Research Article

Numerical Solutions of Stochastic Differential Delay Equations with Poisson Random Measure under the Generalized Khasminskii-Type Conditions

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The Euler method is introduced for stochastic differential delay equations (SDDEs) with Poisson random measure under the generalized Khasminskii-type conditions which cover more classes of such equations than before. The main aims of this paper are to prove the existence of global solutions to such equations and then to investigate the convergence of the Euler method in probability under the generalized Khasminskii-type conditions. Numerical example is given to indicate our results.

1. Introduction

To take into consideration stochastic effects such as corporate defaults, operational failures, market crashes or central bank announcements in financial market, the research on stochastic differential equations (SDEs) with Poisson random measure (see [1, 2]) is important, since Merton initiated the model of such equations in 1976 (see [3]). Due to the rate of change of financial dynamics system depending on its past history, SDDE with Poisson random measure (see [4, 5]), the case we propose and consider in this work, is meaningful.

Since there is no explicit solution for an SDDE with Poisson random measure, one needs, in general, numerical methods which can be classified into strong approximations and weak approximations (see [6–8]).

We here give an overview of the results on the strong approximations of differential equation driven by Wiener process and Poisson random measure. Platen [9] presented a convergence theorem with order $\gamma \in \{0.5, 1, 1.5, \ldots\}$ and originally introduced the jump-adapted methods which are based on all the jump times. Moreover, Bruti-Liberati and Platen

(see [10]) get the jump-adapted order 1.5 scheme, and they also construct the derivative free or implicit jump-adapted schemes with desired order of strong convergence. In [11], for a class of pure jump systems, the order of Taylor schemes is given under weaker conditions than the current literature. In [7, 10], Bruti-Liberati and Platen present the drift-implicit schemes which have order $\gamma \in \{0.5, 1\}$. Recently, [8] develops adaptive time stepping algorithms based on a jump augmented Monte Carlo Euler-Maruyama method, which achieve a prescribed precision. Mao [4] presents the convergence of numerical solutions for variable delay differential equations with Poisson random measure. In [12], the improved Runge-Kutta methods are presented to improve the accuracy behaviour of problems with small noise for SDEs driven by Poisson random measure. Clearly, the results above require that the SDEs with poisson random measure satisfy the global Lipschitz conditions and the linear growth conditions. In [5], the Euler scheme is proved to converge to the analytic solution for SDDEs with Wiener process and Poisson random measure under weaker conditions than the global Lipschitz condition and the linear growth condition.

However, there are many SDDEs with Poisson random measure, especially highly nonlinear equations, which do not satisfy the above-mentioned conditions and classical Khasminskii-type conditions (see [13–15]). And in Section 5, we give such highly nonlinear equation. Our work is motivated by [16] in which the generalized Khasminskii-type conditions are applied to SDDEs with Wiener process. The main contribution in our paper is to present Euler method for SDDEs with Poisson random measure under the generalized Khasminskii-type conditions which cover more classes of these equations than all the mentioned classical conditions above.

Our work is organized as follows. In Section 2, the properties of SDDEs with Poisson random measure are given under the generalized Khasminskii-type conditions. In Section 3, Euler method is analyzed under such conditions. In Section 4, we present the convergence in probability of the Euler method. In Section 5, an example is given.

2. The Generalized Khasminskii-Type Conditions for SDDEs with Poisson Random Measure

2.1. Problem's Setting

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbf{R}^d$, $d \in \mathbf{N}$. Let $u_1 \vee u_2 = \max\{u_1, u_2\}$ and $u_1 \wedge u_2 = \min\{u_1, u_2\}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\tau > 0$ and $\mathbf{R}_+ = [0, \infty)$. Let $C([-\tau, 0]; \mathbf{R}^d)$ denote the family of continuous functions from $[-\tau, 0]$ to $\mathbf{R}^d$ with the norm $|\varphi| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$. Denote by $C(\mathbf{R}^d; \mathbf{R}_+)$ the family of continuous functions from $\mathbf{R}^d$ to $\mathbf{R}_+$. Let $C^2(\mathbf{R}^d; \mathbf{R}_+)$ denote the family of continuously two times differentiable $\mathbf{R}_+$-valued functions from $\mathbf{R}^d$ to $\mathbf{R}_+$. $[z]$ denotes the largest integer which is less than or equal to z in $\mathbf{R}$. $I_{\mathcal{A}}$ denotes the indicator function of a set $\mathcal{A}$.

The following d-dimensional SDDE with Poisson random measure is considered in our paper:

$$
\begin{aligned}
dx(t) = {}& a\big(x(t^-), x((t-\tau)^-)\big)dt + b\big(x(t^-), x((t-\tau)^-)\big)dW(t) \\
& + \int_\varepsilon c\big(x(t^-), x((t-\tau)^-), v\big)\tilde{p}_\phi(dv \times dt),
\end{aligned}
\tag{2.1}
$$

for $t > 0$, where $\tilde{p}_\phi(dv \times dt) := p_\phi(dv \times dt) - \phi(dv)dt$. Here $x(t^-)$ denotes $\lim_{s \uparrow t} x(s)$. The initial data of (2.1) is given by

$$\{x(t) : -\tau \le t \le 0\} = \xi(t) \in C\left([-\tau, 0]; \mathbf{R}^d\right), \tag{2.2}$$

where $x(-\tau^-) = x(-\tau)$.

The drift coefficient $a : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$, the diffusion coefficient $b : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^{d \times m_0}$, and the jump coefficient $c : \mathbf{R}^d \times \mathbf{R}^d \times \varepsilon \to \mathbf{R}^d$ are assumed to be Borel measurable functions and the coefficients are sufficiently smooth.

The randomness in (2.1) is generated by the following (see [8]). An m_0-dimensional Wiener process $W = \{W(t) = (W^1(t), \ldots, W^{m_0}(t))^T\}$ with independent scalar components is defined on a filtered probability space $(\Omega^W, \mathcal{F}^W, (\mathcal{F}_t^W)_{t \ge 0}, \mathbf{P}^W)$. A Poisson random measure $p_\phi(\omega, dv \times dt)$ is on $\Omega^J \times \varepsilon \times [0, \infty)$, where $\varepsilon \subseteq \mathbf{R}^r \setminus \{0\}$ with $r \in \mathbf{N}$, and its deterministic compensated measure $\phi(dv)dt = \lambda f(v)dvdt$. $f(v)$ is a probability density, and we require finite intensity $\lambda = \phi(\varepsilon) < \infty$. The Poisson random measure is defined on a filtered probability space $(\Omega^J, \mathcal{F}^J, (\mathcal{F}_t^J)_{t \ge 0}, \mathbf{P}^J)$. The process $x(t)$ is thus defined on a product space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbf{P})$, where $\Omega = \Omega^W \times \Omega^J$, $\mathcal{F} = \mathcal{F}^W \times \mathcal{F}^J, (\mathcal{F}_t)_{t \ge 0} = (\mathcal{F}_t^W)_{t \ge 0} \times (\mathcal{F}_t^J)_{t \ge 0}$, $\mathbf{P} = \mathbf{P}^W \times \mathbf{P}^J$, and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets. The Wiener process and the Poisson random measure are mutually independent.

To state the generalized Khasminskii-type conditions, we define the operator $LV : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ by

$$
\begin{aligned}
LV(x, y) = {} & V_x(x)a(x, y) + \frac{1}{2}\mathrm{trace}\left(b^{\mathrm{T}}(x, y)V_{xx}(x)b(x, y)\right) \\
& + \int_\varepsilon \left(V(x + c(x, y, v)) - V(x) - V_x(x)c(x, y, v)\right)\phi(dv),
\end{aligned}
\tag{2.3}
$$

where

$$V \in C^2\left(\mathbf{R}^d; \mathbf{R}_+\right), \qquad V_x = \left(\frac{\partial V(x)}{\partial x_1}, \ldots, \frac{\partial V(x)}{\partial x_d}\right), \qquad V_{xx} = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\right)_{d \times d}. \tag{2.4}$$

Now the generalized Khasminskii-type conditions are given by the following assumptions.

Assumption 2.1. For each integer $k \ge 1$, there exists a positive constant C_k, dependent on k, such that

$$\left|a(x, y) - a(\overline{x}, \overline{y})\right|^2 \vee \left|b(x, y) - b(\overline{x}, \overline{y})\right|^2 \le C_k\left(|x - y|^2 + |\overline{x} - \overline{y}|^2\right), \tag{2.5}$$

for $x, y \in \mathbf{R}^d$ with $|x| \vee |y| \le k$. And there exists a positive constant C such that

$$\int_\varepsilon \left|c(x, y, v) - c(\overline{x}, \overline{y}, v)\right|^2 \phi(dv) \le C\left(|x - y|^2 + |\overline{x} - \overline{y}|^2\right), \tag{2.6}$$

for $x, y \in \mathbf{R}^d$.

Assumption 2.2. There are two functions $V \in C^2(\mathbf{R}^d; \mathbf{R}_+)$ and $U \in C(\mathbf{R}^d; \mathbf{R}_+)$ as well as two positive constants μ_1 and μ_2 such that

$$\lim_{|x| \to \infty} V(x) = \infty, \tag{2.7}$$

$$LV(x, y) \leq \mu_1\big(1 + V(x) + V(y) + U(y)\big) - \mu_2 U(x), \tag{2.8}$$

for all $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$.

Assumption 2.3. There exists a positive constant $\overline{C}$ such that

$$\int_\varepsilon |c(0, 0, v)|^2 \phi(dv) \leq \overline{C}. \tag{2.9}$$

Assumption 2.4. There exists a positive constant L such that the initial data (2.2) satisfies

$$|\xi(t) - \xi(s)| \leq L|t - s|^{1/2}, \quad \text{for } -\tau \leq t, \ s \leq 0. \tag{2.10}$$

2.2. The Existence of Global Solutions

In this section, we analyze the existence and the property of the global solution to (2.1) under Assumptions 2.1, 2.2, and 2.4.

In order to demonstrate the existence of the global solution to (2.1), we redefine the following concepts mainly according to [17, 18].

Definition 2.5. Let $\{x(t)\}_{t \geq -\tau}$ be an $\mathbf{R}^d$-valued stochastic process. The process is said to be càdlàg if it is right continuous and for almost all $\omega \in \Omega$ the left limit $\lim_{s \uparrow t} x(s)$ exists and is finite for all $t > -\tau$.

Definition 2.6. Let σ_∞ be a stopping time such that $0 \leq \sigma_\infty \leq T$ a.s. An $\mathbf{R}^d$-valued, $\mathcal{F}_t$-adapted, and càdlàg process $\{x(t) : -\tau \leq t < \sigma_\infty\}$ is called a local solution of (2.1) if $x(t) = \xi(t)$ on $t \in [-\tau, 0]$ and, moreover, there is a nondecreasing sequence $\{\sigma_k\}_{k \geq 1}$ of stopping times such that $0 \leq \sigma_k \uparrow \sigma_\infty$ a.s. and

$$
\begin{aligned}
x(t \wedge \sigma_k) = x(0) &+ \int_0^{t \wedge \sigma_k} a\big(x(s^-), x((s - \tau)^-)\big) ds + \int_0^{t \wedge \sigma_k} b\big(x(s^-), x((s - \tau)^-)\big) dW(s) \\
&+ \int_0^{t \wedge \sigma_k} \int_\varepsilon c\big(x(s^-), x((s - \tau)^-), v\big) \widetilde{p}_\phi(dv \times ds)
\end{aligned}
\tag{2.11}
$$

holds for any $t \in [0, T]$ and $k \geq 1$ with probability 1. If, furthermore,

$$\lim_{t \to \sigma_\infty} \sup |x(t)| = \infty \quad \text{whenever } \sigma_\infty < T, \tag{2.12}$$

then it is called a maximal local solution of (2.1) and σ_∞ is called the explosion time. A local solution $\{x(t) : -\tau \leq t < \sigma_\infty\}$ to (2.1) is called a global solution if $\sigma_\infty = \infty$.

Lemma 2.7. *Under Assumptions 2.1 and 2.4, for any given initial data (2.2), there is a unique maximal local solution to (2.1).*

Proof. From Assumption 2.4, for the initial data (2.2), we have

$$\max_{-\tau \le t \le 0} |\xi(t)| \le \max_{-\tau \le t \le 0} |\xi(t) - \xi(0)| + |\xi(0)| \le L\sqrt{\tau} + |\xi(0)|. \tag{2.13}$$

For each integer $k \ge [L\sqrt{\tau} + |\xi(0)|] + 1$, we define

$$z^{[k]} = \frac{|z| \wedge k}{|z|} z, \qquad 0^{[k]} = 0, \tag{2.14}$$

for $z \in \mathbf{R}^d$. And then we define the truncation functions

$$a_k(x,y) = a\left(x^{[k]}, y^{[k]}\right), \qquad b_k(x,y) = b\left(x^{[k]}, y^{[k]}\right), \qquad c_k(x,y,v) = c(x,y,v), \tag{2.15}$$

for $x, y \in \mathbf{R}^d$ and each $k \ge [L\sqrt{\tau} + |\xi(0)|] + 1$. Moreover, we define the following equation:

$$dx_k(t) = a_k\left(x_k(t^-), x_k((t-\tau)^-)\right)dt + b_k\left(x_k(t^-), x_k((t-\tau)^-)\right)dW(t)$$
$$+ \int_{\varepsilon} c_k\left(x_k(t^-), x_k((t-\tau)^-), v\right)\widetilde{p}_{\phi}(dv \times dt), \tag{2.16}$$

on $t \in [0, T]$ with initial data $x_k(t) = \xi(t)$ on $t \in [-\tau, 0]$. Obviously, the equation satisfies the global Lipschitz conditions and the linear growth conditions. Therefore according to [4], there is a unique global solution $x_k(t)$ to (2.16) and its solution is a càdlàg process (see [17]). We define the stopping time

$$\sigma_k = T \wedge \inf\{t \in [0,T] : |x_k(t)| \ge k\}, \tag{2.17}$$

for $k \ge [L\sqrt{\tau} + |\xi(0)|] + 1$, and

$$\sigma_1 = \cdots = \sigma_{[L\sqrt{\tau} + |\xi(0)|]} = \sigma_{[L\sqrt{\tau} + |\xi(0)|]+1}, \tag{2.18}$$

where we set $\inf \phi = \infty$ (as usual ϕ denotes the empty set) throughout our paper. We can easily get

$$x_k(t) = x_{k+1}(t), \quad -\tau \le t \le \sigma_k, \tag{2.19}$$

which means $\{\sigma_k\}_{k \ge 1}$ is a nondecreasing sequence and then let $\lim_{k \to \infty} \sigma_k = \sigma_{\infty}$ a.s. Now, we define $\{x(t) : -\tau \le t < \sigma_{\infty}\}$ with $x(t) = \xi(t)$ on $t \in [-\tau, 0]$ and

$$x(t) = x_k(t), \quad t \in [\sigma_{k-1}, \sigma_k), \ k \ge 1, \tag{2.20}$$

where $\sigma_0 = 0$. And from (2.16) and (2.19), we can also obtain

$$
\begin{aligned}
x(t \wedge \sigma_k) &= x_k(t \wedge \sigma_k) \\
&= x_0 + \int_0^{t \wedge \sigma_k} a(x(s^-), x((s - \tau)^-)) ds + \int_0^{t \wedge \sigma_k} b(x(s^-), x((s - \tau)^-)) dW(s) \\
&\quad + \int_0^{t \wedge \sigma_k} \int_\varepsilon c(x(s^-), x((s - \tau)^-), v) \tilde{p}_\phi(dv \times ds),
\end{aligned}
\tag{2.21}
$$

for any $t \in [0, T]$ and $k \geq 1$ with probability 1. Moreover, if $\sigma_\infty < T$, then

$$
\lim_{t \to \sigma_\infty} \sup |x(t)| \geq \lim_{k \to \infty} \sup |x(\sigma_k)| = \lim_{k \to \infty} \sup |x_k(\sigma_k)| = \infty.
\tag{2.22}
$$

Hence $\{x(t) : -\tau \leq t < \sigma_\infty\}$ is a maximal local solution to (2.1).

To show the uniqueness of the solution to (2.1), let $\{\bar{x}(t) : -\tau \leq t < \bar{\sigma}_\infty\}$ be another maximal local solution. As the same proof as Theorem 2.8 in [17], we infer that

$$
\mathbf{P}(x(t, \omega) = \bar{x}(t, \omega), (t, \omega) \in [-\tau, \sigma_k \wedge \bar{\sigma}_k) \times \Omega) = 1, \quad k \geq 1.
\tag{2.23}
$$

Hence by $k \to \infty$, we get

$$
\mathbf{P}(x(t, \omega) = \bar{x}(t, \omega), (t, \omega) \in [-\tau, \sigma_\infty \wedge \bar{\sigma}_\infty) \times \Omega) = 1.
\tag{2.24}
$$

Therefore $x(t)$ is a unique local solution and then it is a unique maximal local solution to (2.1).

So we complete the whole proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, the existence of the global solution to (2.1) is shown in the following theorem.

Theorem 2.8. *Under Assumptions 2.1, 2.2, and 2.4, for any given initial data* (2.2)*, there is a unique global solution $x(t)$ to* (2.1) *on $t \in [-\tau, \infty)$.*

Proof. According to Lemma 2.7, there exists a unique maximal local solution to (2.1) on $[-\tau, \sigma_\infty)$. Hence in order to show that this local solution is a global one, we only need to demonstrate $\sigma_\infty = \infty$ a.s. Using Itô's formula (see [1]) to $V(x(t))$, we have

$$
\begin{aligned}
dV(x(t)) &= \Bigg(V_x(x(t^-)) a(x(t^-), x((t - \tau)^-)) \\
&\quad + \frac{1}{2} \text{trace}(b^{\mathrm{T}}(x(t^-), x((t - \tau)^-)) V_{xx}(x(t^-)) b(x(t^-), x((t - \tau)^-))) \Bigg) dt \\
&\quad + \int_\varepsilon (V(x(t^-) + c(x(t^-), x((t - \tau)^-), v)) \\
&\quad\quad - V(x(t^-)) - V_x(x(t^-)) c(x(t^-), x((t - \tau)^-), v)) \phi(dv) dt \\
&\quad + V_x(x(t^-)) b(x(t^-), x((t - \tau)^-)) dW(t) \\
&\quad + \int_\varepsilon (V(x(t^-) + c(x(t^-), x((t - \tau)^-), v)) - V(x(t^-))) \tilde{p}_\phi(dv \times dt)
\end{aligned}
$$

$$= LV(x(t^-), x((t-\tau)^-))dt + V_x(x(t^-))b(x(t^-), x((t-\tau)^-))dW(t)$$
$$+ \int_\varepsilon (V(x(t^-) + c(x(t^-), x((t-\tau)^-), v)) - V(x(t^-)))\tilde{p}_\phi(dv \times dt),$$

$$(2.25)$$

for $t \in [0, \sigma_\infty)$.

Our proof is divided into the following steps.

Step 1. For any integer $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$ and $0 \leq t \leq \tau$, by taking integration and expectations and using Assumption 2.2 to (2.25), we get

$$\mathbf{E}V(x(t \wedge \sigma_k)) - \mathbf{E}V(x(0))$$
$$\leq \mathbf{E} \int_0^{t \wedge \sigma_k} (\mu_1(1 + V(x(s^-)) + V(x((s-\tau)^-)) + U(x((s-\tau)^-))) - \mu_2 U(x(s^-)))ds,$$

$$(2.26)$$

which means

$$\mathbf{E}V(x(t \wedge \sigma_k)) \leq C_1 + \mu_1 \mathbf{E} \int_0^{t \wedge \sigma_k} V(x(s^-))ds - \mu_2 \mathbf{E} \int_0^{t \wedge \sigma_k} U(x(s^-))ds, \qquad (2.27)$$

where

$$C_1 = V(x(0)) + \mu_1 \tau + \mu_1 \int_{-\tau}^0 V(\xi(s))ds + \mu_1 \int_{-\tau}^0 U(\xi(s))ds < \infty. \qquad (2.28)$$

From (2.27), we obtain

$$\mathbf{E}V(x(t \wedge \sigma_k)) \leq C_1 + \mu_1 \mathbf{E} \int_0^{t \wedge \sigma_k} V(x(s^-))ds$$
$$\leq C_1 + \mu_1 \int_0^t \mathbf{E}V(x(s \wedge \sigma_k^-))ds, \qquad (2.29)$$

by the Gronwall inequality (see [18]), which leads to

$$\mathbf{E}V(x(t \wedge \sigma_k)) \leq C_1 e^{\mu_1 \tau}, \qquad (2.30)$$

for $0 \leq t \leq \tau$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$. Let

$$\varsigma_k = \inf_{|x| \geq k, 0 \leq t < \infty} V(x), \quad \text{for } k \geq [L\sqrt{\tau} + |\xi(0)|] + 1. \qquad (2.31)$$

Therefore, from (2.30), we have

$$\varsigma_k \mathbf{P}(\sigma_k \leq \tau) \leq \mathbf{E}(V(x(\sigma_k))I_{\{\sigma_k \leq \tau\}}) \leq \mathbf{E}V(x(\tau \wedge \sigma_k)) \leq C_1 e^{\mu_1 \tau}, \qquad (2.32)$$

by taking $k \rightarrow \infty$, which gives

$$\mathbf{P}(\sigma_\infty \leq \tau) = 0. \tag{2.33}$$

Hence we get

$$\mathbf{P}(\sigma_\infty > \tau) = 1. \tag{2.34}$$

It thus follows from (2.30) and (2.34) that

$$\mathbf{E}V(x(t)) \leq C_1 e^{\mu_1 \tau}, \quad 0 \leq t \leq \tau, \tag{2.35}$$

by taking $k \rightarrow \infty$.

Moreover, from (2.27), we get

$$\begin{aligned}
\mathbf{E}\int_0^{\tau \wedge \sigma_k} U(x(s^-))ds &\leq \mu_2^{-1}\left(C_1 + \mu_1 \mathbf{E}\int_0^{\tau \wedge \sigma_k} V(x(s^-))ds\right) \\
&\leq \mu_2^{-1}\left(C_1 + \mu_1 \int_0^\tau \mathbf{E}V(x(s \wedge \sigma_k^-))ds\right),
\end{aligned} \tag{2.36}$$

by taking $k \rightarrow \infty$, which gives

$$\mathbf{E}\int_0^\tau U(x(s))ds \leq \mu_2^{-1}(C_1 + \tau\mu_1 C_1 e^{\mu_1 \tau}) < \infty, \tag{2.37}$$

where (2.34) and (2.35) are used.

Step 2. For any integer $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$ and $0 \leq t \leq 2\tau$, the similar analysis as above gives

$$\mathbf{E}V(x(t \wedge \sigma_k)) \leq C_2 + \mu_1 \mathbf{E}\int_0^{t \wedge \sigma_k} V(x(s^-))ds - \mu_2 \mathbf{E}\int_0^{t \wedge \sigma_k} U(x(s^-))ds, \tag{2.38}$$

where

$$\begin{aligned}
C_2 &= V(x(0)) + 2\mu_1 \tau + \mu_1 \int_{-\tau}^0 V(\xi(s))ds + \mu_1 \int_{-\tau}^0 U(\xi(s))ds \\
&\quad + \mu_1 \mathbf{E}\int_0^\tau V(x(s))ds + \mu_1 \mathbf{E}\int_0^\tau U(x(s))ds < \infty,
\end{aligned} \tag{2.39}$$

from (2.35) and (2.37).

Thus,

$$\mathbf{E}V(x(t \wedge \sigma_k)) \leq C_2 + \mu_1 \mathbf{E}\int_0^t V(x(s \wedge \sigma_k^-))ds, \tag{2.40}$$

which gives

$$EV(x(t \wedge \sigma_k)) \leq C_2 e^{2\mu_1 \tau}, \tag{2.41}$$

for $0 \leq t \leq 2\tau$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$. Hence we get

$$\varsigma_k P(\sigma_k \leq 2\tau) \leq EV(x(2\tau \wedge \sigma_k)) \leq C_2 e^{2\mu_1 \tau}, \tag{2.42}$$

by taking $k \to \infty$, which implies

$$P(\sigma_\infty \leq 2\tau) = 0, \tag{2.43}$$

that is,

$$P(\sigma_\infty > 2\tau) = 1. \tag{2.44}$$

Moreover, by taking $k \to \infty$ to (2.41), we then get

$$EV(x(t)) \leq C_2 e^{2\mu_1 \tau}, \quad 0 \leq t \leq 2\tau. \tag{2.45}$$

Therefore, from (2.38), (2.44), and (2.45), we have

$$E \int_0^{2\tau} U(x(s))ds \leq \mu_2^{-1}\left(C_2 + 2\tau\mu_1 C_1 e^{2\mu_1 \tau}\right) < \infty. \tag{2.46}$$

Step 3. So for any $i \in \mathbf{N}$, we repeat the similar analysis as above and then obtain

$$P(\sigma_\infty > i\tau) = 1,$$
$$EV(x(t)) \leq C_i e^{i\mu_1 \tau}, \quad 0 \leq t \leq i\tau, \tag{2.47}$$
$$E \int_0^\tau U(x(s))ds \leq \mu_2^{-1}\left(C_i + i\tau\mu_1 C_i e^{i\mu_1 \tau}\right) < \infty,$$

where

$$C_i = V(x(0)) + \mu_1 E \int_{-\tau}^{(i-1)\tau} (1 + V(x(s)) + U(x(s)))ds < \infty. \tag{2.48}$$

So we can get $P(\sigma_\infty = \infty) = 1$ and the required result follows. $\qquad\square$

In the following lemma, we show that the solution of (2.1) remains in a compact set with a large probability.

Lemma 2.9. *Under Assumptions 2.1, 2.2, and 2.4, for any pair of $\epsilon \in (0, 1)$ and $T > 0$, there exists a sufficiently large integer k^*, dependent on ϵ and T, such that*

$$\mathbf{P}(\sigma_k \leq T) \leq \epsilon, \quad \forall k \geq k^*, \tag{2.49}$$

where σ_k is defined in Lemma 2.7.

Proof. According to Theorem 2.8, we can get

$$\mathbf{E}V(x(T \wedge \sigma_k)) \leq C_i e^{i\mu_1 \tau}, \tag{2.50}$$

for i large enough to $i\tau \geq T$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$. Therefore, we have

$$\varsigma_k \mathbf{P}(\sigma_k \leq T) \leq \mathbf{E}\big(V(x(\sigma_k))I_{\{\sigma_k \leq T\}}\big) \leq \mathbf{E}V(x(T \wedge \sigma_k)) \leq C_i e^{i\mu_1 \tau}, \tag{2.51}$$

where

$$\varsigma_k = \inf_{|x| \geq k} V(x) \quad \text{for } k \geq [L\sqrt{\tau} + |\xi(0)|] + 1. \tag{2.52}$$

Under (2.7) in Assumption 2.2, there exists a sufficiently large integer k^* such that

$$\mathbf{P}(\sigma_k \leq T) \leq \frac{C_i e^{i\mu_1 \tau}}{\varsigma_k} \leq \epsilon, \quad \forall k \geq k^*. \tag{2.53}$$

So we complete the proof. □

3. The Euler Method

In this section, we introduce the Euler method to (2.1) under Assumptions 2.1, 2.2, 2.3, and 2.4.

Given a step size $\Delta t = \tau/m \in (0, 1)$, $m \in \mathbf{N}$, the Euler method applied to (2.1) computes approximation $X_n \approx x(t_n)$, where $t_n = n\Delta t$ for $n = -m, -(m-1), \ldots, -1, 0, 1, \ldots$, by setting

$$X_n = \xi(n\Delta t) \quad \text{for } n = -m, -(m-1), \ldots, -1, 0, \tag{3.1}$$

and forming

$$X_{n+1} = X_n + a(X_n, X_{n-m})\Delta t + b(X_n, X_{n-m})\Delta W_n$$
$$+ \int_{t_n}^{t_{n+1}} \int_{\varepsilon} c(X_n, X_{n-m}, v)\widetilde{p}_\phi(dv \times dt), \tag{3.2}$$

for $n = 0, 1, \ldots$, where $\Delta W_n = W(t_{n+1}) - W(t_n)$.

The continuous-time Euler method $\overline{X}(t)$ on $t \in [-\tau, \infty)$ is then defined by

$$\overline{X}(t) := \xi(t) \quad \text{for } t \in [-\tau, 0], \tag{3.3}$$

$$\overline{X}(t) := X_0 + \int_0^t a(Z(s), Z(s - \tau))ds + \int_0^t b(Z(s), Z(s - \tau))dW(s)$$
$$+ \int_0^t \int_\varepsilon c(Z(s), Z(s - \tau), v)\tilde{p}_\phi(dv \times ds), \tag{3.4}$$

for $t \geq 0$, where

$$Z(t) = \sum_{n=-m}^\infty X_n I_{[n\Delta t, (n+1)\Delta t)}(t) \quad \text{for } t \in [-\tau, \infty). \tag{3.5}$$

Actually, we can see in [11] that $p_\phi = \{p_\phi(t) := p_\phi(\varepsilon \times [0, t])\}$ is a process that counts the number of jumps until some given time. The Poisson random measure $p_\phi(dv \times dt)$ generates a sequence of pairs $\{(\iota_i, \xi_i), i \in \{1, 2, \ldots, p_\phi(T)\}\}$ for a given finite positive constant T if $\lambda < \infty$. Here $\{\iota_i : \Omega \to \mathbf{R}_+, i \in \{1, 2, \ldots, p_\phi(T)\}\}$ is a sequence of increasing nonnegative random variables representing the jump times of a standard Poisson process with intensity λ, and $\{\xi_i : \Omega \to \varepsilon, i \in \{1, 2, \ldots, p_\phi(T)\}\}$ is a sequence of independent identically distributed random variables, where ξ_i is distributed according to $\phi(dv)/\phi(\varepsilon)$. Then (3.2) can equivalently be of the following form:

$$X_{n+1} = X_n + \left(a(X_n, X_{n-m}) - \int_\varepsilon c(X_n, X_{n-m}, v)\phi(dv) \right)\Delta t$$
$$+ b(X_n, X_{n-m})\Delta W_n + \sum_{i=p_\phi(t_n)+1}^{p_\phi(t_{n+1})} c(X_n, X_{n-m}, \xi_i). \tag{3.6}$$

In order to analyze the Euler method, we will give two lemmas.

The first lemma shows the close relation between the continuous-time Euler solution (3.4) and its step function $Z(t)$.

Lemma 3.1. *Suppose Assumptions 2.1 and 2.3 hold. Then for any $T > 0$, there exists a positive constant $K_1(k)$, dependent on integer k and independent of Δt, such that for all $\Delta t \in (0, 1)$ the continuous-time Euler method (3.4) satisfies*

$$\mathbf{E}\left|\overline{X}(t) - Z(t)\right|^2 \leq K_1(k)\Delta t, \tag{3.7}$$

for $0 \leq t \leq T \wedge \sigma_k \wedge \rho_k$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$, where σ_k is defined in Lemma 2.7 and $\rho_k = \inf\{t \geq 0 : |\overline{X}(t)| \geq k\}$.

Proof. For $0 \leq t \leq T \wedge \sigma_k \wedge \rho_k$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$, there is an integer n such that $t \in [t_n, t_{n+1})$. Thus it follows from (3.2) that

$$\overline{X}(t) - Z(t) = X_n + \int_{t_n}^{t} a(Z(s), Z(s - \tau))ds + \int_{t_n}^{t} b(Z(s), Z(s - \tau))dW(s)$$

$$+ \int_{t_n}^{t} \int_{\varepsilon} c(Z(s), Z(s - \tau), v)\widetilde{p}_{\phi}(dv \times ds) - X_n. \tag{3.8}$$

Therefore, by taking expectations and the Cauchy-Schwarz inequality and using the martingale properties of $dW(t)$ and $\widetilde{p}_{\phi}(dv \times dt)$, we get

$$\mathbf{E}\left|\overline{X}(t) - Z(t)\right|^2$$

$$\leq 3\mathbf{E}\left|\int_{t_n}^{t} a(Z(s), Z(s - \tau))ds\right|^2 + 3\mathbf{E}\left|\int_{t_n}^{t} b(Z(s), Z(s - \tau))dW(s)\right|^2$$

$$+ 3\mathbf{E}\left|\int_{t_n}^{t} \int_{\varepsilon} c(Z(s), Z(s - \tau), v)\widetilde{p}_{\phi}(dv \times ds)\right|^2 \tag{3.9}$$

$$\leq 3\Delta t \mathbf{E}\int_{t_n}^{t} |a(Z(s), Z(s - \tau))|^2 ds + 3\mathbf{E}\int_{t_n}^{t} |b(Z(s), Z(s - \tau))|^2 ds$$

$$+ 3\mathbf{E}\int_{t_n}^{t} \int_{\varepsilon} |c(Z(s), Z(s - \tau), v)|^2 \phi(dv)ds,$$

where the inequality $|u_1 + u_2 + u_3|^2 \leq 3|u_1|^2 + 3|u_2|^2 + 3|u_3|^2$ for $u_1, u_2, u_3 \in \mathbf{R}^d$ is used. Therefore, by applying Assumption 2.1, we get

$$\mathbf{E}\int_{t_n}^{t} |a(Z(s), Z(s - \tau))|^2 ds$$

$$\leq 2\mathbf{E}\int_{t_n}^{t} |a(Z(s), Z(s - \tau)) - a(0, 0)|^2 ds + 2\mathbf{E}\int_{t_n}^{t} |a(0, 0)|^2 ds$$

$$\leq 2C_k \mathbf{E}\int_{t_n}^{t} \left(|Z(s)|^2 + |Z(s - \tau)|^2\right)ds + 2|a(0, 0)|^2 \Delta t \tag{3.10}$$

$$\leq 4k^2 C_k \Delta t + 2|a(0, 0)|^2 \Delta t,$$

$$\mathbf{E}\int_{t_n}^{t} |b(Z(s), Z(s - \tau))|^2 ds \leq 4k^2 C_k \Delta t + 2|b(0, 0)|^2 \Delta t,$$

$$\mathbf{E}\int_{t_n}^{t} \int_{\varepsilon} |c(Z(s), Z(s - \tau), v)|^2 \phi(dv)ds \leq 4k^2 C \Delta t + 2\Delta t \int_{\varepsilon} |c(0, 0, v)|^2 \phi(dv).$$

Hence by substituting (3.10) into (3.9), we get

$$\mathbf{E}\left|\overline{X}(t) - Z(t)\right|^2 \leq \Delta t \left(24k^2 C_k + 12k^2 C + 6|a(0,0)|^2 + 6|b(0,0)|^2 + 6\int_\varepsilon |c(0,0,v)|^2 \phi(dv)\right), \tag{3.11}$$

for $0 \leq t \leq T \wedge \sigma_k \wedge \rho_k$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$.

So from Assumption 2.3, we can get the result (3.7) by choosing

$$K_1(k) = 24k^2 C_k + 12k^2 C + 6|a(0,0)|^2 + 6|b(0,0)|^2 + 6\int_\varepsilon |c(0,0,v)|^2 \phi(dv). \tag{3.12}$$

$\square$

In the following lemma, we demonstrate that the solution of continuous-time Euler method (3.4) remains in a compact set with a large probability.

Lemma 3.2. *Under Assumptions 2.1, 2.2, 2.3, and 2.4, for any pair of $\epsilon \in (0,1)$ and $T > 0$, there exist a sufficiently large integer k^* and a sufficiently small Δt_1^* such that*

$$\mathbf{P}\left(\rho_{k^*} \leq T\right) \leq \epsilon, \quad \forall \Delta t \leq \Delta t_1^*, \tag{3.13}$$

where ρ_{k^*} is defined in Lemma 3.1.

Proof. Our proof is completed by the following steps.
Step 1. Using Itô's formula (see [1]) to $V(\overline{X}(t))$, for $t \geq 0$, we have

$$\begin{aligned}
dV\left(\overline{X}(t)\right) &= \left(V_x\left(\overline{X}(t)\right)a(Z(t), Z(t-\tau))\right. \\
&\qquad \left. + \frac{1}{2}\text{trace}\left(b^\mathsf{T}(Z(t), Z(t-\tau))V_{xx}\left(\overline{X}(t)\right)b(Z(t), Z(t-\tau))\right)\right)dt \\
&\quad + \int_\varepsilon \left(V\left(\overline{X}(t) + c(Z(t), Z(t-\tau), v)\right)\right. \\
&\qquad \left. -V\left(\overline{X}(t)\right) - V_x\left(\overline{X}(t)\right)c(Z(t), Z(t-\tau), v)\right)\phi(dv)dt \\
&\quad + V_x\left(\overline{X}(t)\right)b(Z(t), Z(t-\tau))dW(t) \\
&\quad + \int_\varepsilon \left(V\left(\overline{X}(t) + c(Z(t), Z(t-\tau), v)\right) - V\left(\overline{X}(t)\right)\right)\tilde{p}_\phi(dv \times dt) \\
&= LV\left(\overline{X}(t), \overline{X}(t-\tau)\right)dt + f\left(\overline{X}(t), \overline{X}(t-\tau), Z(t), Z(t-\tau)\right)dt \\
&\quad + V_x\left(\overline{X}(t)\right)b(Z(t), Z(t-\tau))dW(t) \\
&\quad + \int_\varepsilon \left(V\left(\overline{X}(t) + c(Z(t), Z(t-\tau), v)\right) - V\left(\overline{X}(t)\right)\right)\tilde{p}_\phi(dv \times dt),
\end{aligned} \tag{3.14}$$

where $f(x, y, Z_1, Z_2) : \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ is defined by

$$
\begin{aligned}
f&(x, y, Z_1, Z_2) \\
&= V_x(x)\big(a(Z_1, Z_2) - a(x, y)\big) + \frac{1}{2}\mathrm{trace}\big(b^{\mathrm{T}}(Z_1, Z_2)V_{xx}(x)b(Z_1, Z_2)\big) \\
&\quad - \frac{1}{2}\mathrm{trace}\big(b^{\mathrm{T}}(x, y)V_{xx}(x)b(x, y)\big) \\
&\quad + \int_\varepsilon \big(V_x(x)c(x, y, v) - V_x(x)c(Z_1, Z_2, v)\big)\phi(dv) \\
&\quad + \int_\varepsilon \big(V(x + c(Z_1, Z_2, v)) - V(x + c(x, y, v))\big)\phi(dv).
\end{aligned}
\tag{3.15}
$$

Moreover, for $(x, y, Z_1, Z_2) \in \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d$ with $|x| \vee |y| \vee |Z_1| \vee |Z_2| \le k$, we have

$$
\begin{aligned}
f(x, y, Z_1, Z_2) &= V_x(x)\big(a(Z_1, Z_2) - a(x, y)\big) \\
&\quad + \frac{1}{2}\mathrm{trace}\big(\big(b^{\mathrm{T}}(Z_1, Z_2) - b^{\mathrm{T}}(x, y)\big)V_{xx}(x)b(Z_1, Z_2)\big) \\
&\quad + \frac{1}{2}\mathrm{trace}\big(b^{\mathrm{T}}(x, y)V_{xx}(x)\big(b(Z_1, Z_2) - b(x, y)\big)\big) \\
&\quad + \int_\varepsilon V_x(x)\big(c(x, y, v) - c(Z_1, Z_2, v)\big)\phi(dv) \\
&\quad + \int_\varepsilon \big(V(x + c(Z_1, Z_2, v)) - V(x + c(x, y, v))\big)\phi(dv) \\
&\le L_k\big(|x - Z_1| + |y - Z_2|\big),
\end{aligned}
\tag{3.16}
$$

where Assumptions 2.1 and 2.2 are used and L_k is a positive constant dependent on integer k, intensity λ and independent of Δt. Therefore, from (3.16), Assumption 2.4, and (3.7) in Lemma 3.1, we obtain

$$
\begin{aligned}
\mathbf{E}&\int_0^{t \wedge \rho_k} f\big(\overline{X}(s), \overline{X}(s - \tau), Z(s), Z(s - \tau)\big)ds \\
&\le L_k\mathbf{E}\int_0^{t \wedge \rho_k} \big|\overline{X}(s) - Z(s)\big|ds + L_k\mathbf{E}\int_0^{t \wedge \rho_k} \big|\overline{X}(s - \tau) - Z(s - \tau)\big|ds \\
&\le 2L_k\mathbf{E}\int_0^{t \wedge \rho_k} \big|\overline{X}(s) - Z(s)\big|ds + L_k\int_{-\tau}^0 |\xi(s) - Z(s)|ds \\
&\le 2L_k\int_0^t \Big(\mathbf{E}\big|\overline{X}(s \wedge \rho_k) - Z(s \wedge \rho_k)\big|^2\Big)^{1/2}ds + L_k\sum_{n=-m}^{-1}\int_{t_n}^{t_{n+1}} |\xi(s) - \xi(t_n)|ds \\
&\le 2L_kT\sqrt{K_1(k)\Delta t} + L_kL\tau\sqrt{\Delta t},
\end{aligned}
\tag{3.17}
$$

for $0 \leq t \leq T$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$. Hence by taking expectations and integration to (3.14), applying the martingale properties of $dW(t)$ and $\tilde{p}_\phi(dv \times dt)$, and then using (3.17) and Assumption 2.2, we obtain

$$
\begin{aligned}
\mathbf{E}V&\left(\overline{X}(t \wedge \rho_k)\right) \\
&= V(X_0) + \mathbf{E}\int_0^{t \wedge \rho_k} LV\left(\overline{X}(s), \overline{X}(s-\tau)\right)ds \\
&\quad + \mathbf{E}\int_0^{t \wedge \rho_k} f\left(\overline{X}(s), \overline{X}(s-\tau), Z(s), Z(s-\tau)\right)ds \\
&\leq V(X_0) + 2L_k T\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} \\
&\quad + \mathbf{E}\int_0^{t \wedge \rho_k} \left(\mu_1\left(1 + V\left(\overline{X}(s)\right) + V\left(\overline{X}(s-\tau)\right) + U\left(\overline{X}(s-\tau)\right)\right) - \mu_2 U\left(\overline{X}(s)\right)\right)ds,
\end{aligned}
\tag{3.18}
$$

for $0 \leq t \leq T$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$.

Step 2. For $0 \leq t \leq \tau$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$, it follows from (3.18) that

$$
\begin{aligned}
\mathbf{E}V\left(\overline{X}(t \wedge \rho_k)\right) &\leq V(X_0) + 2L_k T\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_1 \\
&\quad + \mu_1 \mathbf{E}\int_0^{t \wedge \rho_k} V\left(\overline{X}(s)\right)ds - \mu_2 \mathbf{E}\int_0^{t \wedge \rho_k} U\left(\overline{X}(s)\right)ds,
\end{aligned}
\tag{3.19}
$$

where $\alpha_1 = \mu_1 \int_{-\tau}^0 (1 + V(\xi(s)) + U(\xi(s)))ds$. Thus from (3.19), we get

$$
\begin{aligned}
\mathbf{E}V\left(\overline{X}(t \wedge \rho_k)\right) &\leq V(X_0) + 2L_k T\sqrt{K_1(k)\Delta t} \\
&\quad + \tau L_k L\sqrt{\Delta t} + \alpha_1 + \mu_1 \mathbf{E}\int_0^t V\left(\overline{X}(s \wedge \rho_k)\right)ds,
\end{aligned}
\tag{3.20}
$$

by the Gronwall inequality (see [18]), which gives

$$
\mathbf{E}V\left(\overline{X}(t \wedge \rho_k)\right) \leq \left(V(X_0) + 2L_k T\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_1\right)e^{\mu_1 t},
\tag{3.21}
$$

for $0 \leq t \leq \tau$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$. Moreover, from (3.19) and (3.21), we have

$$
\begin{aligned}
\mathbf{E}\int_0^{\tau \wedge \rho_k} &U\left(\overline{X}(s)\right)ds \\
&\leq \mu_2^{-1}\left(V(X_0) + 2L_k T\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_1 + \mu_1 \int_0^\tau \mathbf{E}V\left(\overline{X}(s \wedge \rho_k)\right)ds\right) \\
&\leq \left(V(X_0) + 2L_k T\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_1\right)e^{\mu_1 \tau}\mu_2^{-1},
\end{aligned}
\tag{3.22}
$$

for $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$.

Step 3. For $0 \leq t \leq 2\tau$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$, it follows from (3.18) that

$$
\begin{aligned}
\mathbf{E}V\Big(\overline{X}(t \wedge \rho_k)\Big) \\
\leq V(X_0) + 2L_kT\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_2 \\
+ \mu_1 \mathbf{E} \int_0^{t \wedge \rho_k} V\Big(\overline{X}(s)\Big)ds - \mu_2 \mathbf{E} \int_0^{t \wedge \rho_k} U\Big(\overline{X}(s)\Big)ds.
\end{aligned}
\tag{3.23}
$$

As the same way as Step 2, we can obtain

$$
\begin{aligned}
\mathbf{E}V\Big(\overline{X}(t \wedge \rho_k)\Big) \\
\leq \Big(V(X_0) + 2L_kT\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_2\Big)e^{\mu_1 t}, \quad 0 \leq t \leq 2\tau,
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\mathbf{E} \int_0^{2\tau \wedge \rho_k} U\Big(\overline{X}(s)\Big)ds \\
\leq \Big(V(X_0) + 2L_kT\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_2\Big)e^{2\mu_1 \tau}\mu_2^{-1},
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
\alpha_2 &= \mu_1 \mathbf{E} \int_0^{2\tau \wedge \rho_k} \Big(1 + V\Big(\overline{X}(s-\tau)\Big) + U\Big(\overline{X}(s-\tau)\Big)\Big)ds \\
&= \mu_1 \mathbf{E} \int_{-\tau}^{2\tau \wedge \rho_k - \tau} \Big(1 + V\Big(\overline{X}(s)\Big) + U\Big(\overline{X}(s)\Big)\Big)ds \\
&\leq \mu_1 \mathbf{E} \int_{-\tau}^{\tau \wedge \rho_k} \Big(1 + V\Big(\overline{X}(s)\Big) + U\Big(\overline{X}(s)\Big)\Big)ds \\
&= \alpha_1 + \mu_1 \mathbf{E} \int_0^{\tau \wedge \rho_k} \Big(1 + V\Big(\overline{X}(s)\Big) + U\Big(\overline{X}(s)\Big)\Big)ds \\
&\leq \alpha_1 + \mu_1 \tau + \mu_1 \int_0^{\tau} \mathbf{E}V\Big(\overline{X}(s \wedge \rho_k)\Big)ds + \mu_1 \mathbf{E} \int_0^{\tau \wedge \rho_k} U\Big(\overline{X}(s)\Big)ds \\
&\leq \mu_1 \tau + \Big(V(X_0) + 2L_kT\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_1\Big)\Big(e^{\mu_1 \tau} + e^{\mu_1 \tau}\mu_1\mu_2^{-1}\Big),
\end{aligned}
\tag{3.26}
$$

from (3.21) and (3.22). So (3.24) becomes

$$
\mathbf{E}V\Big(\overline{X}(t \wedge \rho_k)\Big) \leq \Big(V(X_0) + 2L_kT\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_1\Big)\beta_{1,0} + \beta_{2,0},
\tag{3.27}
$$

for $0 \leq t \leq 2\tau$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$, where

$$
\beta_{1,0} = \Big(1 + e^{\mu_1 \tau} + e^{\mu_1 \tau}\mu_1\mu_2^{-1}\Big)e^{2\mu_1 \tau}, \qquad \beta_{2,0} = (\mu_1 \tau - \alpha_1)e^{2\mu_1 \tau}.
\tag{3.28}
$$

Step 4. By repeating the same way in Steps 2 and 3, we get

$$\mathbf{E}V\left(\overline{X}(T \wedge \rho_k)\right) \le \left(V(X_0) + 2L_kT\sqrt{K_1(k)\Delta t} + \tau L_k L\sqrt{\Delta t} + \alpha_1\right)\beta_1 + \beta_2, \tag{3.29}$$

for $k \ge [L\sqrt{\tau} + |\xi(0)|] + 1$, where β_1 and β_2 are two constants dependent on μ_1, μ_2, τ, T and independent of k and Δt. Therefore, we have

$$\mathbf{P}(\rho_k \le T) \le \frac{V(X_0)\beta_1 + \alpha_1\beta_1 + \beta_2 + 2\beta_1 L_k T\sqrt{K_1(k)\Delta t} + \beta_1 \tau L_k L\sqrt{\Delta t}}{\vartheta_k}, \tag{3.30}$$

where

$$\vartheta_k = \inf_{|\overline{X}| \ge k} V\left(\overline{X}\right), \quad k \ge [L\sqrt{\tau} + |\xi(0)|] + 1. \tag{3.31}$$

Now, for any $\epsilon \in (0, 1)$, we can choose sufficiently large integer k^* such that

$$\frac{V(X_0)\beta_1 + \alpha_1\beta_1 + \beta_2}{\vartheta_{k^*}} \le \frac{\epsilon}{2} \tag{3.32}$$

and sufficiently small Δt_1^* such that

$$\frac{2\beta_1 L_{k^*} T\sqrt{K_1(k^*)\Delta t_1^*} + \beta_1 \tau L_{k^*} L\sqrt{\Delta t_1^*}}{\vartheta_{k^*}} \le \frac{\epsilon}{2}. \tag{3.33}$$

So from (3.30), we can obtain

$$\mathbf{P}(\rho_{k^*} \le T) \le \epsilon, \quad \forall \Delta t \le \Delta t_1^*. \tag{3.34}$$

$\square$

4. Convergence in Probability

In this section, we show the convergence in probability of the Euler method to (2.1) over a finite time interval $[0, T]$, which is based on the following lemma.

Lemma 4.1. *Under Assumptions 2.1, 2.3, and 2.4, for any $T > 0$, there exists a positive constant $K_2(k)$, dependent on k and independent of Δt, such that for all $\Delta t \in (0, 1)$ the solution of (2.1) and the continuous-time Euler method (3.4) satisfy*

$$\mathbf{E}\left(\sup_{0 \le t \le T}\left|x(t \wedge \sigma_k \wedge \rho_k) - \overline{X}(t \wedge \sigma_k \wedge \rho_k)\right|^2\right) \le K_2(k)\Delta t, \tag{4.1}$$

where σ_k and ρ_k are defined in Lemmas 2.7 and 3.1, respectively, and $k \ge [L\sqrt{\tau} + |\xi(0)|] + 1$.

Proof. From (2.1) and (3.4), for any $0 \leq t' \leq T$ and $k \geq [L\sqrt{\tau} + |\xi(0)|] + 1$, we have

$$\mathbf{E}\left(\sup_{0 \leq t \leq t'}\left|x(t \wedge \sigma_k \wedge \rho_k) - \overline{X}(t \wedge \sigma_k \wedge \rho_k)\right|^2\right)$$

$$\leq 3\mathbf{E}\left(\sup_{0 \leq t \leq t'}\left|\int_0^{t \wedge \sigma_k \wedge \rho_k}(a(x(s^-), x((s-\tau)^-)) - a(Z(s), Z(s-\tau)))ds\right|^2\right)$$

$$+ 3\mathbf{E}\left(\sup_{0 \leq t \leq t'}\left|\int_0^{t \wedge \sigma_k \wedge \rho_k}(b(x(s^-), x((s-\tau)^-)) - b(Z(s), Z(s-\tau)))dW(s)\right|^2\right)$$

$$+ 3\mathbf{E}\left(\sup_{0 \leq t \leq t'}\left|\int_0^{t \wedge \sigma_k \wedge \rho_k}\int_\varepsilon (c(x(s^-), x((s-\tau)^-), v) - c(Z(s), Z(s-\tau), v))\widetilde{p}_\phi(dv \times ds)\right|^2\right),$$

$$(4.2)$$

where the inequality $|u_1 + u_2 + u_3|^2 \leq 3|u_1|^2 + 3|u_2|^2 + 3|u_3|^2$ for $u_1, u_2, u_3 \in \mathbf{R}^d$ is used. Therefore, by using the Cauchy-Schwarz inequality, Assumptions 2.1 and 2.4, Fubini's Theorem, and Lemma 3.1, we obtain

$$\mathbf{E}\left(\sup_{0 \leq t \leq t'}\left|\int_0^{t \wedge \sigma_k \wedge \rho_k}(a(x(s^-), x((s-\tau)^-)) - a(Z(s), Z(s-\tau)))ds\right|^2\right)$$

$$\leq \mathbf{E}\left(\sup_{0 \leq t \leq t'}\int_0^{t \wedge \sigma_k \wedge \rho_k}1^2 ds \int_0^{t \wedge \sigma_k \wedge \rho_k}\left|a(x(s^-), x((s-\tau)^-)) - a(Z(s), Z(s-\tau))\right|^2 ds\right)$$

$$\leq T\mathbf{E}\left(\int_0^{t' \wedge \sigma_k \wedge \rho_k}\left|a(x(s^-), x((s-\tau)^-)) - a(Z(s), Z(s-\tau))\right|^2 ds\right)$$

$$\leq TC_k \mathbf{E}\left(\int_0^{t' \wedge \sigma_k \wedge \rho_k}\left|x(s^-) - Z(s)\right|^2 ds\right) + TC_k\mathbf{E}\left(\int_0^{t' \wedge \sigma_k \wedge \rho_k}\left|x((s-\tau)^-) - Z(s-\tau)\right|^2 ds\right)$$

$$\leq 2TC_k\mathbf{E}\left(\int_0^{t' \wedge \sigma_k \wedge \rho_k}\left|x(s^-) - Z(s)\right|^2 ds\right) + TC_k\int_{-\tau}^0 |\xi(s) - Z(s)|^2 ds$$

$$\leq 4TC_k\mathbf{E}\left(\int_0^{t' \wedge \sigma_k \wedge \rho_k}\left|\overline{X}(s) - Z(s)\right|^2 ds\right)$$

$$+ 4TC_k\mathbf{E}\left(\int_0^{t' \wedge \sigma_k \wedge \rho_k}\left|x(s^-) - \overline{X}(s)\right|^2 ds\right)$$

$$+ TC_k\sum_{n=-m}^{-1}\int_{t_n}^{t_{n+1}}|\xi(s) - \xi(t_n)|^2 ds$$

$$\leq 4TC_k \left(\int_0^{t'} \mathbf{E} \left| \overline{X}(s \wedge \sigma_k \wedge \rho_k) - Z(s \wedge \sigma_k \wedge \rho_k) \right|^2 ds \right)$$

$$+ 4TC_k \int_0^{t'} \mathbf{E} \left(\sup_{0 \leq u \leq s} \left| x(u \wedge \sigma_k \wedge \rho_k^-) - \overline{X}(u \wedge \sigma_k \wedge \rho_k) \right|^2 \right) ds + TC_k L^2 \tau \Delta t$$

$$\leq \left(4T^2 C_k K_1(k) + TC_k L^2 \tau \right) \Delta t$$

$$+ 4TC_k \int_0^{t'} \mathbf{E} \left(\sup_{0 \leq u \leq s} \left| x(u \wedge \sigma_k \wedge \rho_k^-) - \overline{X}(u \wedge \sigma_k \wedge \rho_k) \right|^2 \right) ds.$$

$$(4.3)$$

Moreover, by using the martingale properties of $dW(t)$ and $\widetilde{p}_\phi(dv \times dt)$, Assumptions 2.1 and 2.4, Fubini's Theorem, and Lemma 3.1, we have

$$\mathbf{E} \left(\sup_{0 \leq t \leq t'} \left| \int_0^{t \wedge \sigma_k \wedge \rho_k} \left(b(x(s^-), x((s - \tau)^-)) - b(Z(s), Z(s - \tau)) \right) dW(s) \right|^2 \right)$$

$$\leq 4\mathbf{E} \int_0^{t' \wedge \sigma_k \wedge \rho_k} \left| b(x(s^-), x((s - \tau)^-)) - b(Z(s), Z(s - \tau)) \right|^2 ds$$

$$\leq 4C_k \mathbf{E} \int_0^{t' \wedge \sigma_k \wedge \rho_k} \left| x(s^-) - Z(s) \right|^2 ds + 4C_k \mathbf{E} \int_0^{t' \wedge \sigma_k \wedge \rho_k} \left| x((s - \tau)^-) - Z(s - \tau) \right|^2 ds$$

$$\leq 8C_k \mathbf{E} \int_0^{t' \wedge \sigma_k \wedge \rho_k} \left| x(s^-) - Z(s) \right|^2 ds + 4C_k \int_{-\tau}^0 \left| \xi(s) - Z(s) \right|^2 ds$$

$$\leq 16C_k \mathbf{E} \left(\int_0^{t' \wedge \sigma_k \wedge \rho_k} \left| \overline{X}(s) - Z(s) \right|^2 ds \right) + 16C_k \mathbf{E} \left(\int_0^{t' \wedge \sigma_k \wedge \rho_k} \left| x(s^-) - \overline{X}(s) \right|^2 ds \right)$$

$$+ 4C_k \sum_{n=-m}^{-1} \int_{t_n}^{t_{n+1}} \left| \xi(s) - \xi(t_n) \right|^2 ds$$

$$\leq \left(16TC_k K_1(k) + 4C_k L^2 \tau \right) \Delta t$$

$$+ 16C_k \int_0^{t'} \mathbf{E} \left(\sup_{0 \leq u \leq s} \left| x(u \wedge \sigma_k \wedge \rho_k^-) - \overline{X}(u \wedge \sigma_k \wedge \rho_k) \right|^2 \right) ds,$$

$$\mathbf{E} \left(\sup_{0 \leq t \leq t'} \left| \int_0^{t \wedge \sigma_k \wedge \rho_k} \int_\varepsilon \left(c(x(s^-), x((s - \tau)^-), v) - c(Z(s), Z(s - \tau), v) \right) \widetilde{p}_\phi(dv \times ds) \right|^2 \right)$$

$$
\leq 4\mathbf{E}\left|\int_0^{t'\wedge\sigma_k\wedge\rho_k}\int_\varepsilon (c(x(s^-),x((s-\tau)^-),v)-c(Z(s),Z(s-\tau),v))\widetilde{p}_\phi(dv\times ds)\right|^2
$$

$$
= 4\mathbf{E}\int_0^{t'\wedge\sigma_k\wedge\rho_k}\int_\varepsilon |c(x(s^-),x((s-\tau)^-),v)-c(Z(s),Z(s-\tau),v)|^2\phi(dv)ds
$$

$$
\leq \left(16TCK_1(k)+4CL^2\tau\right)\Delta t
$$

$$
+16C\int_0^{t'}\mathbf{E}\left(\sup_{0\leq u\leq s}\left|x(u\wedge\sigma_k\wedge\rho_k^-)-\overline{X}(u\wedge\sigma_k\wedge\rho_k)\right|^2\right)ds.
$$

$$(4.4)$$

Therefore by substituting (4.3) and (4.4) into (4.2), we get

$$
\mathbf{E}\left(\sup_{0\leq t\leq t'}\left|x(t\wedge\sigma_k\wedge\rho_k)-\overline{X}(t\wedge\sigma_k\wedge\rho_k)\right|^2\right)
$$

$$
\leq \Delta t\left(12T^2C_kK_1(k)+3TC_kL^2\tau+48TC_kK_1(k)+48TCK_1(k)+12C_kL^2\tau+12CL^2\tau\right) \quad (4.5)
$$

$$
+(12TC_k+48C_k+48C)\int_0^{t'}\mathbf{E}\left(\sup_{0\leq u\leq s}\left|x(u\wedge\sigma_k\wedge\rho_k^-)-\overline{X}(u\wedge\sigma_k\wedge\rho_k)\right|^2\right)ds.
$$

So by using the Gronwall inequality (see [18]), we have the result (4.1) by choosing

$$
K_2(k) = \left(12T^2C_kK_1(k)+3TC_kL^2\tau+48TC_kK_1(k)+48TCK_1(k)\right.
$$

$$
\left.+12C_kL^2\tau+12CL^2\tau\right)\exp\left(12T^2C_k+48TC_k+48TC\right).
$$

$$(4.6)$$

$\square$

Now, we state our main theorem which shows the convergence in probability of the continuous-time Euler method (3.4).

Theorem 4.2. *Under Assumptions 2.1, 2.2, 2.3, and 2.4, for sufficiently small $\epsilon,\varsigma\in(0,1)$, there is a Δt^* such that for all $\Delta t < \Delta t^*$*

$$
\mathbf{P}\left(\sup_{0\leq t\leq T}\left|x(t)-\overline{X}(t)\right|^2\geq\varsigma\right)\leq\epsilon, \quad (4.7)
$$

for any $T > 0$.

Proof. For sufficiently small $\epsilon,\varsigma\in(0,1)$, we define

$$
\overline{\Omega} = \left\{\omega:\sup_{0\leq t\leq T}\left|x(t)-\overline{X}(t)\right|^2\geq\varsigma\right\}. \quad (4.8)
$$

By Lemmas 2.9 and 3.2, there exists a pair of $k^\star$ and $\Delta t_1^\star$ such that

$$\mathbf{P}(\sigma_{k^\star} \leq T) \leq \frac{\epsilon}{3},$$

$$\mathbf{P}(\rho_{k^\star} \leq T) \leq \frac{\epsilon}{3}, \quad \forall \Delta t \leq \Delta t_1^\star. \tag{4.9}$$

We then have

$$\mathbf{P}\left(\overline{\Omega}\right) \leq \mathbf{P}\left(\overline{\Omega} \cap \{\sigma_{k^\star} \wedge \rho_{k^\star} > T\}\right) + \mathbf{P}(\sigma_{k^\star} \wedge \sigma_{k^\star} \leq T)$$

$$\leq \mathbf{P}\left(\overline{\Omega} \cap \{\sigma_{k^\star} \wedge \sigma_{k^\star} > T\}\right) + \mathbf{P}(\sigma_{k^\star} \leq T) + \mathbf{P}(\rho_{k^\star} \leq T) \tag{4.10}$$

$$\leq \mathbf{P}\left(\overline{\Omega} \cap \{\sigma_{k^\star} \wedge \rho_{k^\star} > T\}\right) + \frac{2\epsilon}{3},$$

for $\Delta t \leq \Delta t_1^\star$. Moreover, from Lemma 4.1, we have

$$\varsigma \mathbf{P}\left(\overline{\Omega} \cap \{\sigma_{k^\star} \wedge \rho_{k^\star} > T\}\right) \leq \mathbf{E}\left(I_{\{\sigma_{k^\star} \wedge \rho_{k^\star} > T\}} \sup_{0 \leq t \leq T} \left|x(t) - \overline{X}(t)\right|^2\right)$$

$$\leq \mathbf{E}\left(\sup_{0 \leq t \leq T} \left|x(t \wedge \sigma_{k^\star} \wedge \rho_{k^\star}) - \overline{X}(t \wedge \sigma_{k^\star} \wedge \rho_{k^\star})\right|^2\right) \tag{4.11}$$

$$\leq K_2(k^\star)\Delta t,$$

which gives

$$\mathbf{P}\left(\overline{\Omega} \cap \{\sigma_{k^\star} \wedge \rho_{k^\star} > T\}\right) \leq \frac{\epsilon}{3}, \tag{4.12}$$

for $\Delta t \leq \Delta t_2^\star$. Hence, from the inequalities above, we get

$$\mathbf{P}\left(\overline{\Omega}\right) \leq \epsilon, \tag{4.13}$$

for $\Delta t \leq \Delta t^\star$, where $\Delta t^\star = \min\{\Delta t_1^\star, \Delta t_2^\star\}$.

We remark that the continuous-time Euler solution $\overline{X}(t)$ cannot be computed, since it requires knowledge of the entire Brownian motion and Poisson random measure paths. So the last theorem demonstrates the convergence in probability of the discrete Euler solution (3.2). □

Theorem 4.3. *Under Assumptions 2.1, 2.2, 2.3, and 2.4, for sufficiently small $\epsilon, \varsigma \in (0, 1)$, there is a Δt^* such that for all $\Delta t < \Delta t^*$*

$$\mathbf{P}\left(|x(t) - Z(t)|^2 \geq \varsigma, \, 0 \leq t \leq T \right) \leq \epsilon, \tag{4.14}$$

for any $T > 0$.

Proof. For sufficiently small $\epsilon, \varsigma \in (0, 1)$, we define

$$\widetilde{\Omega} = \left\{ \omega : |x(t) - Z(t)|^2 \geq \varsigma, \, 0 \leq t \leq T \right\}. \tag{4.15}$$

As the same analysis as the proof in Theorem 4.2, we have

$$\mathbf{P}\left(\widetilde{\Omega} \right) \leq \mathbf{P}\left(\widetilde{\Omega} \cap \{ \sigma_{k^*} \wedge \rho_{k^*} > T \} \right) + \frac{2\epsilon}{3}. \tag{4.16}$$

Recalling that

$$\begin{aligned}
\varsigma \mathbf{P}\left(\widetilde{\Omega} \cap \{ \sigma_{k^*} \wedge \rho_{k^*} > T \} \right) &\leq \mathbf{E}\left(|x(T) - Z(T)|^2 I_{\{\sigma_{k^*} \wedge \rho_{k^*} > T\}} \right) \\
&\leq \mathbf{E}\left| x(T \wedge \sigma_{k^*} \wedge \rho_{k^*}) - Z(T \wedge \sigma_{k^*} \wedge \rho_{k^*}) \right|^2 \\
&\leq 2\mathbf{E}\left(\sup_{0 \leq t \leq T} \left| x(t \wedge \sigma_{k^*} \wedge \rho_{k^*}) - \overline{X}(t \wedge \sigma_{k^*} \wedge \rho_{k^*}) \right|^2 \right) \\
&\quad + 2\mathbf{E}\left| \overline{X}(T \wedge \sigma_{k^*} \wedge \rho_{k^*}) - Z(T \wedge \sigma_{k^*} \wedge \rho_{k^*}) \right|^2 \\
&\leq 2K_1(k^*)\Delta t + 2K_2(k^*)\Delta t
\end{aligned} \tag{4.17}$$

and using Lemmas 3.1 and 4.1, we obtain

$$\mathbf{P}\left(\widetilde{\Omega} \cap \{ \sigma_{k^*} \wedge \rho_{k^*} > T \} \right) \leq \frac{\epsilon}{3}, \tag{4.18}$$

for sufficiently small Δt. So the inequalities above demonstrate

$$\mathbf{P}\left(\widetilde{\Omega} \right) \leq \epsilon, \tag{4.19}$$

for all sufficiently small Δt.

Hence we complete the result (4.14). □

5. Numerical Example

In this section, a numerical example is analyzed under Assumptions 2.1, 2.2, 2.3, and 2.4 which cover many highly nonlinear SDDEs driven by Poisson random measure.

Now, we consider the following equation:

$$dx(t) = \left(2x^2\left((t - 0.05)^-\right) - 4x^3(t^-)\right)dt + 3x^2\left((t - 0.05)^-\right)dW(t)$$

$$+ \int_\varepsilon vx^2\left((t - 0.05)^-\right)\widetilde{p}_\phi(dv \times dt), \quad t > 0, \tag{5.1}$$

with $x(t) = t^2$, $t \in [-0.05, 0]$, where $d = m = r = 1$. The compensated measure of the Poisson random measure $p_\phi(dv \times dt)$ is given by $\phi(dv)dt = \lambda f(v)dv\,dt$, where $\lambda = 5$ and

$$f(v) = \frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(\ln v)^2}{2}\right), \quad 0 \le v < \infty, \tag{5.2}$$

is the density function of a lognormal random variable.

Clearly, the equation cannot satisfy the global Lipschitz conditions, the linear growth conditions and the classical Khasminskii-type conditions. But, the local Lipschitz conditions are satisfied. On the other hand, for $V(x) = |x|^2$, we have

$$LV(x, y) = 2x\left(2y^2 - 4x^3\right) + 9y^4 + \int_\varepsilon v^2 y^4 \lambda \frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(\ln v)^2}{2}\right)dv$$

$$\le 2x^2 + 60y^4 - 8x^4 \tag{5.3}$$

$$= 60\left(1 + x^2 + y^2 + y^4\right) - 8x^4,$$

where $U(x) = x^4$, $\mu_1 = 60$, $\mu_2 = 8$. In other words, the equation satisfies Assumptions 2.1, 2.2, 2.3, and 2.4.

So according to Theorem 2.8, (5.1) has a unique global solution $x(t)$ on $t \in [-0.05, \infty)$. Given the stepsize $\Delta t = 0.0025$, we can have the Euler method to (5.1)

$$X_n = (n\Delta t)^2, \quad \text{for } n = -20, -19, \ldots, -1, 0,$$

$$X_{n+1} = X_n + \left(2X_{n-20}^2 - 4X_n^3\right)\Delta t + 3X_{n-20}^2\Delta W_n + \int_{t_n}^{t_{n+1}}\int_\varepsilon vX_{n-20}^2\widetilde{p}_\phi(dv \times dt), \tag{5.4}$$

for $n = 0, 1, \ldots$, where $\Delta W_n = W(t_{n+1}) - W(t_n)$.

And in Matlab experiment, we actually obtain the discrete Euler

$$X_{n+1} = X_n + \left(2X_{n-20}^2 - 4X_n^3 - 5\sqrt{e}X_{n-20}^2\right)\Delta t$$

$$+ 3X_{n-20}^2\Delta W_n + X_{n-20}^2\sum_{i=p_\phi(t_n)+1}^{p_\phi(t_{n+1})}\xi_i, \tag{5.5}$$

where ξ_i is distributed according to $f(v)$. Subsequently, we can get the result in Theorems 4.2 and 4.3.
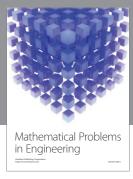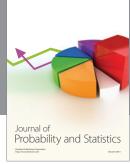
Acknowledgment

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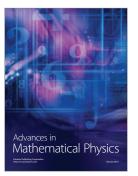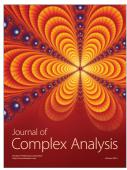[18] X. R. Mao and C. G. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, UK, 2006.