then there exists a subset E of positive measure and $\varepsilon > 0$ with $|T(r)^*| \geq (1+\varepsilon)r$ on E. Since

$$\lim_{n\to\infty} \frac{\ln\left(1+(1+\varepsilon)^n r^n\right)}{\ln(1+r^n)} = \infty, \tag{3.2}$$

there is a positive integer n_0 such that

$$\int_E \ln\left(1+(1+\varepsilon)^{n_0} r^{n_0}\right) d\sigma > \int_{\partial X} \ln(1+r^{n_0})\, d\sigma. \tag{3.3}$$

From this and since T is a multiplicative isometry on $N_*(X)$ we have that

$$\begin{aligned} \int_{\partial X} \ln(1+r^{n_0})\, d\sigma &= \int_{\partial X} \ln\left(1+\left|T(r)^*\right|^{n_0}\right) d\sigma \\ &\geq \int_E \ln\left(1+(1+\varepsilon)^{n_0} r^{n_0}\right) d\sigma > \int_{\partial X} \ln(1+r^{n_0})\, d\sigma, \end{aligned} \tag{3.4}$$

which is a contradiction proving $|T(r)^*| \leq r$ almost everywhere on ∂X. Hence $|T(1/r)^*| \geq 1/r$ holds almost everywhere on ∂X as $T(r)T(1/r) = T(1) = 1$ almost everywhere on ∂X. Since

$$\ln\left(1+\frac{1}{r}\right) = \int_{\partial X} \ln\left(1+\frac{1}{r}\right) d\sigma = \int_{\partial X} \ln\left(1+\left|T\left(\frac{1}{r}\right)^*\right|\right) d\sigma, \tag{3.5}$$

we have that $|T(1/r)^*| = 1/r$ and $|T(r)^*| = r$ almost everywhere on ∂X.

Since $\ln(1+(1-r)) = d(r,1) = d(T(r),1)$ and

$$d(T(r),1) = \int_{\partial X} \ln\left(1+\left|1-T(r)^*\right|\right) d\sigma, \tag{3.6}$$

it is easy to check that $T(1/2)^* = 1/2$ almost everywhere on ∂X. Hence $T(1/2) = 1/2$ holds. As T is multiplicative, T is 1/2-homogeneous in the sense that $T(f/2) = T(f)/2$ holds for every $f \in N_*(X)$.

Let $f, g \in H^1(X)$. It requires only elementary calculation applying the 1/2-homogeneity of T to check that

$$\int_{\partial X} \ln\left(1 + \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right) d\sigma = \int_{\partial X} \ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right) d\sigma \tag{3.7}$$

holds. Multiplying (3.7) by 2^m and then letting $m \to \infty$ we get

$$\int_{\partial X} |f^* - g^*| d\sigma = \int_{\partial X} |T(f)^* - T(g)^*| d\sigma \tag{3.8}$$

by the monotone convergence theorem, since $2^m \ln(1 + (t/2^m))$ nondecreases monotonically to t as $m \to \infty$ for any $t \geq 0$, which can be easily proved by considering the function $g_t(x) = x \ln(1 + (t/x))$. From (3.8) for $g = 0$, we obtain $T(H^1(X)) \subseteq H^1(X)$ and the restricted map $T|_{H^1(X)}$ is an isometry with respect to the metric induced by the H^1-norm $\|\cdot\|_1$.

Let the function θ on the interval $[0, \infty)$ be defined as

$$\theta(x) = \begin{cases} \dfrac{1}{2}, & x = 0 \\ \dfrac{x - \ln(1 + x)}{x^2}, & x > 0. \end{cases} \tag{3.9}$$

It is easy to check that θ is positive and continuous on $[0, \infty)$ and $\lim_{x\to\infty}\theta(x) = 0$. Hence θ is bounded on $[0, \infty)$, so that

$$M_\theta := \sup_{x\geq 0} \theta(x) < \infty. \tag{3.10}$$

We claim that the inclusion $T(H^2(X)) \subseteq H^2(X)$ and $T|_{H^2(X)}$ is isometric with respect to the metric induced by the H^2-norm. For this purpose let $f, g \in H^2(X)$. Now note that since $H^2(X) \subset H^1(X)$, equality (3.7) holds and as well as the next equality

$$\int_{\partial X} \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right| d\sigma = \int_{\partial X} \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right| d\sigma. \tag{3.11}$$

By subtracting (3.7) from (3.11) and then multiplying such obtained equation by 2^m we obtain

$$\int_{\partial X} |f^* - g^*|^2 \theta\left(\left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right) d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^2 \theta\left(\left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right) d\sigma. \tag{3.12}$$

As θ is bounded the function $M_\theta |f^* - g^*|^2$ is an integrable function dominating the integrand in the left-hand side integral in (3.12). Letting $m \to \infty$ and applying the Lebesgue theorem on dominated convergence to the left-hand side and Fatou's lemma to the right-hand side (as θ is positive on $[0, \infty)$) we obtain

$$\int_{\partial X} |f^* - g^*|^2 \theta(0) d\sigma \geq \int_{\partial X} |T(f)^* - T(g)^*|^2 \theta(0) d\sigma. \tag{3.13}$$

From this and since $\theta(0) = 1/2$ we get that the function $|T(f)^* - T(g)^*|^2$ is integrable. Letting again $m \to \infty$ in (3.12) we have that

$$\int_{\partial X} |f^* - g^*|^2 d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^2 d\sigma \tag{3.14}$$

by the Lebesgue theorem on dominated convergence now applied to both integrals in (3.12). Hence $\|f - g\|_2 = \|T(f) - T(g)\|_2$ for every pair of $f, g \in H^2(X)$. For $g = 0$, we get $\|f\|_2 = \|T(f)\|_2$ and consequently $T(H^2(X)) \subseteq H^2(X)$, as claimed.

Since $H^2(X)$ is a Hilbert space, it is uniformly convex. Hence by Lemma 2.1 the restriction $T|_{H^2(X)}$ is real-linear. Since the operations of scalar multiplication and addition on N_* are continuous and $H^2(X)$ is dense in $N_*(X)$ we see that T is real-linear on $N_*(X)$.

First assume $T(i) = i$. As T is real-linear and multiplicative, T is complex-linear in this case. Then by [3, Theorem 2.2] and since $T(1) = 1$, there is an inner map ψ such that $T(f) = f \circ \psi$ for every $f \in N_*(X)$.

Now assume $T(i) = -i$. Let $\widetilde{T} : N_*(X) \to N_*(X)$ be defined as $\widetilde{T}(f) = T(\widetilde{f})$ for every $f \in N_*(X)$, where

$$\widetilde{f}(z_1, \ldots, z_n) = \overline{f(\overline{z}_1, \ldots, \overline{z}_n)} \tag{3.15}$$

for $f \in N_*(X)$. Then $\widetilde{T}$ is well defined and a complex-linear isometry from $N_*(X)$ into itself. Again by [3, Theorem 2.2] we have that there is an inner map ψ on X whose boundary map ψ^* is measure preserving such that $\widetilde{T}(f) = f \circ \psi$ for every $f \in N_*$. This implies that $T(f) = \overline{f \circ \overline{\psi}}$ for every $f \in N_*(X)$. □

Corollary 3.2 (see [1]). *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N_*(X) \to N_*(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism ψ on X such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N_*(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N_*(X), \end{aligned} \tag{3.16}$$

where ψ is a unitary transformation for $X = \mathbb{B}_n$, while for $X = \mathbb{D}^n$, $\psi(z_1, \ldots, z_n) = (e^{i\theta_1} z_{j_1}, \ldots, e^{i\theta_n} z_{j_n})$ for some real numbers θ_j for $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.

Proof. By Theorem 3.1, T is complex-linear or conjugate linear. If T is complex-linear, then the result holds by [3, Corollary 2.3]. If T is conjugate linear, then put $\widetilde{T}(f) = T(\widetilde{f})$ for $f \in N_*(X)$, where $\widetilde{f}$ is defined as in (3.15). Then $\widetilde{T}(f) = f \circ \psi$, for every $f \in N_*(X)$, and for an inner map ψ on X whose boundary map ψ^* is measure preserving. Since $\widetilde{T}$ is a surjective isometry, the desired property of ψ again follows from [3, Corollary 2.3]. □

3.2. Multiplicative Isometries on $N^p(X)$

The next result concerns the Privalov class.

Theorem 3.3. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$ and $1 < p < \infty$. Suppose that $T : N^p(X) \to N^p(X)$ is a (not necessarily linear) multiplicative isometry. Then there is an inner map ψ on X whose boundary map ψ^* is measure preserving and such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N^p(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N^p(X). \end{aligned} \tag{3.17}$$

Proof. Since T is multiplicative we see by the same way as in the proof of Theorem 3.1 that $T(0) = 0$, $T(1) = 1$ and $T(i) = i$ or $T(i) = -i$. Also we see that $T(1/2) = 1/2$. It follows by the proof of Theorem 3.1 that for every pair f and g in $H^p(X)$,

$$\int_{\partial X} \left(\ln\left(1 + \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right)\right)^p d\sigma = \int_{\partial X} \left(\ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right)\right)^p d\sigma \tag{3.18}$$

holds. Multiplying (3.18) by 2^{mp} and then letting $m \to \infty$ we get

$$\int_{\partial X} |f^* - g^*|^p d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^p d\sigma. \tag{3.19}$$

Thus $T(H^p(X)) \subseteq H^p(X)$. The Hardy space $H^p(X)$ can be seen as a subspace of $L^p(\partial X)$. Since $L^p(\partial X)$ is uniformly convex, so is $H^p(X)$ for $1 < p < \infty$. Then by Lemma 2.1 the operator T is real-linear on $H^p(X)$. Since $H^p(X)$ is a dense subspace of $N^p(X)$ we see that T is real-linear on $N^p(X)$. As we have already learnt that $T(i) = i$ or $T(i) = -i$, we obtain that T is complex-linear or conjugate linear on $N^p(X)$. The rest of the proof is similar to the last part of the proof of Theorem 3.1 applying [7, Theorem 1] instead of [3, Theorem 2.2]. We omit the details. □

Corollary 3.4. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$ and $1 < p < \infty$. Suppose that $T : N^p(X) \to N^p(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism ψ on X such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N^p(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N^p(X), \end{aligned} \tag{3.20}$$

where ψ is a unitary transformation for $X = \mathbb{B}_n$, while for $X = \mathbb{D}^n$, $\psi(z_1, \ldots, z_n) = (e^{i\theta_1} z_{j_1}, \ldots, e^{i\theta_n} z_{j_n})$ for some real numbers θ_j for $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.

Proof. By Theorem 3.3, T is complex-linear or conjugate linear. If T is complex-linear, then the result follows directly from [7, Corollary and Remark 3]. If T is conjugate linear, then put $\widetilde{T}(f) = T(\widetilde{f})$ for $f \in N^p(X)$, where $\widetilde{f}$ is defined as in (3.15). Then $\widetilde{T}$ is a complex-linear isometric surjection from $N^p(X)$ onto itself. Hence by [7, Corollary and Remark 3] there is a desired automorphism on X such that $\mathrm{T}(f) = \overline{f \circ \overline{\varphi}}$ for every $f \in N^p(X)$. □

3.3. Multiplicative Isometries on $AN^p_\alpha(X)$

The next result concerns the Bergman-Privalov class.

Theorem 3.5. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$, $1 \le p < \infty$ and $\alpha > -1$. Suppose that $T : AN^p_\alpha(X) \to AN^p_\alpha(X)$ is a (not necessarily linear) multiplicative isometry. Then there is a holomorphic self-map φ on X with the property that*

$$\int_X h \circ \varphi(z) dV_\alpha(z) = \int_X h(z) dV_\alpha(z) \tag{3.21}$$

for every bounded or positive Borel function h on X such that either of the following formulas holds:

$$\begin{aligned} T(f) &= f \circ \varphi \quad \text{for every } f \in AN^p_\alpha(X), \\ T(f) &= \overline{f \circ \overline{\varphi}} \quad \text{for every } f \in AN^p_\alpha(X). \end{aligned} \tag{3.22}$$

Proof. We can prove the theorem in a way similar to that in the proofs of Theorem 3.1 for $p = 1$ and Theorem 3.3 for $1 < p < \infty$. For the case of $p = 1$, instead of using the Hardy spaces $H^1(X)$ and $H^2(X)$ we make use of the weighted Bergman spaces $A^1_\alpha(X)$ and $A^2_\alpha(X)$. For the case of $1 < p < \infty$, instead of using the Hardy space $H^p(X)$ we make use of the weighted Bergman space $A^p_\alpha(X)$. We also apply [10, Theorem 1] for $X = \mathbb{B}_n$ and [2, Theorem 2] for $X = \mathbb{D}^n$ to represent complex-linear isometries instead of [3, Theorem 2.2]. □

Corollary 3.6 (see [2]). *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$, $1 \le p < \infty$ and $\alpha > -1$. Suppose that $T : AN^p_\alpha(X) \to AN^p_\alpha(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism φ on X such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \varphi \quad \text{for every } f \in AN^p_\alpha(X), \\ T(f) &= \overline{f \circ \overline{\varphi}} \quad \text{for every } f \in AN^p_\alpha(X), \end{aligned} \tag{3.23}$$

where φ is a unitary transformation for $X = \mathbb{B}_n$, while for $X = \mathbb{D}^n$, $\varphi(z_1, \dots, z_n) = (e^{i\theta_1} z_{j_1}, \dots, e^{i\theta_n} z_{j_n})$ for some real numbers θ_j for $j = 1, \dots, n$ and a permutation $(j_1, \dots, j_n)$ of the integers from 1 to n.

Proof. By Theorem 3.5, T is complex-linear or conjugate linear. Suppose that T is complex-linear. If $X = \mathbb{B}_n$, then the conclusion follows by [10, Theorem 2], while for $X = \mathbb{D}^n$ the conclusion follows similar to the corresponding part of the proof of [2, Theorem 3]. If T is conjugate linear, then the conclusion follows from the similar argument in the proof of Corollary 3.2. □

3.4. Isometries on $N\log^{\beta}N(X)$

In [12] Ueki characterized complex-linear isometries on the Zygmund F-algebra on $\mathbb{B}_n$. For $\mathbb{D}^n$ the following result is proved similar to [12, Theorem 1]. Hence it is omitted.

Theorem 3.7. *Let $\beta > 0$. If T is a complex-linear isometry of $N\log^{\beta}N(\mathbb{D}^n)$ into itself, then there exist an inner function Ψ and an inner map ψ on $\mathbb{D}^n$ whose boundary map ψ^* is measure preserving on $\mathbb{T}^n$ such that*

$$T(f) = \Psi C_{\psi}(f) = \Psi(f \circ \psi) \quad \text{for every } f \in N\log^{\beta}N(\mathbb{D}^n). \tag{3.24}$$

Conversely, for given such Ψ and ψ, the weighted composition operator ΨC_{ψ} is an injective linear isometry of $N\log^{\beta}N(\mathbb{D}^n)$.

For the surjective isometries the result is as follows.

Corollary 3.8. *An isometry T of $N\log^{\beta}N(\mathbb{D}^n)$ is surjective if and only if $T = aC_{\mathcal{U}}$ where $a \in \mathbb{C}$ with $|a| = 1$ and $\mathcal{U}(z_1, \ldots, z_n) = (e^{i\theta_1}z_{j_1}, \ldots, e^{i\theta_n}z_{j_n})$ for some real numbers θ_j, $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.*

To prove Corollary 3.8 we need the next auxiliary result.

Lemma 3.9. *For any function $f \in N(\mathbb{D}^n)$, $f \in N\log^{\beta}N(\mathbb{D}^n)$ if and only if $\varphi_{\beta}(\ln^+|f^*|) \in L^1(\mathbb{T}^n)$ and*

$$\varphi_{\beta}(\ln^+|f(z)|) \le \int_{\mathbb{T}^n} P(z,\zeta)\varphi_{\beta}(\ln^+|f^*(\zeta)|)d\sigma(\zeta) \quad \text{for } z \in \mathbb{D}^n, \tag{3.25}$$

where $P(z,\zeta)$ denotes the Poisson kernel for $\mathbb{D}^n$;

$$P(z,\zeta) = P_{r_1}(\theta_1 - \phi_1)\cdots P_{r_n}(\theta_n - \phi_n) \tag{3.26}$$

for $z = (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})$, $\zeta = (e^{i\phi_1}, \ldots, e^{i\phi_n})$ and

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \tag{3.27}$$

is the Poisson kernel for the unit disk $\mathbb{D}$.

Proof. If $f \in N\log^{\beta}N(\mathbb{D}^n)$, then Fatou's lemma shows that $\varphi_{\beta}(\ln^+|f^*|) \in L^1(\mathbb{T}^n)$. The inclusion (2.18) implies $f \in N_*(\mathbb{D}^n)$, and so we see that $\ln^+|f|$ has the least n-harmonic majorant. Since the least n-harmonic majorant of $\ln^+|f|$ is the Poisson integral $P[\ln^+|f^*|]$, we obtain the following inequality:

$$\ln^+|f(z)| \le \int_{\mathbb{T}^n} P(z,\zeta)\ln^+|f^*(\zeta)|d\sigma(\zeta) \quad \text{for } z \in \mathbb{D}^n. \tag{3.28}$$

Note that $\varphi_\beta(t)$ is strictly increasing and convex on $[0,\infty)$, and the measures $d\mu_z(\zeta) = P(z,\zeta)d\sigma(\zeta)$ are normalized on $\mathbb{T}^n$, which follows from the well-known equality

$$\int_{\mathbb{T}^n} P(z,\zeta)d\sigma(\zeta) = 1. \tag{3.29}$$

Applying Jensen's inequality to (3.28), we obtain the desired inequality (3.25).

Conversely we put $z = r\eta(0 \le r < 1, \eta \in \mathbb{T}^n)$ in (3.25). By integrating with respect to η and applying Fubini's theorem, we have that

$$\int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f(r\eta)|)d\sigma(\eta) \le \int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f^*(\zeta)|)d\sigma(\zeta)\int_{\mathbb{T}^n} P(r\eta,\zeta)d\sigma(\eta). \tag{3.30}$$

By the symmetric property $P(r\eta,\zeta) = P(r\zeta,\eta)$ and the normalization property of the Poisson kernel, we obtain that

$$\sup_{0\le r<1}\int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f(r\eta)|)d\sigma(\eta) \le \int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f^*(\zeta)|)d\sigma(\zeta). \tag{3.31}$$

Hence the condition $\varphi_\beta(\ln^+|f^*|) \in L^1(\mathbb{T}^n)$ implies that $f \in N\log^\beta N(\mathbb{D}^n)$. □

Now we give a proof of Corollary 3.8.

Proof of Corollary 3.8. Suppose that T is surjective. Then Theorem 3.7 gives that $T = \Psi C_\psi$. A standard argument shows that ψ is an automorphism of $\mathbb{D}^n$. So there are conformal maps φ_j $(j = 1,\dots,n)$ of $\mathbb{D}$ onto $\mathbb{D}$ and there is a permutation $(j_1,\dots,j_n)$ of the integers from 1 to n such that

$$\psi(z_1,\dots,z_n) = (\varphi_1(z_{j_1}),\dots,\varphi_n(z_{j_n})). \tag{3.32}$$

The mean value theorem shows that

$$\int_{\mathbb{T}^n} \varphi_k(\zeta_{j_k})d\sigma(\zeta) = \int_{\mathbb{T}} \varphi_k(\zeta_{j_k})d\sigma_1(\zeta_{j_k}) = \varphi_k(0) \tag{3.33}$$

for each $k \in \{1,\dots,n\}$. Here $d\sigma_1$ denotes the one-dimensional normalized Lebesgue measure on the unit circle $\mathbb{T}$.

On the other hand, the measure-preserving property of ψ^* gives that

$$\int_{\mathbb{T}^n} \varphi_k(\zeta_{j_k})d\sigma(\zeta) = \int_{\mathbb{T}^n} \langle\psi^*(\zeta), e_k\rangle d\sigma(\zeta) = \int_{\mathbb{T}^n} \langle\zeta, e_k\rangle d\sigma(\zeta) = \int_{\mathbb{T}^n} \zeta_k d\sigma(\zeta) = 0. \tag{3.34}$$

By (3.33) and (3.34) we see that ψ fixes the origin, and so each φ_k is the rotation transform.

Next we prove that Ψ is a unimodular constant. If $f \in N\log^{\beta} N(\mathbb{D}^n)$ is such that $1 = T(f) = \Psi C_{\varphi}(f)$, then $1/\Psi = f \circ \varphi \in N\log^{\beta} N(\mathbb{D}^n)$. Inequality (3.25) in Lemma 3.9 gives that

$$\varphi_{\beta}\left(\ln^{+}\frac{1}{|\Psi(z)|}\right) \leq \int_{\mathbb{T}^n} P(z,\zeta)\varphi_{\beta}\left(\ln^{+}\frac{1}{|\Psi^{*}(\zeta)|}\right)d\sigma(\zeta) = 0, \tag{3.35}$$

and so we have $1/|\Psi| \leq 1$ on $\mathbb{D}^n$. Since Ψ is inner, Ψ is a unimodular constant. □

Now we show results on multiplicative isometries on the Zygmund F-algebras on $\mathbb{B}_n$ and $\mathbb{D}^n$.

Theorem 3.10. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N\log^{\beta} N(X) \to N\log^{\beta} N(X)$ is a (not necessarily linear) multiplicative isometry. Then there exists an inner map φ on X whose boundary map φ^{*} is measure preserving on ∂X, such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \varphi \quad \text{for every } f \in N\log^{\beta} N(X), \\ T(f) &= \overline{f \circ \overline{\varphi}} \quad \text{for every } f \in N\log^{\beta} N(X). \end{aligned} \tag{3.36}$$

Note that multiplicative isometries of the Privalov class and the Zygmund F-algebra have the same form as multiplicative isometries of the Smirnov class.

Proof of Theorem 3.10. As T is multiplicative we obtain $T(1) = 1$, $T(0) = 0$, $T(-1) = -1$ and $T(i) = i$ or $T(i) = -i$. Since

$$\lim_{n\to\infty} \frac{\varphi_{\beta}(((1+\varepsilon)/2)^n)}{\varphi_{\beta}((1/2^n))} = \infty \tag{3.37}$$

holds for every $\varepsilon > 0$, the equation $T(1/2) = 1/2$ is proved similarly as in Theorem 3.1.

Let $f, g \in H^1(X)$. Then we can prove that

$$\int_{\partial X} 2^m \varphi_{\beta}\left(\ln\left(1 + \left|\frac{f^{*}}{2^m} - \frac{g^{*}}{2^m}\right|\right)\right)d\sigma = \int_{\partial X} 2^m \varphi_{\beta}\left(\ln\left(1 + \left|\frac{T(f)^{*}}{2^m} - \frac{T(g)^{*}}{2^m}\right|\right)\right)d\sigma, \tag{3.38}$$

following the lines of the corresponding part of the proof in Theorem 3.1. By some calculation we see that

$$\varphi_{\beta}(\ln(1+x)) \leq (\ln \gamma_{\beta})^{\beta} x \tag{3.39}$$

holds for every $x \geq 0$. Hence we get

$$2^m \varphi_{\beta}\left(\ln\left(1 + \left|\frac{f^{*}}{2^m} - \frac{g^{*}}{2^m}\right|\right)\right) \leq (\ln \gamma_{\beta})^{\beta} |f^{*} - g^{*}|, \tag{3.40}$$

almost everywhere on ∂X and $(\ln \gamma_\beta)^\beta |f^* - g^*|$ is an integrable function dominating $2^m \varphi_\beta(\ln(1 + |(f^*/2^m) - (g^*/2^m)|))$. We get

$$\lim_{m\to\infty} \int_{\partial X} 2^m \varphi_\beta\left(\ln\left(1 + \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right)\right) d\sigma = (\ln \gamma_\beta)^\beta \int_{\partial X} |f^* - g^*| d\sigma \tag{3.41}$$

by the Lebesgue dominated convergence theorem since

$$\lim_{m\to\infty} 2^m \varphi_\beta\left(\ln\left(1 + \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right)\right) = (\ln \gamma_\beta)^\beta |f^* - g^*|. \tag{3.42}$$

On the other hand, applying Fatou's lemma we get

$$\begin{aligned}
&(\ln \gamma_\beta)^\beta \int_{\partial X} \left|T(f)^* - T(g)^*\right| d\sigma \\
&\quad \le \liminf_{m\to\infty} \int_{\partial X} 2^m \varphi_\beta\left(\ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right)\right) d\sigma \\
&\quad = \liminf_{m\to\infty} \int_{\partial X} 2^m \varphi_\beta\left(\ln\left(1 + \left|\frac{f^*}{2^m} - \frac{g^*}{2^m}\right|\right)\right) d\sigma \\
&\quad = (\ln \gamma_\beta)^\beta \int_{\partial X} |f^* - g^*| d\sigma < \infty,
\end{aligned} \tag{3.43}$$

from which for $g = 0$ we get $T(H^1(X)) \subseteq H^1(X)$. Since

$$2^m \varphi_\beta\left(\ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right)\right) \le (\ln \gamma_\beta)^\beta \left|T(f)^* - T(g)^*\right| \tag{3.44}$$

follows from (3.40), the function $(\ln \gamma_\beta)^\beta |T(f)^* - T(g)^*|$ is an integrable function dominating $2^m \varphi_\beta\left(\ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right)\right)$. Hence

$$(\ln \gamma_\beta)^\beta \int_{\partial X} \left|T(f)^* - T(g)^*\right| d\sigma = \lim_{m\to\infty} \int_{\partial X} 2^m \varphi_\beta\left(\ln\left(1 + \left|\frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m}\right|\right)\right) d\sigma \tag{3.45}$$

holds by the Lebesgue dominated convergence theorem. Consequently

$$\int_{\partial X} |f^* - g^*| d\sigma = \int_{\partial X} \left|T(f)^* - T(g)^*\right| d\sigma \tag{3.46}$$

holds. As f and g are arbitrary elements of $H^1(X)$ we obtain that $T|_{H^1(X)}$ is isometric on $H^1(X)$ with respect to the metric induced by the H^1-norm.

We also obtain that there exists a bounded positive continuous function θ_1 on $[0, \infty)$ such that $\theta_1(0) \neq 0$ and

$$x^2\theta_1(x) = \{\ln \gamma_\beta\}^\beta x - \varphi_\beta(\ln(1+x)). \tag{3.47}$$

Applying this equality we obtain that $T(H^2(X)) \subseteq H^2(X)$ and $T|_{H^2(X)}$ is a real-linear isometry on $H^2(X)$, hence T is a complex-linear (if $T(i) = i$) or conjugate linear isometry (if $T(i) = -i$) on $N\log^\beta N(X)$, similar as in the proof of Theorem 3.1. The rest of the proof is similar to the last part of the proof of Theorem 3.1 applying [12, Theorem 1] for $X = \mathbb{B}_n$ and Theorem 3.7 for $X = \mathbb{D}^n$ instead of [3, Theorem 2.2]. We omit the details. □

Corollary 3.11. *Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N\log^\beta N(X) \to N\log^\beta N(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there exists a holomorphic automorphism ψ on X such that either of the following formulas holds:*

$$\begin{aligned} T(f) &= f \circ \psi \quad \text{for every } f \in N\log^\beta N(X), \\ T(f) &= \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N\log^\beta N(X), \end{aligned} \tag{3.48}$$

where ψ is a unitary transformation for $X = \mathbb{B}_n$, while for $X = \mathbb{D}^n$, $\psi(z_1, \ldots, z_n) = (e^{i\theta_1} z_{j_1}, \ldots, e^{i\theta_n} z_{j_n})$ for some real numbers θ_j, $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to n.

Note that surjective multiplicative isometries of the Privalov class, the Bergman-Privalov class, and the Zygmund F-algebra have the same form as surjective multiplicative isometries of the Smirnov class.

Proof of Corollary 3.11. By Theorem 3.10, T is complex-linear or conjugate linear. Suppose that T is complex-linear. Applying [12, Corollary 1] for $X = \mathbb{B}_n$ and Corollary 3.8 for $X = \mathbb{D}^n$ the result follows in this case. If T is conjugate linear, then the result follows by similar arguments as in the proof of Corollary 3.2. □

Acknowledgments

The first and fourth authors are partly supported by the Grants-in-Aid for Scientific Research, Japan Society for the Promotion of Science. The second author is partly supported by the Grant from Keiryokai Research Foundation no. 97. The third author is partially supported by the Serbian Ministry of Science (Projects III41025 and III44006).

References

[1] O. Hatori and Y. Iida, "Multiplicative isometries on the Smirnov class," *Central European Journal of Mathematics*, vol. 9, no. 5, pp. 1051–1056, 2011.

[2] S. Stević, "On some isometries on the Bergman-Privalov class on the unit ball," *Nonlinear Analysis*, vol. 75, pp. 2448–2454, 2012.

[3] K. Stephenson, "Isometries of the Nevanlinna class," *Indiana University Mathematics Journal*, vol. 26, no. 2, pp. 307–324, 1977.

[4] I. I. Privalov, *Boundary Properties of Analytic Functions*, Moscow University Press, Moscow, Russia, 1950.

[5] I. I. Privalov, *Randeigenshaften Analytischer Funktionen*, Deutscher Verlag der Wiss., Berlin, Germany, 1956.

[6] A. V. Subbotin, "Functional properties of Privalov spaces of holomorphic functions of several vari%ables," *Matematicheskie Zametki*, vol. 65, no. 2, pp. 280–288, 1999.

[7] A. V. Subbotin, "Isometries of Privalov spaces of holomorphic functions of several variables," *Journal of Mathematical Sciences*, vol. 135, no. 1, pp. 2794–2802, 2006.

[8] Y. Iida and N. Mochizuki, "Isometries of some F-algebras of holomorphic functions," *Archiv der Mathematik*, vol. 71, no. 4, pp. 297–300, 1998.

[9] M. Stoll, "Mean growth and Taylor coefficients of some topological algebras of analytic functions," *Annales Polonici Mathematici*, vol. 35, no. 2, pp. 139–158, 1978.

[10] Y. Matsugu and S. Ueki, "Isometries of weighted Bergman-Privalov spaces on the unit ball of $\mathbb{C}^n$," *Journal of the Mathematical Society of Japan*, vol. 54, no. 2, pp. 341–347, 2002.

[11] O. M. Eminyan, "Zygmund F-algebras of holomorphic functions in the ball and in the polydisk," *Doklady Mathematics*, vol. 65, no. 3, pp. 353–355, 2002.

[12] S. Ueki, "Isometries of the Zygmund F-algebra," *Proceedings of the American Mathematical Society*. In press.