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Research Article

On Semilinear Integro-Differential Equations with Nonlocal Conditions in Banach Spaces

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We study the abstract Cauchy problem for a class of integrodifferential equations in a Banach space with nonlinear perturbations and nonlocal conditions. By using MNC estimates, the existence and continuous dependence results are proved. Under some additional assumptions, we study the topological structure of the solution set.

1. Introduction

In this paper, we investigate the following problem:

$$x'(t) = A\left[x(t) + \int_0^t F(t-s)x(s)ds\right] + g(t,x(t)), \qquad t \in J := [0,T], \tag{1.1}$$

$$x(0) + h(x) = x_0. \tag{1.2}$$

Here $x(t)$ takes values in a Banach space X; $F(t)$, for each $t \in J$, is a linear operator on X; maps $g : J \times X \to X$ and $h : C(J;X) \to X$ are given. In this model, A is the generator of a strongly continuous semigroup $S(\cdot)$ on X.

It is known that (1.1) with $g = g(t)$ arises from some real applications. For example, the classical heat equation for medium with memory can be written as

$$x_t(t, y) = \frac{\partial^2}{\partial y^2}\left[x(t, y) + \int_0^t b(t-s)x(s, y)ds\right] + g(t, y), \quad x(0, y) = x_0, \tag{1.3}$$

where $t \in \mathbb{R}^+$ and $y \in [0, a] \subset \mathbb{R}^+$ (for more details, see [1, 2]). In addition, if we replace the initial condition $x(0, y) = x_0$ by the nonlocal condition (1.2), it allows to describe the model more effectively. As an example of h, the following function can be considered:

$$h(x) = \sum_{i=1}^{p} c_i x(t_i), \tag{1.4}$$

where c_i $(i = 1, \ldots, p)$ are given constants and $0 \leq t_1 < \cdots < t_p \leq T$. As another example, one can take

$$h(x) = \sum_{i=1}^{p} K_i x(t_i), \tag{1.5}$$

where $K_i : X \to X$ are given linear operators. Regarding to (1.3), in the case $X = L^2(0, a)$, the operators K_i can be given by

$$K_i x(t_i, y) = \int_0^a k_i(\xi, y)x(t_i, \xi)d\xi, \tag{1.6}$$

where k_i $(i = 1, \ldots, p)$ are continuous kernel functions.

Semilinear problem (1.1)-(1.2) with $F = 0$ was studied extensively. In [3–5], the existence and uniqueness results were obtained by using the contraction mapping principle, under the Lipschitz conditions imposed on g and h. Supposing Carathéodory-type conditions on g, the authors in [6] proved the global existence result with the assumption that the semigroup $S(t)$ is compact. However, as it was indicated in [7], if the Lipschitz condition is relaxed, one may get difficulties in proving the compactness of the solution map since the map $t \mapsto S(t)$, in general, is not uniformly continuous in $[0, T]$, even in case when $S(t)$ is compact. Recently, Fan and Li [8] gave an asymptotical method to solve this problem for the case when $S(t)$ is a compact strongly continuous semigroup and the nonlocal function h is supposed to be continuous only.

It is known that, in the case $F = 0$, the mild solution of (1.1)-(1.2) on J is defined via the integral equation

$$x(t) = S(t)[x_0 - h(x)] + \int_0^t S(t-s)g(s, x(s))ds, \quad t \in J. \tag{1.7}$$

Problem (1.1)-(1.2) involving integro-differential equations was introduced in [2]. The complete references to integro-differential equations can be found in [1, 9, 10]. For some additional problems on solvability and controllability of integro-differential equations, we refer the reader to [11–13]. In order to represent the mild solutions via the variation of constants formula for this case, the notion of so-called resolvent for the corresponding linear equation

$$x'(t) = A\left[x(t) + \int_0^t F(t-s)x(s)ds\right], \quad t \in J \tag{1.8}$$

can be applied. More precisely, an operator-valued function $R(\cdot) : J \mapsto L(X)$ is called the resolvent of (7) if it satisfies the following:

(1) $R(0) = I$, the identity operator on X,

(2) for each $v \in X$, the map $t \mapsto R(t)v$ is continuous on J,

(3) if Y is the Banach space formed from $D(A)$, the domain of A, endowed with the graph norm, then $R(t) \in L(Y)$, $R(\cdot)y \in C^1(J;X) \cap C(J;Y)$ for $y \in Y$ and

$$\begin{aligned}\frac{d}{dt}R(t)y &= A\left[R(t)y + \int_0^t F(t-s)R(s)yds\right] \\ &= R(t)Ay + \int_0^t R(t-s)AF(s)ds, \quad t \in J.\end{aligned} \tag{1.9}$$

For the existence of resolvent operators, we refer the reader to [14].

It is worth noting that, from definition of resolvent operator and the uniform boundedness principle, there exists $C_R < +\infty$ such that

$$\sup_{t\in J}\|R(t)\|_{L(X)} \le C_R. \tag{1.10}$$

Then the mild solution on J can be represented as

$$x(t) = R(t)[x_0 - h(x)] + \int_0^t R(t-s)g(s,x(s))ds, \quad t \in J. \tag{1.11}$$

By a similar approach as in [3], the authors in [2] obtained the existence and uniqueness of solutions for (1.11) with the assumptions of the Lipschitz conditions on g and h.

In this work, instead of the Lipschitz conditions posed on g and h, we assume the regularity of g and h expressed in terms of the measure of noncompactness. The mentioned regularity can be considered as a generalization of the Lipschitz condition. We first prove the existence of solutions for (1.1)-(1.2) in Section 2. Our method is to find fixed points of a corresponding condensing map, which yields the existence but does not provide the uniqueness of solutions. The arguments in this work are mainly based on the estimates with measure of noncompactness (MNC estimates). It should be noted that this technique was developed in [15], and it has been employed widely for differential inclusions. In Section 3, we prove that the solution set of our problem is continuously dependent on initial data. Section 4 is devoted to a special case when h is a Lipschitz function and $R(t)$ is compact for $t > 0$. We show that, in this case, the solution set to (1.1)-(1.2) has the so-called R_δ-set structure. We end this paper with an example in Section 5.

2. Existence Results

We start with the recalling of some notions and facts (see, e.g. [15, 16]).

Definition 2.1. Let $\mathcal{E}$ be a Banach space with power set $\mathcal{P}(\mathcal{E})$, and $(\mathcal{A}, \geq)$ a partially ordered set. A function $\beta : \mathcal{P}(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{\text{co}}\ \Omega) = \beta(\Omega) \text{ for every } \Omega \in \mathcal{P}(\mathcal{E}), \tag{2.1}$$

where $\overline{\text{co}}\ \Omega$ is the closure of convex hull of Ω. An MNC β is called

(i) *monotone*, if $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$ such that $\Omega_0 \subset \Omega_1$, then $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) *nonsingular*, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}$, $\Omega \in \mathcal{P}(\mathcal{E})$;

(iii) *invariant with respect to union with compact sets*, if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{P}(\mathcal{E})$.

If, in addition, $\mathcal{A}$ is a cone in a normed space, we say that β is

(iv) *algebraically semiadditive*, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$;

(v) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is *the Hausdorff* MNC, which satisfies all properties given in the previous definition:

$$\chi(\Omega) = \inf\{\varepsilon : \Omega \text{ has a finite } \varepsilon\text{-net}\}. \tag{2.2}$$

Other examples of MNC defined on the space $C(J;\ X)$ of continuous functions on an interval $J = [0, T]$ with values in a Banach space X are the following:

(i) the modulus of fiber noncompactness:

$$\gamma(\Omega) = \sup_{t\in J} \chi(\Omega(t)), \tag{2.3}$$

where χ is the Hausdorff MNC on X and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(ii) the modulus of equicontinuity:

$$\text{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|<\delta} \left\| y(t_1) - y(t_2) \right\|. \tag{2.4}$$

As indicated in [15], these MNCs satisfy all properties mentioned in Definition 2.1 except the regularity.

Let $\mathcal{T} \in L(\mathcal{E})$, that is, $\mathcal{T}$ is a bounded linear operator from $\mathcal{E}$ into $\mathcal{E}$. We recall the notion of χ-norm (see e.g., [16]) as follows:

$$\|\mathcal{T}\|_\chi := \inf\{M : \chi(\mathcal{T}\Omega) \leq M\chi(\Omega),\ \Omega \subset \mathcal{E} \text{ is a bounded set}\}. \tag{2.5}$$

The χ-norm of $\mathcal{T}$ can be evaluated as

$$\|\mathcal{T}\|_{\chi} = \chi(\mathcal{T}\mathbf{S}_1) = \chi(\mathcal{T}\mathbf{B}_1), \tag{2.6}$$

where $\mathbf{S}_1$ and $\mathbf{B}_1$ are the unit sphere and the unit ball in $\mathcal{E}$, respectively. It is easy to see that

$$\|\mathcal{T}\|_{\chi} \leq \|\mathcal{T}\|_{L(X)}. \tag{2.7}$$

Definition 2.2. A continuous map $\mathcal{F} : Z \subseteq \mathcal{E} \to \mathcal{E}$ is said to be condensing with respect to a MNC β (β-condensing) if for every bounded set $\Omega \subset Z$ that is not relatively compact, we have

$$\beta(\mathcal{F}(\Omega)) \ngeq \beta(\Omega). \tag{2.8}$$

Let β be a monotone nonsingular MNC in $\mathcal{E}$. The application of the topological degree theory for condensing maps (see, e.g., [15, 16]) yields the following fixed point principles.

Theorem 2.3 (cf. [15, Corollary 3.3.1]). *Let $\mathcal{M}$ be a bounded convex closed subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ a β- condensing map. Then* $\operatorname{Fix} \mathcal{F} = \{x = \mathcal{F}(x)\}$ *is a nonempty compact set.*

Theorem 2.4 (cf. [15, Corollary 3.3.3]). *Let $\mathcal{U} \subset \mathcal{E}$ be a bounded open neighborhood of zero, and $\mathcal{F} : \overline{\mathcal{U}} \to \mathcal{E}$ a β- condensing map satisfying the following boundary condition:*

$$x \neq \lambda \mathcal{F}(x) \tag{2.9}$$

for all $x \in \partial\mathcal{U}$ and $0 < \lambda \leq 1$. Then the fixed point set $\operatorname{Fix}(\mathcal{F}) = \{x = \mathcal{F}(x)\} \subset \mathcal{U}$ *is nonempty and compact.*

Now, returning to problem (1.1)-(1.2), we impose the following assumptions for g and h:

(G1) the map $g : J \times X \to X$ is continuous;

(G2) there exist function $\mu \in L^1(J)$ and nondecreasing function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(t,\eta)\|_X \leq \mu(t)\Upsilon(\|\eta\|_X) \tag{2.10}$$

for a.e. $t \in J$ and for all $\eta \in X$;

(G3) there exists a function $k \in L^1(J)$ such that for each nonempty, bounded set $\Omega \subset X$ we have

$$\chi(g(t,\Omega)) \leq k(t)\chi(\Omega) \tag{2.11}$$

for a.e. $t \in J$, where χ is the Hausdorff MNC in X;

(H1) $h : C(J;\ X) \to X$ is a continuous function and there is a nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|h(x)\|_X \leq \Theta(\|x\|_C), \tag{2.12}$$

for all $x \in C(J;\ X)$, where $\|x\|_C = \|x\|_{C(J;X)}$;

(H2) there is a constant C_h such that

$$\chi(h(\Omega)) \leq C_h \gamma(\Omega) \tag{2.13}$$

for any bounded subset $\Omega \subset C(J; X)$, where γ is defined in (2.3).

(H3) if $\Omega \subset C(J; X)$ is a bounded set, then

$$\mathrm{mod}_C(R(\cdot)h(\Omega)) = 0. \tag{2.14}$$

Remark 2.5. (1) If X is a finite dimensional space, one can exclude the hypothesis (G3) since it can be deduced from (G2).

(2) It is known (see, e.g, [15, 16]) that condition (G3) is fulfilled if

$$g(t,\eta) = g_1(t,\eta) + g_2(t,\eta), \tag{2.15}$$

where g_1 is Lipschitz with respect to the second argument:

$$\|g_1(t,\xi) - g_1(t,\eta)\|_X \leq k(t)\|\xi - \eta\|_X \tag{2.16}$$

for a.e. $t \in J$ and $\xi, \eta \in X$ with $k \in L^1(J)$ and g_2 is compact in second argument; that is, for each $t \in J$ and bounded $\Omega \subset X$, the set $g_2(t,\Omega)$ is relatively compact in X.

(3) If we assume that h is completely continuous, that is, it is continuous and compact on bounded sets, then (H2)-(H3) will be satisfied. It is obvious that if the function h in (1.4) obeys (H1)-(H2) and function $t \mapsto R(t)$ is uniformly continuous, (H3) is also satisfied. It is worth noting that the function h given by (1.5)-(1.6) obeys (H1)–(H3).

As in [2], we assume in the sequel that

(F1) $F(t) \in L(X)$ for $t \in J$ and for $x(\cdot)$ continuous with values in $Y = D(A)$, $AF(\cdot)x(\cdot) \in L^1(J; X)$;

(F2) for each $x \in X$, the function $t \mapsto F(t)x$ is continuously differentiable on J.

It is known that under conditions (F1)-(F2), the resolvent operator for (1.8) exists. We assume, in addition, that

(HA) $t \mapsto R(t)$ is uniformly norm continuous for $t > 0$.

We define the following operator:

$$\Phi : L^1(J;X) \longrightarrow C(J;X),$$
$$\Phi(f)(t) = \int_0^t R(t-s)f(s)ds. \tag{2.17}$$

Before collecting some properties of Φ, we recall the following definitions.

Definition 2.6. A subset Q of $L^1(J;X)$ is said to be integrably bounded if there exists a function $\mu \in L^1(J)$ such that

$$\|f(t)\|_X \leq \mu(\mathrm{t}) \quad \text{for a.e. } t \in J, \tag{2.18}$$

for all $f \in Q$.

Definition 2.7. The sequence $\{\xi_n\} \subset L^1(J;X)$ is called semicompact if it is integrably bounded and the set $\{\xi_n(t)\}$ is relatively compact in X for a.e. $t \in J$.

By using hypothesis (HA) and the same arguments as those in [15, Lemma 4.2.1, Theorem 4.2.2, Proposition 4.2.1, and Theorem 5.1.1], one can verify the following properties for Φ:

(Φ1) the operator Φ sends any integrably bounded set in $L^1(J;X)$ to equicontinuous set in $C(J;X)$;

(Φ2) the following inequality holds:

$$\|\Phi(\xi)(t) - \Phi(\eta)(t)\|_X \leq C_R \int_0^t \|\xi(s) - \eta(s)\|_X ds \tag{2.19}$$

for every $\xi, \eta \in L^1(J;X)$, $t \in J$;

(Φ3) for any compact $K \subset X$ and sequence $\{\xi_{\mathrm{n}}\} \subset L^1(J;X)$ such that $\{\xi_n(t)\} \subset K$ for a.e. $t \in J$, the weak convergence $\xi_n \rightharpoonup \xi$ implies the uniform convergence $\Phi(\xi_n) \to \Phi(\xi)$;

(Φ4) if $\{\xi_{\mathrm{n}}\} \subset L^1(J;X)$ is an integrably bounded sequence and $q \in L^1(J)$ is a nonnegative function such that $\chi(\{\xi_n(t)\}) \leq q(t)$, for a.e. $t \in J$, then

$$\chi(\{\Phi(\xi_n)(t)\}) \leq 2C_R \int_0^t q(s)ds, \qquad t \in J; \tag{2.20}$$

(Φ5) if $\{\xi_n\} \subset L^1(J;X)$ is a semicompact sequence, then $\{\xi_n\}$ is weakly compact in $L^1(J;X)$ and $\{\Phi(\xi_n)\}$ is relatively compact in $C(J;X)$. Moreover, if $\xi_n \rightharpoonup \xi_0$, then $\Phi(\xi_n) \to \Phi(\xi_0)$.

Denote

$$\Phi^*(x)(t) = R(t)[x_0 - h(x)] \tag{2.21}$$

for $t \in J$ and $x \in C(J;X)$. By N_g we denote the Nemytskii operator corresponding to the nonlinearity g, that is,

$$N_g(x)(t) = g(t, x(t)) \quad \text{for } t \in J,\ x \in C(J;X). \tag{2.22}$$

We see that x is a solution of (1.1)-(1.2) if and only if

$$x = \Phi^*(x) + \Phi N_g(x). \tag{2.23}$$

Let

$$\Psi(x) = \Phi^*(x) + \Phi N_g(x). \tag{2.24}$$

Then the solutions of (1.1)-(1.2) can be considered as the fixed points of Ψ, the operator defined on $C(J;X)$.

It follows from (G1) and (H1) that Ψ is continuous on $C(J;X)$. Consider the function

$$\begin{gathered} \nu : \mathcal{P}(C(J;\ X)) \longrightarrow \mathbb{R}^2_+, \\ \nu(\Omega) = \max_{D\in\Delta(\Omega)} \big(\gamma(D), \operatorname{mod}_C(D)\big), \end{gathered} \tag{2.25}$$

where γ and mod_C are defined in (2.3) and (2.4), respectively, $\Delta(\Omega)$ denotes the collection of all countable subsets of Ω, and the maximum is taken in the sense of the ordering in the cone $\mathbb{R}^2_+$. By applying the same arguments as in [15], we have that ν is well defined. That is, the maximum is archiving in $\Delta(\Omega)$ and so ν is an MNC in the space $C(J;X)$, which satisfies all properties in Definition 2.1 (see [15, Example 2.1.3] for details).

Theorem 2.8. *Let F satisfy (F1)-(F2). Assume that conditions (G1)–(G3) and (H1)–(H3) are fulfilled. If*

$$\ell := C_R\left(C_h + 2\int_0^T k(s)ds\right) < 1, \tag{2.26}$$

then Ψ is ν-condensing.

Proof. Let $\Omega \subset C(J;X)$ be such that

$$\nu(\Psi(\Omega)) \geq \nu(\Omega). \tag{2.27}$$

We will show that Ω is relatively compact in $C(J;\ X)$. By the definition of ν, there exists a sequence $\{z_n\} \subset \Psi(\Omega)$ such that

$$\nu(\Psi(\Omega)) = \big(\gamma(\{z_n\}), \operatorname{mod}_C(\{z_n\})\big). \tag{2.28}$$

Following the construction of Ψ, one can take a sequence $\{x_n\} \subset \Omega$ such that

$$z_n = \Phi^*(x_n) + \Phi(g_n), \tag{2.29}$$

where

$$\begin{aligned} g_n(t) &= g(t, x_n(t)), \quad t \in J, \\ \Phi^*(x_n)(t) &= R(t)[x_0 - h(x_n)], \\ \Phi(g_n)(t) &= \int_0^t R(t-s) g_n(s) ds. \end{aligned} \tag{2.30}$$

Using assumption (G3), we have

$$\begin{aligned} \chi(\{g_n(s)\}) &= \chi(\{g(s, x_n(s))\}) \\ &\le k(s)\chi(\{x_n(s)\}) \\ &\le k(s)\gamma(\{x_n\}), \end{aligned} \tag{2.31}$$

for all $s \in J$. Then by (Φ4), we obtain

$$\chi(\{\Phi(g_n)(t)\}) \le 2C_R \left(\int_0^t k(s) ds \right) \gamma(\{x_n\}). \tag{2.32}$$

Noting that

$$\Phi^*(x_n)(t) = R(t)x_0 - R(t)h(x_n), \tag{2.33}$$

we have

$$\begin{aligned} \chi(\{\Phi^*(x_n)(t)\}) &= \chi(\{R(t)h(x_n)\}) \\ &\le C_R C_h \gamma(\{x_n\}) \end{aligned} \tag{2.34}$$

due to (2.5)-(2.7) and (H2). Combining (2.29), (2.31), and (2.32), we get

$$\gamma(\{z_n\}) \le \ell \gamma(\{x_n\}). \tag{2.35}$$

Combining the last inequality with (2.27), we have

$$\gamma(\{x_n\}) \le \ell \gamma(\{x_n\}), \tag{2.36}$$

and therefore

$$\gamma(\{x_n\}) = 0. \tag{2.37}$$

But then (2.35) implies

$$\gamma(\{z_n\}) = 0. \tag{2.38}$$

Putting (2.37) together with (2.31), we obtain that $\{g_n\}$ is semicompact. Hence, by (Φ5) one that has $\{\Phi(g_n)\}$ is relatively compact. This yields

$$\operatorname{mod}_C(\{\Phi(g_n)\}) = 0. \tag{2.39}$$

By (H3), we have

$$\operatorname{mod}_C(\{\Phi^*(x_n)\}) = 0. \tag{2.40}$$

Taking (2.29) into account again, we obtain

$$\operatorname{mod}_C(\{z_n\}) = 0. \tag{2.41}$$

Now it follows from (2.38)-(2.41) that

$$\nu(\Omega) = (0,0). \tag{2.42}$$

By regularity of ν, we come to the conclusion that Ω is relatively compact. □

Remark 2.9. If $R(t)$ is compact for $t > 0$, we can drop assumption (G3) in the foregoing theorem. Indeed, for any bounded sequence $\{x_n\} \subset C(J;X)$, by setting $\xi_n(t,s) = R(t-s)g(s,x_n(s))$, one sees that under hypothesis (G2), $\{\xi_n(t,\cdot)\}$ is an integrably bounded sequence in $L^1(0,t;X)$. In addition, since $R(t)$, $t > 0$, is compact, we have

$$\chi(\{\xi_n(t,s)\}) = 0, \quad \text{for a.e. } s \in [0,t]. \tag{2.43}$$

Then by [15, Corollary 4.2.5], we obtain

$$\chi\left(\left\{\int_0^t \xi_n(t,s)ds\right\}\right) = 0, \tag{2.44}$$

for each $t \in J$. By this reason, inequality (2.32) becomes

$$\chi(\{\Phi(g_n)(t)\}) = 0, \tag{2.45}$$

without the reference to (G3).

We now can formulate the existence result in the following way.

Theorem 2.10. *Under assumptions of Theorem 2.8, if one has*

$$\liminf_{r\to\infty}\frac{C_R}{r}\left(\Theta(r)+\Upsilon(r)\int_0^T\mu(s)ds\right)<1, \tag{2.46}$$

then the solution set to problem (1.1)-(1.2) *is nonempty and compact.*

Proof. We will use Theorem 2.3. Applying the results of Theorem 2.8, we only need to verify the existence of a number $r>0$ such that

$$\Psi(B_r)\subseteq B_r, \tag{2.47}$$

where B_r is the closed ball in $C(J;X)$ centered at origin with radius r. Indeed, assume to the contrary that for each $n\in\mathbb{N}\setminus\{0\}$, there is $x_n\in C(J;X)$ such that

$$\|x_n\|_C\le n,\qquad \text{but } \|\Psi(x_n)\|_c>n. \tag{2.48}$$

Recalling that

$$\Psi(x_n)(t)=R(t)[x_0-h(x_n)]+\int_0^t R(t-s)g(s,x_n(s))ds, \tag{2.49}$$

we have

$$\|\Psi(x_n)(t)\|_X\le C_R(\|x_0\|_X+\Theta(\|x_n\|_C))+C_R\int_0^t\mu(s)\Upsilon(\|x_n(s)\|_X)ds, \tag{2.50}$$

due to (H1) and (G2). Then

$$n<\|\Psi(x_n)\|_C\le C_R(\|x_0\|_X+\Theta(n))+C_R\Upsilon(n)\int_0^T\mu(s)ds. \tag{2.51}$$

Equivalently,

$$1<\frac{\|\Psi(x_n)\|_C}{n}\le\frac{1}{n}\left(C_R(\|x_0\|_X+\Theta(n))+C_R\Upsilon(n)\int_0^T\mu(s)ds\right). \tag{2.52}$$

Passing in the last inequality to the limit as $n\to+\infty$, one gets a contradiction due to assumption (2.46). Thus the proof is completed. □

We have some special cases related to the growth of Υ and Θ.

Corollary 2.11. *Assume hypotheses of Theorem 2.8, in which (G2) and (H1) are replaced by*

(G2′) $\|g(t,\eta)\|_X\le\mu(t)(1+\|\eta\|^p)$, $\mu\in L^1(J)$, $0\le p<1$, *for all* $(t,\eta)\in J\times X$;

(H1′) $h : C(J; X) \to X$ *is continuous and*

$$\|h(x)\|_X \le h_0 + h_1 \|x\|_c^q, \quad h_0, h_1 > 0, \ 0 \le q < 1, \tag{2.53}$$

for all $x \in C(J;\ X)$, *respectively.Then the solution set to problem* (1.1)-(1.2) *is nonempty and compact.*

Proof. Since $p < 1$ and $q < 1$, condition (2.46) in Theorem 2.10 is testified obviously. Then we get the conclusion. □

Corollary 2.12. *Assume hypotheses of Theorem 2.8, in which (G2) and (H1) are replaced by*

(G2″) $\|g(t,\eta)\|_X \le \mu(t)(1 + \|\eta\|)$, $\mu \in L^1(J)$, *for all* $(t,\eta) \in J \times X$;

(H1″) $h : C(J; X) \to X$ *is continuous and*

$$\|h(x)\|_X \le h_0 + h_1 \|x\|_C, \quad \textit{for some } h_0, h_1 > 0, \tag{2.54}$$

for all $x \in C(J;\ X)$, *respectively. If one has*

$$C_R \left(h_0 + \int_0^T \mu(s)ds \right) < 1, \tag{2.55}$$

then the solution set to problem (1)-(2) is nonempty and compact.

Proof. Under (G2″) and (H1″), condition (2.55) is equivalent to (2.46) and the conclusion of Theorem 2.10 holds. □

It should be mentioned that if $q = 0$ in (H1′), that is, the nonlocal function h is uniformly bounded, then one can relax the growth of Υ, by the arguments similar to those in [17].

Theorem 2.13. *Assume the hypotheses of Theorem 2.8, in which (H1) is replaced by*

(H1b) h *is a continuous function and* $\|h(x)\|_X \le M_h$ *for all* $x \in C(J; X)$, *where* M_h *is a positive constant.*

If one has

$$C_R \int_0^T \mu(s)ds < \int_{\widetilde{M}}^{\infty} \frac{dz}{\Upsilon(z)}, \tag{2.56}$$

where $\widetilde{M} = C_R(\|x_0\|_X + M_h)$, *then the solution set to problem* (1.1)-(1.2) *is nonempty and compact.*

Proof. In this case we employ Theorem 2.4. It suffices to verify the boundary condition in Theorem 2.4. We show that if $x = \lambda\Psi(x)$ for $\lambda \in (0,1]$, then x must belong to a bounded set. Indeed, suppose

$$x(t) = \lambda R(t)[x_0 - h(x)] + \lambda \int_0^t R(t-s)g(s,x(s))ds. \tag{2.57}$$

It follows that

$$\|x(t)\|_X \le C_R(\|x_0\|_X + M_h) + C_R \int_0^t \mu(s)\Upsilon(\|x(s)\|_X)ds. \tag{2.58}$$

Putting

$$v(t) = C_R(\|x_0\|_X + M_h) + C_R \int_0^t \mu(s)\Upsilon(\|x(s)\|_X)ds, \tag{2.59}$$

we have $\|x(t)\|_X \le v(t)$, for all $t \in J$, and

$$\begin{aligned} v'(t) &= C_R\mu(t)\Upsilon(\|x(t)\|_X) \\ &\le C_R\mu(t)\Upsilon(v(t)), \end{aligned} \tag{2.60}$$

due to the fact that Υ is nondecreasing. Then, by using (2.56), we have

$$\int_{\widetilde{M}}^{v(t)} \frac{dz}{\Upsilon(z)} \le C_R \int_0^T \mu(s)ds < \int_{\widetilde{M}}^{\infty} \frac{dz}{\Upsilon(z)}, \tag{2.61}$$

for all $t \in J$. The last inequalities imply that $\sup_{t\in J} v(t)$ is bounded, so is $\|x\|_C$. □

3. Continuous Dependence Result

We start with some notions from the theory of multivalued maps (see, e.g. [15] for details).

Let (Y, ϱ_Y) and (Z, ϱ_Z) be metric spaces; $\mathcal{K}(Z)$ denotes the collection of all nonempty compact subsets of Z. A multivalued map (multimap) $G : Y \to \mathcal{K}(Z)$ is said to be (i) *upper semicontinuous (u.s.c.)* if for each $y \in Y$ and $\epsilon > 0$ there exists $\delta = \delta(y,\epsilon) > 0$ such that condition $\varrho_Y(y,y') < \delta$ implies $G(y') \subset U_\epsilon(G(y))$, where $U_\epsilon(G(y))$ denotes the ϵ-neighborhood of the set $G(y)$ induced by the metric ϱ_Z; (ii) *closed* if its graph $\{(y,z) \in Y \times Z : z \in G(y)\}$ is a closed subset of $Y \times Z$; (iii) *compact* if $G(Y)$ is relatively compact in Z; (iv) *quasicompact* if its restriction to any compact set is compact.

The following assertion gives a sufficient condition for upper semicontinuity.

Lemma 3.1 (see[15]). *Let $G : Y \to \mathcal{K}(Z)$ be a closed quasicompact multimap. Then G is u.s.c.*

Consider the solution multimap

$$\begin{gathered} W: X \multimap C(J;X), \\ W(v) = \{x : x \text{ is a solution of (1.1)-(1.2) with initial value } x_0 = v\}. \end{gathered} \tag{3.1}$$

Notice that, as we demonstrated previously, under conditions of our existence theorems, the solution multimap W has compact values. We will study the continuity properties of W.

Theorem 3.2. *Under the assumptions of Theorem 2.10, the solution map W defined in* (3.1) *is u.s.c.*

Proof. We first prove that W is a quasicompact multimap. Let $Q \subset X$ be a compact set. We will show that $W(Q)$ is relatively compact in $C(J;X)$. Suppose that $\{x_n\} \subset W(Q)$. Then there exists a sequence $\{v_n\} \subset Q$ such that

$$x_n(t) = R(t)v_n - R(t)h(x_n) + \Phi g_n(t), \tag{3.2}$$

where $g_n(t) = g(t, x_n(t))$.

Notice that the sequence $\{x_n\}$ is bounded. In fact, from (3.2) we have the estimate

$$\|x_n\|_c \leq C_R(\|v_n\|_x + \Theta(\|x_n\|_c)) + C_R\Upsilon(\|x_n\|_c)\int_0^T \mu(s)ds. \tag{3.3}$$

Supposing to the contrary that the sequence $\|x_n\|_C$ is unbounded, by dividing the last inequality over $\|x_n\|_C$ and using condition (2.46) and the boundedness of the sequence $\{v_n\}$, we get a contradiction.

Since $\{v_n\}$ is relatively compact, we obtain from (3.2) that

$$\chi(\{x_n(t)\}) \leq \chi(\{R(t)h(x_n)\}) + \chi\big(\{\Phi g_n(t)\}\big). \tag{3.4}$$

Using (G3) we have

$$\chi\big(\{g_n(s)\}\big) \leq k(s)\chi(\{x_n(s)\}) \leq k(s)\gamma(\{x_n\}) \tag{3.5}$$

for all $s \in J$. Referring to (Φ4), one gets

$$\chi\big(\{\Phi(g_n)(s)\}\big) \leq 2C_R\left(\int_0^t k(s)ds\right)\gamma(\{x_n\}), \tag{3.6}$$

and then

$$\gamma\big(\{\Phi(g_n)\}\big) \leq 2C_R\left(\int_0^T k(s)ds\right)\gamma(\{x_n\}). \tag{3.7}$$

On the other hand, by (H2) one has

$$\chi(\{R(t)h(x_n)\}) \leq C_R C_h \gamma(\{x_n\}). \tag{3.8}$$

Combining the last inequality with (3.4)-(3.7), we have

$$\gamma(\{x_n\}) \leq \ell\gamma(\{x_n\}). \tag{3.9}$$

This leads to the conclusion that $\gamma(\{x_n\}) = 0$.

Now, condition (G2) implies that $\{g_n\}$ is integrably bounded in $L^1(J;X)$. Thus (Φ1) ensures that $\{\Phi(g_n)\}$ is equicontinuous. Then applying condition (H3), we obtain

$$\mathrm{mod}_C(\{x_n\}) \leq \mathrm{mod}_C(\{\Phi^*(x_n)\}) + \mathrm{mod}_C\left(\{\Phi(g_n)\}\right) = 0. \tag{3.10}$$

So we have $\nu(\{x_n\}) = (0,0)$ and therefore $\{x_n\}$ is relatively compact in $C(J;X)$.

In order to prove that W is u.s.c., it remains, according to Lemma 3.1, to show that W is closed. Let $v_n \to v$ in X and $x_n \in W(v_n), x_n \to x$ in $C(J;X)$. We claim that $x \in W(v)$. Indeed, one has

$$x_n(t) = \Phi^*(x_n)(t) + \int_0^t R(t-s)g(s,x_n(s))ds. \tag{3.11}$$

We first observe that

$$\Phi^*(x_n) = R(\cdot)[v_n - h(x_n)] \longrightarrow R(\cdot)[v - h(x)] = \Phi^*(x) \tag{3.12}$$

in $C(J;X)$ in accordance with (H1). In addition, since g is a continuous function, we have $g(s,x_n(s)) \to g(s,x(s))$ a.e. $s \in J$. The Lebesgue dominated convergence theorem implies that

$$g(\cdot,x_n(\cdot)) - g(\cdot,x(\cdot)) \longrightarrow 0 \quad \text{in } L^1(J;X) \tag{3.13}$$

due to the fact that $\{g(\cdot,x_n(\cdot))\}$ is integrably bounded. Therefore, taking (3.11) into account, we arrive at

$$x(t) = \Phi^*(x)(t) + \int_0^t R(t-s)g(s,x(s))ds, \qquad t \in J. \tag{3.14}$$

The proof is completed. □

4. Lipschitz Assumption for the Function from Nonlocal Condition

4.1. Existence Result

In this section, we assume that h is a Lipschitz function.

(H2′) There exists a constant $h_0 > 0$ such that

$$\|h(x) - h(y)\|_X \leq h_0\|x - y\|_C. \tag{4.1}$$

This implies the growth of h:

$$\|h(x)\|_X \leq h_0\|x\|_C + \|h(0)\|_X, \tag{4.2}$$

and the last inequality covers (H1).

Let χ_C be the Hausdorff MNC in $C(J;\ X)$. We have

$$\begin{aligned}\left\|\Phi^*(x)(t) - \Phi^*(y)(t)\right\|_X &= \left\|R(t)\left[h(x) - h(y)\right]\right\|_X \\ &\leq C_R h_0\left\|x - y\right\|_C,\end{aligned} \tag{4.3}$$

for all $t \in J$, where Φ^* is given in (2.21). Thus

$$\left\|\Phi^*(x) - \Phi^*(y)\right\|_C \leq C_R h_0\left\|x - y\right\|_C. \tag{4.4}$$

Then we know that (see [15]) condition (H2′) implies

$$\chi_C(\Phi^*(\Omega)) \leq C_R h_0 \chi_C(\Omega), \tag{4.5}$$

for any bounded set $\Omega \subset C(J;\ X)$. We recall the following facts, which will be used in the sequel: for each bounded set $\Omega \subset C(J;\ X)$, one has the following:

(i) $\chi(\Omega(t)) \leq \chi_C(\Omega)$, for all $t \in J$;

(ii) if Ω is an equicontinuous set ($\mathrm{mod}_C(\Omega) = 0$), then

$$\chi_C(\Omega) = \sup_{t\in J}\chi(\Omega(t)) \ \left(= \gamma(\Omega)\right). \tag{4.6}$$

Theorem 4.1. *Assume that g satisfies (G1)–(G3) and h obeys (H2′). If the following relations*

$$\ell := C_R\left(h_0 + 2\int_0^T k(s)ds\right) < 1, \tag{4.7}$$

$$C_R h_0 + C_R\int_0^T \mu(s)ds \liminf_{r\to\infty}\frac{\Psi(r)}{r} < 1, \tag{4.8}$$

hold true, then problem (1.1)-(1.2) *has at least one solution.*

Proof. As we know from the proof of Theorem 2.10, condition (4.8) implies that there exists a ball $B_r \subset C(J, X)$, $r > 0$, such that

$$\Psi(B_r) \subset B_r. \tag{4.9}$$

To apply Theorem 2.3, we verify that Ψ is χ_C-condensing. Let $\Omega \subset C(J; X)$ be a bounded set satisfying the inequality

$$\chi_C(\Psi(\Omega)) \geq \chi_C(\Omega). \tag{4.10}$$

We will show that Ω is relatively compact. Notice that

$$\Psi(\Omega) \subset \Phi^*(\Omega) + \Phi \circ N_g(\Omega), \tag{4.11}$$

where

$$\begin{gathered}\Phi^*(\Omega)(t) = R(t)[x_0 - h(\Omega)], \\ \Phi \circ N_g(\Omega)(t) = \left\{ \int_0^t R(t-s) g(s, y(s)) ds : y \in \Omega \right\}.\end{gathered} \tag{4.12}$$

Then we have

$$\chi_C(\Psi(\Omega)) \le \chi_C(\Phi^*(\Omega)) + \chi_C\big(\Phi \circ N_g(\Omega)\big). \tag{4.13}$$

The boundedness of Ω in $C(J;X)$ implies that $N_g(\Omega)$ is a bounded set in $L^1(J;X)$. By (Φ1), the set $\Phi \circ N_g(\Omega)$ is equicontinuous and therefore we have

$$\begin{aligned}\chi_C\big(\Phi \circ N_g(\Omega)\big) &= \sup_{t\in J} \chi\big(\Phi \circ N_g(\Omega)(t)\big) \\ &\le \sup_{t\in J} 2C_R \int_0^t k(s)\chi(\Omega(s))ds,\end{aligned} \tag{4.14}$$

due to (G3) and (Φ4). Thus

$$\chi_C\big(\Phi \circ N_g(\Omega)\big) \le 2C_R \chi_C(\Omega) \int_0^T k(s)ds. \tag{4.15}$$

Combining (4.5), (4.13), and (4.15), we obtain

$$\chi_C(\Psi(\Omega)) \le C_R\left(h_0 + 2\int_0^T k(s)ds \right)\chi_C(\Omega). \tag{4.16}$$

Relations (4.7) and (4.10) yield

$$\chi_C(\Omega) \le \ell \chi_C(\Omega). \tag{4.17}$$

Since $\ell < 1$, we have $\chi_C(\Omega) = 0$. The regularity of χ_C ensures that Ω is relatively compact. □

Remark 4.2. (1) Assumption (H2$'$) allows us to drop (H1)–(H3).

(2) As indicated in Remark 2.9, in the case when $R(t)$ is compact for $t > 0$, condition (G3) can be dropped and condition (4.7) is reduced to

$$C_R h_0 < 1, \tag{4.18}$$

which is covered by (4.8). Recall that in this case we have

$$\chi\big(\Phi \circ N_g(\{x_n\})(t)\big) = 0 \tag{4.19}$$

for any bounded sequence $\{x_n\} \subset C(J; X)$ and for all $t \in J$.

4.2. The Structure of the Solution Set

We are in a position to study the structure of the solution set to (1.1)-(1.2) under the hypotheses of Theorem 4.1 and the assumption that $R(t),\ t > 0$, is compact. At first, let us recall some notions.

Definition 4.3. A subset B of a metric space Y is said to be contractible in Y if the inclusion map $i_B : B \to Y$ is null-homotopic; that is, there exist $y_0 \in Y$ and a continuous map $h : B \times [0,1] \to Y$ such that $h(y, 0) = y$ and $h(y, 1) = y_0$ for every $y \in B$.

The following notion [18] is important for our purposes.

Definition 4.4. Let Y be a metric space; a subset $B \subset Y$ is called an R_δ-set if B can be represented as the intersection of a decreasing sequence of compact contractible sets.

The next lemma gives us a tool for the recognition of R_δ-set.

Lemma 4.5 (see [19]). *Let X be a metric space, E a Banach space, and $V : X \to E$ a proper map; that is, V is continuous and $V^{-1}(K)$ is compact for each compact set $K \subset E$. Suppose that there exists a sequence $\{V_n\}$ of mappings from X into E such that*

(1) *V_n is proper and $\{V_n\}$ converges to V uniformly on X;*

(2) *for a given point $y_0 \in E$ and for all y in a neighborhood $\mathcal{N}(y_0)$ of y_0 in E, there exists exactly one solution x_n of the equation $V_n(x) = y$.*

Then $V^{-1}(y_0)$ is an R_δ-set.

In order to use this lemma, we need the following result, which is called Lasota-Yorke Approximation Theorem (see, e.g. [20]).

Lemma 4.6. *Let E be a normed space, X a metric space, and $f : X \to E$ a continuous map. Then for each $\epsilon > 0$, there is a locally Lipschitz map $f_\epsilon : X \to E$ such that*

$$\left\| f_\epsilon(x) - f(x) \right\|_E < \epsilon, \quad \text{for every } x \in X. \tag{4.20}$$

We now can formulate the main result of this section.

Theorem 4.7. *Assume that g satisfies (G1)-(G2) and h obeys (H2'). If $R(t)$ is compact for $t > 0$ and*

$$C_R h_0 + C_R \int_0^T \mu(s) ds \liminf_{r \to \infty} \frac{\Upsilon(r)}{r} < 1, \tag{4.21}$$

then the solution set of problem (1.1)-(1.2) *is an R_δ-set.*

Proof. By Theorem 4.1 and Remark 4.2, the hypotheses of Theorem 4.7 provide the existence result, that is,

$$\mathrm{Fix}(\Psi) \neq \emptyset. \tag{4.22}$$

We will show that $\mathrm{Fix}(\Psi)$ is an R_δ-set.

Consider the nonlinearity g. By Lemma 4.6, there exists a sequence of functions $\{g_n\}$ such that

(i) $g_n : J \times X \to X$ is continuous and locally Lipschitz map;

(ii) $\|g_n(t,\eta) - g(t,\eta)\|_X < \epsilon_n$ for all $(t,\eta) \in J \times X$, where $\epsilon_n \to 0$ as $n \to \infty$.

One can assume, without loss of generality, that

$$\|g_n(t,\eta)\|_X \le \mu(t)\Upsilon(\|\eta\|_X) + 1 \tag{4.23}$$

for all $(t,\eta) \in J \times X$ and $n \ge 1$. Let us consider the following equation:

$$x(t) = y(t) + R(t)[x_0 - h(x)] + \int_0^t R(t-s)g_n(s,x(s))ds, \quad t \in J, \tag{4.24}$$

where $y \in C(J;X)$ is a given function. Define $\Psi_n : C(J;X) \to C(J;X)$ by

$$\Psi_n(x)(t) = y(t) + R(t)[x_0 - h(x)] + \int_0^t R(t-s)g_n(s,x(s))ds, \quad t \in J. \tag{4.25}$$

By applying the same arguments as in the proof of Theorem 4.1 and in Remark 4.2, we can see that Ψ_n is χ_C-condensing. In addition, using the similar estimates as in Theorem 2.10, one can find a ball B_r, $r > 0$, such that

$$\Psi_n(B_r) \subset B_r, \tag{4.26}$$

due to (4.23). Therefore, Ψ_n has a fixed point due to Theorem 2.3 and then (4.24) has at least one solution. Moreover, since $h(\cdot)$ is Lipschitzian and $g_n(t,\cdot)$ is a locally Lipschitz function, the solution to (4.24) is unique.

Now by setting

$$\begin{aligned} V_n(x)(t) &= x(t) - \left[R(t)(x_0 - h(x)) + \int_0^t R(t-s)g_n(s,x(s))ds\right], \\ V(x)(t) &= x(t) - \left[R(t)(x_0 - h(x)) + \int_0^t R(t-s)g(s,x(s))ds\right], \quad t \in J, \end{aligned} \tag{4.27}$$

we see that $\{V_n\}$ converges to V uniformly on $C(J;X)$. In addition, for a given $y \in C(J;X)$ the equation

$$V_n(x) = y \tag{4.28}$$

has a unique solution, which is the fixed point of Ψ_n mentioned previously.

We now show that V and V_n are proper. We proceed with V_n; the proof for V is similar. Obviously, V_n is continuous. Let $K \subset C(J;X)$ be a compact set and $V_n(\Omega) = K$. We claim that Ω is a compact set in $C(J;\ X)$. Since V_n is continuous and K is closed, we deduce that Ω is closed. Assume that $\{x_j\}$ is a sequence in Ω, then one can take a sequence $\{y_j\} \subset K$ such that

$$V_n(x_j) = y_j. \tag{4.29}$$

Equivalently,

$$x_j(t) = y_j(t) + R(t)\left[x_0 - h(x_j)\right] + \int_0^t R(t-s) g_n(s, x_j(s))\, ds, \qquad t \in J. \tag{4.30}$$

We first show that the sequence $\{x_j\}$ is bounded. We have

$$\left\|x_j(t)\right\|_X \le \left\|y_j(t)\right\|_X + C_R\left(\|x_0\|_x + h_0\left\|x_j\right\|_C + \|h(0)\|_X\right) + C_R\int_0^t \left(\mu(s)\Upsilon\left(\left\|x_j(s)\right\|_X\right) + 1\right) ds, \tag{4.31}$$

due to (H2$'$) and (4.23). Thus

$$\left\|x_j\right\|_C \le \left\|y_j\right\|_C + C_R(\|x_0\|_X + \|h(0)\|_X) + TC_R + C_R h_0\left\|x_j\right\|_C + C_R\Upsilon\left(\left\|x_j\right\|_C\right)\int_0^T \mu(s)\, ds. \tag{4.32}$$

If $\{x_j\}$ is unbounded, then there is a subsequence (still denoted by $\{x_j\}$) such that $\|x_j\|_C \to +\infty$ as $j \to +\infty$. Now from the last inequality, it follows that

$$\begin{aligned} 1 \le{} & \frac{1}{\left\|x_j\right\|_C}\left(\left\|y_j\right\|_C + C_R(\|x_0\|_X + \|h(0)\|_X) + TC_R\right) \\ & + C_R h_0 + C_R\int_0^T \mu(s)\, ds \frac{\Upsilon\left(\left\|x_j\right\|_C\right)}{\left\|x_j\right\|_C}. \end{aligned} \tag{4.33}$$

Passing in the last inequality to limits as $j \to +\infty$, one gets a contradiction due to the hypotheses of the Theorem. Now from (4.30), we have

$$\chi_C(\{x_j\}) \le \chi_C(\{y_j\}) + \chi_C(R(\cdot)h(\{x_j\})) + \chi_C(\Phi \circ N_{g_n}(\{x_j\})). \tag{4.34}$$

Using the same arguments as in the proof of Theorem 4.1 and Remark 4.2, we obtain that

$$\begin{aligned} \chi_C(\Phi \circ N_{g_n}(\{x_j\})) &= \sup_{t \in J} \chi(\Phi \circ N_{g_n}(\{x_j\})(t)) = 0, \\ \chi_C(R(\cdot)h(\{x_j\})) &= \sup_{t \in J} \chi(R(t)h(\{x_j\})) \le C_R h_0 \chi_C(\{x_j\}). \end{aligned} \tag{4.35}$$

Substituting the last inequalities into (4.34) and using the fact that $\{y_j\}$ is a compact sequence, we obtain

$$\chi_C(\{x_j\}) \le C_R h_0 \chi_C(\{x_j\}). \tag{4.36}$$

Noting that $C_R h_0 < 1$, we deduce $\chi_C(\{x_j\}) = 0$. Therefore, $\{x_j\}$ is a relatively compact sequence in $C(J;X)$ and we arrive at the conclusion that Ω is compact and then V_n is proper.

Finally, by the observation that $\operatorname{Fix}\Psi = V^{-1}(0)$, from Lemma 4.5, we obtain that $\operatorname{Fix}\Psi$ is an R_δ-set. □

Remark 4.8. The topological structure of the solution set of problem (1.1)-(1.2) for the case of a non-compact resolvent $R(t)$ is an open problem.

Further Remarks

Some additional remarks can be given in the case when $R(t),\ t > 0$, is compact. Following the technique presented in [8], we can consider the following problem:

$$x'(t) = A\left[x(t) + \int_0^t F(t-s)x(s)ds\right] + g(t,x(t)), \quad t \in J := [0,T], \tag{4.37}$$

$$x(0) + h_n(x) = x_0, \tag{4.38}$$

where $h_n(x) = R(1/n)h(x),\ n \in \mathbb{N} \setminus \{0\}$. Since h is continuous, h_n is a completely continuous function. Then it satisfies (H2)-(H3). Therefore, under the assumptions (G1)–(G3), and (H1) and (2.46), problem (4.37)-(4.38) has at least one mild solution $x_n \in C(J;X)$. Furthermore,

$$\begin{aligned}\chi(\Phi N_g(\Omega)(t)) &= \chi\left(\int_0^t R(t-s)g(s,\Omega(s))ds\right)\\ &\le 2\int_0^t \chi(R(t-s)g(s,\Omega(s)))ds = 0\end{aligned} \tag{4.39}$$

for all bounded subset $\Omega \subset C(J;\ X)$. Thus, one can drop assumption (G3), and then the solution operator for (4.37)-(4.38)

$$\Psi_n(x)(t) = R(t)[x_0 - h_n(x)] + \int_0^t R(t-s)g(s,x(s))ds \tag{4.40}$$

is ν-condensing without assuming the condition

$$C_R\left(C_h + 2\int_0^T k(s)ds\right) < 1. \tag{4.41}$$

In fact, we have the following assertion.

Theorem 4.9. *Let $R(t)$ be compact for $t > 0$. Under assumptions (G1)-(G2) and (H1) and* (2.46)*, the solution set of problem* (4.37)-(4.38) *is a nonempty compact set.*

Using the same arguments as in [8], one can prove that the sequence $\{x_n\}$ of solutions to (4.37)-(4.38) is relatively compact. Finally, passing to the limit as $n \to +\infty$ in the equation

$$x_n(t) = R(t)x_0 - R(t)R\left(\frac{1}{n}\right)h(x_n) + \int_0^t R(t-s)g(s, x_n(s))ds, \tag{4.42}$$

we obtain the solution of problem (1.1)-(1.2).

5. Example

We conclude this note with an example, in which we find a representation for the resolvent operator generated by the linear part and impose suitable conditions to demonstrate the existence result and the structure of the solution set. Precisely, consider the following system:

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left[u(x,t) + \int_0^t a(t-s)u(x,s)ds\right] + \int_0^x f(t, y, u(y,t))dy, \quad x \in (0,\pi),\ t \in (0,b], \tag{5.1}$$

$$u(x,0) + \sum_{i=1}^m \alpha_i u(x,t_i) = u_0, \qquad t_i \in (0,b), \tag{5.2}$$

$$u(0,t) = u(\pi,t) = 0. \tag{5.3}$$

Let $X = L^2(0;\pi)$,

$$D(A) = \{z \in X : \ z \text{ and } z' \text{ are absolutely continuous}, \quad z(0) = z(\pi) = 0\}, \tag{5.4}$$

and $A : D(A) \to X$, $Az = z''$. Then it is known that A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on X. Recall that the functions

$$\left\{\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx : n = 1, 2, \ldots\right\} \tag{5.5}$$

form an orthonormal basis in X and they are the eigenfunctions corresponding to the eigenvalues $\{\lambda_n = n^2 : n = 1, 2, \ldots\}$ of $-A$.

We are in a position to consider the linear part of (5.1)–(5.3):

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left[u(x,t) + \int_0^t a(t-s)u(x,s)ds\right], \quad x \in [0,\pi],\ t \in [0,b], \tag{5.6}$$

$$u(x,0) = u_0, \tag{5.7}$$

$$u(0,t) = u(\pi,t) = 0. \tag{5.8}$$

Assume that $a \in C^1(0;\ b)$ and $u_0 = \sum_{n=1}^{\infty} \gamma_n \phi_n$. We are searching for the resolvent operator generated by (5.6)–(5.8) in the following form:

$$(R(t)u_0)(x) = u(x,t) = \sum_{n=1}^{\infty} \gamma_n T_n(t)\phi_n(x), \tag{5.9}$$

where T_n is the solution of the equation

$$T_n'(t) = -n^2\left[T_n(t) + \int_0^t a(t-s)T_n(s)ds\right] \tag{5.10}$$

subject to $T_n(0) = 1$. We know that

$$\widehat{T_n}(\lambda) = \frac{1}{\lambda + n^2(1+\hat{a}(\lambda))}, \tag{5.11}$$

where $\widehat{T_n}$ and $\hat{a}$ are the Laplace transforms of T_n and a, respectively. For the simple case, when a is constant, $a < 0$, we have

$$\widehat{T_n}(\lambda) = \frac{\lambda}{\lambda^2 + n^2\lambda + n^2 a}. \tag{5.12}$$

By some computations, one gets

$$T_n(t) = \frac{p(n)}{2p(n)+n^2}e^{p(n)t} + \frac{q(n)}{2q(n)+n^2}e^{q(n)t}, \tag{5.13}$$

where $p(n) = (1/2)(-n^2 + n\sqrt{n^2-4a})$ and $q(n) = (1/2)(-n^2 - n\sqrt{n^2-4a})$.

Taking assumption (HA) into account, we conclude that T_n needs to satisfy the following condition:

$$|T_n(t) - T_n(s)| \le c_n|t-s|, \quad \text{for } t,s \in (0,b], \quad \sum_{n=1}^{\infty} c_n^2 < +\infty. \tag{5.14}$$

This condition is obviously fulfilled for T_n given by (5.13).

We now verify the compactness of $R(t)$ for $t > 0$. Since the embedding $H_0^1(0;\pi) \subset X$ is compact, it is sufficient to find a condition providing that the set

$$\frac{\partial}{\partial x} R(t) B_X(0,r) = \left\{ \frac{\partial}{\partial x} R(t) v : \ \|v\|_X \le r \right\} \tag{5.15}$$

is bounded in X. It is easy to verify that this condition follows from

$$\sum_{n=1}^{\infty} |n T_n(t)|^2 < +\infty, \qquad \text{for } t > 0. \tag{5.16}$$

The last inequality also holds for T_n given by (5.13).

As far as nonlinear problem (5.1)–(5.3) is concerned, we see that the nonlocal function $h(u)(x) = \sum_{i=1}^{m} \alpha_i u(x, t_i)$ is a Lipschitz function with the constant $h_0 = \sum_{i=1}^{m} \alpha_i$. Let $g(t,v)(x) = \int_0^x f(t, y, v(y)) dy$. Then the nonlinearity g satisfies (G1)-(G2) if we assume that $f : [0,b] \times [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\mu \in L^1(0;b)$ such that

$$\left| f(t,y,\eta) \right| \le \mu(t)\left(1 + |\eta|\right), \qquad \forall t \in [0,b], \ y \in [0,\pi], \ \eta \in \mathbb{R}. \tag{5.17}$$

As indicated in [16], the Hausdorff MNC of a bounded set $\Omega \subset L^2(0;\pi)$ can be expressed by

$$\chi(\Omega) = \frac{1}{2} \lim_{\delta \to 0} \sup_{v \in \Omega} \max_{0 \le \omega \le \delta} \|v - v_\omega\|_{L^2(0;\pi)} , \tag{5.18}$$

where v_ω denote the ω-translation of v:

$$v_\omega(x) = \begin{cases} v(\omega + x), & 0 \le x \le \pi - \omega, \\ v(\pi), & \pi - \omega \le x \le \pi, \end{cases} \tag{5.19}$$

or, alternatively, the Steklov function:

$$v_\omega(x) = \frac{1}{2\omega} \int_{x-\omega}^{x+\omega} v(z) dz \tag{5.20}$$

(v is extended outside of $[0,\pi]$ by zero). Therefore, condition (G3) is provided by the following inequality:

$$\left| \int_x^{x+\omega} f(t, y, v(y)) dy \right| \le k(t) |v(x) - v_\omega(x)|, \qquad t \in [0,b]; \ x, \omega \in [0,\pi], \tag{5.21}$$

for a nonnegative function $k \in L^1(0; \ b)$.

Applying Theorem 4.1, we obtain that problem (5.1)-(5.2) has at least one solution if (5.14), (5.17), and (5.21) take place together with the following estimates:

$$C_R\left(h_0 + 2\int_0^b k(s)ds\right) < 1,$$
$$C_R\left(h_0 + \int_0^b \mu(s)ds\right) < 1. \tag{5.22}$$

If we assume that (5.16) holds and hence $R(t)$ is compact for $t > 0$, then the solution set of (5.1)–(5.3) is an R_δ-set if (5.17) holds and

$$C_R\left(h_0 + \int_0^b \mu(s)ds\right) < 1, \tag{5.23}$$

due to Theorem 4.7.

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