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Research Article

Analytic Solutions for a Functional Differential Equation Related to a Traffic Flow Model

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We study the existence of analytic solutions of a functional differential equation $(z(s)+\alpha)^2 z'(s) = \beta(z(s+z(s)) - z(s))$ which comes from traffic flow model. By reducing the equation with the Schröder transformation to an auxiliary equation, the author discusses not only that the constant λ at resonance, that is, at a root of the unity, but also those λ near resonance under the Brjuno condition.

1. Introduction

Traffic flow models have found much attention [1–13] in the last few years. They mostly fall into two types: one is "macroscopic" which was introduced by Aw and Rascle [14], and Zhang [13], we also refer the reader to [2–5, 9, 12], and the other is called "microscopic" which has been discussed in [8, 10, 11, 15]. In particular, Illner et al. [4, 7] investigated kinetic models which can be seen as a bridge between macroscopic and microscopic models.

Recently, Illner and McGregor [6] studied

$$(z(s)+\alpha)^2 z'(s) = \beta(z(s+z(s)) - z(s)), \tag{1.1}$$

where α, β are positive parameters arising from a nonlocal traffic flow model in a travelling wave approximation. Analytical and numerical studies of (1.1) exist, in particular on the existence and properties of nonconstant travelling wave solutions.

In this paper, we prove the existence of analytic solutions for (1.1) by locally reducing the equation to another functional differential equation, which we called auxiliary equation.

In fact, if we let $g(s) = s + z(s)$, then $z(s) = g(s) - s$, and (1.1) can be written in

$$\left(g(s) - s + \alpha\right)^2 \left(g'(s) - 1\right) = \beta\left(g(g(s)) - 2g(s) + s\right). \tag{1.2}$$

As in [16, 17], we reduce (1.2) with $g(s) = h(\lambda h^{-1}(s))$ to the auxiliary equation

$$(h(\lambda s) - h(s) + \alpha)^2 \left(h'(\lambda s) - h'(s)\right) = \beta h'(s)\left(h\left(\lambda^2 s\right) - 2h(\lambda s) + h(s)\right), \quad s \in \mathbb{C}. \tag{1.3}$$

If we can prove the existence of analytic solutions for (1.3), then the analytic solutions of (1.1) can be obtained.

Throughout this paper, we will assume that $\alpha, \beta > 0$, and λ in (1.3) satisfies one of the following conditions:

(C1) $0 < |\lambda| < 1$;

(C2) $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and θ is a Brjuno number [18, 19]: $B(\theta) = \sum_{n=0}^{\infty}(\log q_{n+1}/q_n) < \infty$, where $\{p_n/q_n\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ;

(C3) $\lambda = e^{2\pi i q/p}$ for some integer $p \in \mathbb{N}$ with $p \geq 2$ and $q \in \mathbb{Z} \setminus \{0\}$, and $\lambda \neq e^{2\pi i \xi/v}$ for all $1 \leq v \leq p-1$ and $\xi \in \mathbb{Z} \setminus \{0\}$.

We observe that λ is inside the unit circle S^1 in case (C1) but on S^1 in the rest of cases. More difficulties are encountered for λ on S^1 since the small divisor $\lambda^n - 1$ is involved in the latter (2.24). Under Diophantine condition, "$\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and there exist constants $\zeta > 0$ and $\delta > 0$ such that$|\lambda^n - 1| \geq \zeta^{-1} n^{-\delta}$ for all $n \geq 1$," the number $\lambda \in S^1$ is "far" from all roots of the unity. Since then, we have been striving to give a result of analytic solutions for those λ "near" a root of the unity, that is, neither being roots of the unity nor satisfying the Diophantine condition. The Brjuno condition in (C2) provides such a chance for us. Moreover, we also discuss the so-called resonance case, that is, the case of (C3).

2. Analytic Solutions of the Auxiliary Equation

In this section, we discuss local invertible analytic solutions of (1.3) with the initial condition

$$h(s) = \sum_{n=1}^{\infty} a_n s^n, \quad h(0) = 0,\ h'(0) = \eta \neq 0,\ \eta \in \mathbb{C}. \tag{2.1}$$

Lemma 2.1. *Equation* (1.3) *has a formal solution of the form*

$$h(s) = \eta s + \sum_{n=2}^{\infty} a_n s^n, \tag{2.2}$$

where η is as in (2.1).

Proof. If we let $h(s) = \sum_{n=1}^{\infty} a_n s^n$ and substituting into (1.3), we have

$$\begin{aligned}
&\alpha^2 \sum_{n=0}^{\infty} (n+1)(\lambda^n - 1) a_{n+1} s^n \\
&\quad + 2\alpha \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} (i+1) \left(\lambda^{n-i} - 1\right) \left(\lambda^{i} - 1\right) a_{n-i} a_{i+1} \right) s^n \\
&\quad + \sum_{n=2}^{\infty} \left(\sum_{i=0}^{n-2} \sum_{k=1}^{n-i-1} (i+1) \left(\lambda^{n-k-i} - 1\right) \left(\lambda^{k} - 1\right) \left(\lambda^{i} - 1\right) a_{n-k-i} a_k a_{i+1} \right) s^n \\
&= \beta \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} (i+1) \left(\lambda^{n-i} - 1\right)^2 a_{n-i} a_{i+1} \right) s^n.
\end{aligned} \tag{2.3}$$

Comparing coefficients we obtain

$$\alpha^2 \left(\lambda^0 - 1\right) a_1 = 0, \tag{2.4}$$

$$\begin{aligned}
&\alpha^2 (n+1)(\lambda^n - 1) a_{n+1} \\
&\quad = \beta \sum_{i=0}^{n-1} (i+1) \left(\lambda^{n-i} - 1\right)^2 a_{n-i} a_{i+1} \\
&\qquad - 2\alpha \sum_{i=0}^{n-1} (i+1) \left(\lambda^{n-i} - 1\right) \left(\lambda^{i} - 1\right) a_{n-i} a_{i+1} \\
&\qquad - \sum_{i=0}^{n-2} \sum_{k=1}^{n-i-1} (i+1) \left(\lambda^{n-k-i} - 1\right) \left(\lambda^{k} - 1\right) \left(\lambda^{i} - 1\right) a_{n-k-i} a_k a_{i+1}, \quad n \geq 1.
\end{aligned} \tag{2.5}$$

Then for arbitrarily chosen $a_1 = \eta \neq 0$, the sequence $\{a_n\}_{n=2}^{\infty}$ is successively determined by (2.5) in a unique manner. □

This shows that (1.3) has a formal power series solution of the form (2.2).

Theorem 2.2. *Suppose that (C1) holds, then* (1.3) *in a neighborhood of the origin has an analytic solution of the form* (2.2).

Proof. From (C1), we have

$$\lim_{n \to \infty} \frac{1}{|\lambda^n - 1|} = 1. \tag{2.6}$$

There exists $L > 0$ such that $1/|\lambda^n - 1| \leq L$, for all $n \geq 1$. It follows from (2.5) that

$$|a_{n+1}| \leq \frac{8L}{\alpha^2} \left[(\alpha + \beta) \sum_{i=0}^{n-1} |a_{n-i}||a_{i+1}| + \sum_{i=0}^{n-2} \sum_{k=1}^{n-i-1} |a_{n-k-i}||a_k||a_{i+1}| \right] \tag{2.7}$$

for $n \geq 1$.

We consider the implicit function equation:

$$B(s) = |\eta| s + \frac{8L}{\alpha^2}\left[(\alpha+\beta)B^2(s) + B^3(s)\right]. \tag{2.8}$$

Define the function

$$\Theta(s,\omega;L,\eta,\alpha,\beta) = |\eta| s - \omega + \frac{8L}{\alpha^2}\left[(\alpha+\beta)\omega^2 + \omega^3\right] \tag{2.9}$$

for (s,ω) from a neighborhood of $(0,0)$, then the function $B(s)$ satisfies

$$\Theta(s,B(s);L,\eta,\alpha,\beta) = 0. \tag{2.10}$$

In view of $\Theta(0,0;L,\eta,\alpha,\beta) = 0$,

$$\Theta'_\omega(0,0;L,\eta,\alpha,\beta) = -1 \neq 0, \tag{2.11}$$

and the implicit function theorem, there exists a unique function $\Phi(s)$, analytic in a neighborhood of zero, such that

$$\Phi(0) = 0, \qquad \Phi'(0) = -\frac{\Theta'_s(0,0;L,\eta,\alpha,\beta)}{\Theta'_\omega(0,0;L,\eta,\alpha,\beta)} = |\eta|, \tag{2.12}$$

and $\Theta(s,\Phi(s);L,\eta,\alpha,\beta) = 0$. According to (2.10), we have $B(s) = \Phi(s)$.

If we assume that the power series expansion of $B(s)$ is as follows:

$$B(s) = \sum_{n=1}^{\infty} B_n s^n, \qquad B_1 = |\eta|, \tag{2.13}$$

substituting the series in (2.10) and comparing coefficients, we obtain $B_1 = |\eta|$ and

$$B_{n+1} = \frac{8L}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1} B_{n-i}B_{i+1} + \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1} B_{n-k-i}B_kB_{i+1}\right], \quad n \geq 2. \tag{2.14}$$

From (2.7) we obtain immediately that $|a_n| \leq B_n$ for all n by induction. This implies that (2.2) converges in a neighborhood of the origin. This completes the proof. □

Next we devote to the existence of analytic solutions of (1.3) under the Brjuno condition. First, we recall briefly the definition of Brjuno numbers and some basic facts. As stated in [20], for a real number θ, we let $[\theta]$ denote its integer part and $\{\theta\} = \theta - [\theta]$ its fractional part. Then every irrational number θ has a unique expression of the Gauss' continued fraction:

$$\theta = d_0 + \theta_0 = d_0 + \frac{1}{d_1 + \theta_1} = \cdots, \tag{2.15}$$

denoted simply by $\theta = [d_0, d_1, \ldots, d_n, \ldots]$, where d_j's and θ_j's are calculated by the algorithm: (a) $d_0 = [\theta]$, $\theta_0 = \{\theta\}$, and (b) $d_n = [1/\theta_{n-1}]$, $\theta_n = \{1/\theta_{n-1}\}$ for all $n \geq 1$. Define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} q_{-2} &= 1, \qquad q_{-1} = 0, \qquad q_n = d_n q_{n-1} + q_{n-2}, \\ p_{-2} &= 0, \qquad p_{-1} = 1, \qquad p_n = d_n p_{n-1} + p_{n-2}. \end{aligned} \tag{2.16}$$

It is easy to show that $p_n/q_n = [d_0, d_1, \ldots, d_n]$. Thus, for every $\theta \in \mathbb{R} \setminus \mathbb{Q}$ we associate, using its convergence, an arithmetical function $B(\theta) = \sum_{n\geq 0}(\log q_{n+1}/q_n)$. We say that θ is a Brjuno number or that it satisfies Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $d_{n+1} \leq ce^{d_n}$ for all $n \geq 0$, where $c > 0$ is a constant, then $\theta = [d_0, d_1, \ldots, d_n, \ldots]$ is a Brjuno number but is not a Diophantine number. So the case (C2) contains both Diophantine condition and a part of λ "near" resonance. Let $\theta \in \mathbb{R}\setminus\mathbb{Q}$ and $(q_n)_{n\in\mathbb{N}}$ be the sequence of partial denominators of the Gauss's continued fraction for θ. As in [20], let

$$A_k = \left\{ n \geq 0 \mid \|n\theta\| \leq \frac{1}{8q_k} \right\}, \qquad E_k = \max\left(q_k, \frac{q_{k+1}}{4}\right), \qquad \eta_k = \frac{q_k}{E_k}. \tag{2.17}$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \geq 0$, define

$$l_k(n) = \max\left((1 + \eta_k)\frac{n}{q_k} - 2, (m_n\eta_k + n)\frac{1}{q_k} - 1 \right), \tag{2.18}$$

where $m_n = \max\{j \mid 0 \leq j \leq n, j \in A_k^*\}$. We then define function $h_k : \mathbb{N} \to \mathbb{R}_+$ as follows:

$$h_k(n) = \begin{cases} \dfrac{m_n + \eta_k n}{q_k} - 1, & \text{if } m_n + q_k \in A_k^*, \\ l_k(n), & \text{if } m_n + q_k \notin A_k^*. \end{cases} \tag{2.19}$$

Let $g_k(n) := \max(h_k(n), [n/q_k])$, and define $k(n)$ by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Clearly, $k(n)$ is nondecreasing. Then we are able to state the following result.

Lemma 2.3 (Davie's lemma [21]). *Let $K(n) = n\log 2 + \sum_{k=0}^{k(n)} g_k(n)\log(2q_{k+1})$. Then*

(a) *there is a universal constant $\varrho > 0$ (independent of n and θ) such that*

$$K(n) \leq n\left(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \varrho \right), \tag{2.20}$$

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ *for all n_1 and n_2, and*

(c) $-\log|\lambda^n - 1| \leq K(n) - K(n-1)$.

Theorem 2.4. *Suppose that (C2) holds, then* (1.3) *has an analytic solution of the form* (2.2) *in a neighborhood of the origin.*

Proof. As in the Theorem 2.2, we seek a power series solution of the form (2.2). First, we have

$$|a_{n+1}| \le \frac{1}{|\lambda^n - 1|}\frac{8}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}|a_{n-i}||a_{i+1}| + \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}|a_{n-k-i}||a_k||a_{i+1}|\right] \tag{2.21}$$

for $n \ge 2$.

To construct a majorant series, we consider the implicit functional equation:

$$\Theta(s, \psi; L, \eta, \alpha, \beta) = 0, \tag{2.22}$$

where Θ is defined in (2.9) and $L = 1$. Similarly to the proof of Theorem 2.2, using the implicit function theorem we can prove that (2.22) has a unique analytic solution $\psi(s)$ in a neighborhood of the origin such that $\psi(0) = 0$, $\psi'(0) = |\eta|$ and $\Theta(s, \psi(s); L, \eta, \alpha, \beta) = 0$. Thus $\psi(s)$ in (2.22) can be expanded into a convergent series:

$$\psi(s) = \sum_{n=1}^{\infty} B_n s^n, \tag{2.23}$$

in a neighborhood of the origin. Replacing (2.23) into (2.22) and comparing coefficients, we obtain that $B_1 = |\eta|$ and

$$B_n = \frac{8}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}B_{n-i}B_{i+1} + \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}B_{n-k-i}B_kB_{i+1}\right], \quad n \ge 2. \tag{2.24}$$

Note that the series (2.23) converges in a neighborhood of the origin. Now, we can deduce, by induction, that $|a_n| \le B_n e^{K(n-1)}$ for $n \ge 1$, where $K : \mathbb{N} \to \mathbb{R}$ is defined in Lemma 2.3.

In fact, $|a_1| = |\eta| = B_1$. For inductive proof we assume that $|a_j| \le B_j e^{K(j-1)}$, for $j = 1, 2, \ldots, n$. From (2.21) we know

$$\begin{aligned}
|a_{n+1}| &\le \frac{1}{|\lambda^n - 1|}\frac{8}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}|a_{n-i}||a_{i+1}| + \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}|a_{n-k-i}||a_k||a_{i+1}|\right] \\
&\le \frac{1}{|\lambda^n - 1|}\frac{8}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}B_{n-i}B_{i+1}e^{K(n-i-1)+K(i)}\right. \\
&\qquad \left. + \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}B_{n-k-i}B_kB_{i+1}e^{K(n-k-i-1)+K(k-1)+K(i)}\right].
\end{aligned} \tag{2.25}$$

Note that

$$K(n-i-1)+K(i)\leq K(n-1),$$
$$K(n-k-i-1)+K(k-1)+K(i)\leq K(n-1). \tag{2.26}$$

Then from Lemma 2.3, we have

$$|a_{n+1}|\leq\frac{e^{K(n-1)}}{|\lambda^n-1|}\frac{8}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}B_{n-i}B_{i+1}+\sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}B_{n-k-i}B_kB_{i+1}\right],\quad n\geq 1. \tag{2.27}$$

Since $\sum_{n=1}^{\infty}B_ns^n$ is convergent in a neighborhood of the origin, there exists a constant $\Lambda>0$ such that

$$B_n<\Lambda^n,\quad n\geq 1. \tag{2.28}$$

Moreover, from Lemma 2.3, we know that $K(n)\leq n(B(\theta)+\varrho)$ for some universal constant $\varrho>0$. Then

$$|a_n|\leq B_ne^{K(n-1)}\leq\Lambda^ne^{(n-1)(B(\theta)+\varrho)}, \tag{2.29}$$

that is,

$$\limsup_{n\to\infty}(|a_n|)^{1/n}\leq\limsup_{n\to\infty}\left(\Lambda^ne^{(n-1)(B(\theta)+\varrho)}\right)^{1/n}=\Lambda e^{B(\theta)+\varrho}. \tag{2.30}$$

This implies that the convergence radius of (2.2) is at least $(\Lambda e^{B(\theta)+\varrho})^{-1}$. This completes the proof. □

In the case (C3) both the Diophantine condition and Brjuno condition are not satisfied. We need to define a sequence $\{C_n\}_{n=1}^{\infty}$ by $C_1=|\eta|$ and

$$C_{n+1}=\frac{8\Gamma}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}C_{n-i}C_{i+1}+\sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}C_{n-k-i}C_kC_{i+1}\right],\quad n\geq 1, \tag{2.31}$$

where $\Gamma:=\max\{1,1/|1-\lambda|,1/|1-\lambda^2|,\ldots,1/|1-\lambda^{(p-1)}|\}$, and p is defined in (C3).

Theorem 2.5. *Assume that (C3) holds. Let* $\{a_n\}_{n=0}^{\infty}$ *be determined by* $a_1=\eta$ *and*

$$\alpha^2(n+1)(\lambda^n-1)a_{n+1}=\Xi(n,\lambda),\quad n\geq 1, \tag{2.32}$$

where

$$\begin{aligned}\Xi(n,\lambda) = {}& \beta\sum_{i=0}^{n-1}(i+1)\left(\lambda^{n-i}-1\right)^2 a_{n-i}a_{i+1} \\ & - 2\alpha\sum_{i=0}^{n-1}(i+1)\left(\lambda^{n-i}-1\right)\left(\lambda^{i}-1\right)a_{n-i}a_{i+1} \\ & - \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}(i+1)\left(\lambda^{n-k-i}-1\right)\left(\lambda^{k}-1\right)\left(\lambda^{i}-1\right)a_{n-k-i}a_k a_{i+1}.\end{aligned} \tag{2.33}$$

If $\Xi(lp,\lambda) = 0$ *for all* $l = 1,2,\ldots$, *then* (1.3) *has an analytic solution of the form*

$$h(s) = \eta s + \sum_{n=lp,l\in\mathbb{N}}\mu_{lp+1}s^{lp+1} + \sum_{n\neq lp,l\in\mathbb{N}}a_{n+1}s^{n+1}, \quad \mathbb{N} = \{1,2,3,\ldots\} \tag{2.34}$$

in a neighborhood of the origin, where all μ_{lp+1}*'s are arbitrary constants satisfying the inequality* $|\mu_{lp+1}| \le C_{lp+1}$ *and the sequence* $\{C_n\}_{n=1}^{\infty}$ *is defined in* (2.31). *Otherwise, if* $\Xi(lp,\alpha) \neq 0$ *for some* $l = 1,2,\ldots$, *then* (1.3) *has no analytic solutions in any neighborhood of the origin.*

Proof. Analogously to the proof of Lemma 2.1, let (2.2) be the expansion of a formal solution $h(s)$ of (1.3); we also have (2.5) or (2.32). If $\Xi(lp,\lambda) \neq 0$ for some natural number l, then the equality in (2.32) does not hold for $n = lp$ since $\lambda^{lp} - 1 = 0$. In such a circumstance (1.3) has no formal solutions.

If $\Xi(lp,\lambda) = 0$ for all natural numbers l, then there are infinitely many choices of corresponding a_{lp+1} in (2.32) and the formal solutions (2.2) form a family of functions of infinitely many parameters. We can arbitrarily choose $a_{lp+1} = \mu_{lp+1}$ such that $|\mu_{lp+1}| \le C_{lp+1}$, $l = 1,2,\ldots$. In what follows we prove that the formal solution (2.2) converges in a neighborhood of the origin. First of all, note that $|\lambda^n - 1|^{-1} \le \Gamma$, for $n \neq lp$. It follows from (2.32) that

$$|a_{n+1}| \le \frac{8\Gamma}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}|a_{n-i}||a_{i+1}| + \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}|a_{n-k-i}||a_k||a_{i+1}|\right], \tag{2.35}$$

for all $n \neq lp$, $l = 1,2,\ldots$. Further, we can prove that

$$|a_n| \le C_n, \quad n = 1,2,\ldots. \tag{2.36}$$

In fact, for inductive proof we assume that $|a_r| \le C_r$ for all $1 \le r \le n$. When $n = lp$, we have $|a_{n+1}| = |\mu_{n+1}| \le C_{n+1}$. On the other hand, when $n \neq lp$, from (2.36) we get

$$\begin{aligned}|a_{n+1}| &\le \frac{8\Gamma}{\alpha^2}\left[(\alpha+\beta)\sum_{i=0}^{n-1}C_{n-i}C_{i+1} + \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}C_{n-k-i}C_kC_{i+1}\right] \\ &= C_{n+1}\end{aligned} \tag{2.37}$$

as desired. Set

$$F(s) = \sum_{n=1}^{\infty} C_n s^n, \quad C_1 = |\eta|. \tag{2.38}$$

It is easy to check that (2.38) satisfies

$$\Theta(x, F; \Gamma, \eta, \alpha, \beta) = 0, \tag{2.39}$$

where the function Θ is defined in (2.9). Moreover, similarly to the proof of Theorem 2.2, we can prove that (2.39) has a unique analytic solution $F(s)$ in a neighborhood of the origin such that $F(0) = 0$ and $F'(0) = |\eta| \neq 0$. Thus, (2.38) converges in a neighborhood of the origin. By the convergence of (2.38) and inequality (2.36), the series (2.2) converges in a neighborhood of the origin. This completes the proof. □

3. Analytic Solutions of (1.2)

Theorem 3.1. *Suppose that conditions of Theorems 2.2, 2.4, or 2.5 are fulfilled. Then* (1.2) *has an invertible analytic solution of the form*

$$g(s) = h\left(\lambda h^{-1}(s)\right) \tag{3.1}$$

in a neighborhood of the origin, where $h(s)$ is an analytic solutions of (1.3) *satisfying the initial conditions* (2.1).

Proof. In a view of Theorems 2.2–2.5, we may find an analytic solution $h(s)$ of the auxiliary equation (1.3) in the form of (2.2) such that $h(0) = 0$ and $h'(0) = \eta \neq 0$. Clearly the inverse $h^{-1}(s)$ exists and is analytic in a neighborhood of the $h(0) = 0$. Define

$$g(s) := h\left(\lambda h^{-1}(s)\right). \tag{3.2}$$

Then $g(s)$ is invertible analytic in a neighborhood of $s = 0$. From (3.2) it is easy to see

$$\begin{gathered} g(0) = h\left(\lambda h^{-1}(0)\right) = h(0) = 0, \\ g'(0) = \lambda h'\left(\lambda h^{-1}(0)\right)\left(h^{-1}\right)'(0) = \frac{\lambda h'(\lambda h^{-1}(0))}{h'(h^{-1}(0))} = \frac{\lambda h'(0)}{h'(0)} = \lambda \neq 0. \end{gathered} \tag{3.3}$$

From (1.3), we have

$$
\begin{aligned}
&(g(s)-s+\alpha)^2(g'(s)-1)\\
&\quad=\left(h\left(\lambda h^{-1}(s)\right)-h\left(h^{-1}(s)\right)+\alpha\right)^2\left(\frac{h'(\lambda h^{-1}(s))}{h'(h^{-1}(s))}-1\right)\\
&\quad=\beta\left(h\left(\lambda^2 h^{-1}(s)\right)-2h\left(\lambda h^{-1}(s)\right)+h\left(h^{-1}(s)\right)\right)\\
&\quad=\beta(g(g(s))-2g(s)+s)
\end{aligned}
\tag{3.4}
$$

as required. This completes the proof. □

By Theorem 3.1, we have shown that under the conditions of Theorems 2.2, 2.4, or 2.5, (1.2) has an analytic solution $g(s)=h(\lambda h^{-1}(s))$ in a neighborhood of the number 0, where $h(s)$ is an analytic solution of (1.3). Since the function $h(s)$ in (2.2) can be determined by (2.5), it is possible to calculate, at least in theory, the explicit form of $h(s)$, an analytic solution of (1.3), in a neighborhood of the fixed point 0 of $h(s)$. However, knowing that an analytic solution of (1.3) exists, we can take an alternative route as follows.

Example 3.2. Consider

$$
(z(s)+1)^2 z'(s)=z(s+z(s))-z(s),
\tag{3.5}
$$

where

$$
\alpha=\beta=1.
\tag{3.6}
$$

If taking $\lambda=1/2$, then by (1.2) and (1.3), we have

$$
(g(s)-s+1)^2(g'(s)-1)=g(g(s))-2g(s)+s,
\tag{3.7}
$$

and the auxiliary equation is

$$
\left(h\left(\frac{1}{2}s\right)-h(s)+1\right)^2\left(h'\left(\frac{1}{2}s\right)-h'(s)\right)=h'(s)\left(h\left(\frac{1}{4}s\right)-2h\left(\frac{1}{2}s\right)+h(s)\right),\quad s\in\mathbb{C}.
\tag{3.8}
$$

From Lemma 2.1, $a_1 = \eta \neq 0$ is given arbitrarily, and a_n, $n \geq 2$ can be determined by

$$\begin{aligned}
&(n+1)\left(\frac{1}{2^n}-1\right)a_{n+1} \\
&\quad = \sum_{i=0}^{n-1}(i+1)\left(\frac{1}{2^{n-i}}-1\right)^2 a_{n-i}a_{i+1} - 2\sum_{i=0}^{n-1}(i+1)\left(\frac{1}{2^{n-i}}-1\right)\left(\frac{1}{2^i}-1\right)a_{n-i}a_{i+1} \\
&\qquad - \sum_{i=0}^{n-2}\sum_{k=1}^{n-i-1}(i+1)\left(\frac{1}{2^{n-k-i}}-1\right)\left(\frac{1}{2^k}-1\right)\left(\frac{1}{2^i}-1\right)a_{n-k-i}a_k a_{i+1}, \quad n \geq 1.
\end{aligned} \tag{3.9}$$

Now, by (3.9),

$$\begin{aligned}
a_2 &= \frac{h''(0)}{2!} = -\frac{1}{4}\eta^2, \\
a_3 &= \frac{h'''(0)}{3!} = -\frac{1}{36}a_1 a_2 = \frac{1}{144}\eta^3, \\
&\cdots.
\end{aligned} \tag{3.10}$$

Because $h(0) = 0$, $h'(0) = \eta \neq 0$, and the inverse $h^{-1}(s)$ is analytic near the origin, we can calculate that

$$\begin{aligned}
\left(h^{-1}\right)'(0) &= \frac{1}{h'\left(h^{-1}(0)\right)} = \frac{1}{\eta}, \\
\left(h^{-1}\right)''(0) &= -\frac{h''\left(h^{-1}(0)\right)\left(h^{-1}\right)'(0)}{\left(h'\left(h^{-1}(0)\right)\right)^2} = \frac{1}{2\eta}, \\
\left(h^{-1}\right)'''(0) &= -\frac{\left[h'''\left(h^{-1}(0)\right)\left(\left(h^{-1}\right)'(0)\right)^2 + h''\left(h^{-1}(0)\right)\left(h^{-1}\right)''(0)\right]\left(h'\left(h^{-1}(0)\right)\right)^2}{\left(h'\left(h^{-1}(0)\right)\right)^4} \\
&\quad + \frac{h''\left(h^{-1}(0)\right)\left(h^{-1}\right)'(0)\cdot 2h'\left(h^{-1}(0)\right)h''\left(h^{-1}(0)\right)\left(h^{-1}\right)'(0)}{\left(h'\left(h^{-1}(0)\right)\right)^4} \\
&= \frac{(5/24)\eta^3 + (1/2)\eta^3}{\eta^4} \\
&= \frac{17}{24}\cdot\frac{1}{\eta} \\
&\cdots.
\end{aligned} \tag{3.11}$$

Now, we have

$$
\begin{aligned}
g(0) &= h\left(\frac{1}{2}h^{-1}(0)\right) = h(0) = 0, g'(0) \\
&= h'\left(\frac{1}{2}h^{-1}(0)\right)\frac{1}{2}\left(h^{-1}\right)'(0) = \frac{1}{2}h'(0)\left(h^{-1}\right)'(0) = \frac{1}{2}, g''(0) \\
&= \frac{1}{4}h''\left(\frac{1}{2}h^{-1}(0)\right)\left[\left(h^{-1}\right)'(0)\right]^2 + \frac{1}{2}h'\left(\frac{1}{2}h^{-1}(0)\right)\left(h^{-1}\right)''(0) = \frac{1}{8}, g'''(0) \\
&= \frac{1}{8}h'''\left(\frac{1}{2}h^{-1}(0)\right)\left[\left(h^{-1}\right)'(0)\right]^3 + \frac{1}{2}h''\left(\frac{1}{2}h^{-1}(0)\right)\left(h^{-1}\right)'(0)\left(h^{-1}\right)''(0) \\
&\quad + \frac{1}{4}h''\left(\frac{1}{2}h^{-1}(0)\right)\left(h^{-1}\right)'(0)\left(h^{-1}\right)''(0) + \frac{1}{2}h'\left(\frac{1}{2}h^{-1}(0)\right)\left(h^{-1}\right)'''(0) \\
&= \frac{1}{8}h'''(0)\left[\left(h^{-1}\right)'(0)\right]^3 + \frac{3}{4}h''(0)\left(h^{-1}\right)'(0)\left(h^{-1}\right)''(0) \\
&\quad + \frac{1}{2}h'(0)\left(h^{-1}\right)'''(0) \\
&= \frac{11}{64}, \\
&\cdots .
\end{aligned}
\tag{3.12}
$$

Thus, near the origin, (3.7) has an analytic solution:

$$
\begin{aligned}
g(s) &= \frac{1}{2}s - \frac{1}{16}s^2 + \frac{11}{384}s^3 + \cdots, \\
z(s) = g(s) - s &= -\frac{1}{2}s - \frac{1}{16}s^2 + \frac{11}{384}s^3 + \cdots
\end{aligned}
\tag{3.13}
$$

is the analytic solution of (3.5).

Remark 3.3. If we restrict our arguments to the real number field, then by Theorem 3.1, (1.1) has an invertible analytic real solution. We can define a real sequence $\{b_m\}_{m=0}^{\infty}$ and obtain a solution $h(s)$ of the form of (2.2) with real coefficients. Restricted on $\mathbb{R}$ both the function $h(s)$ and its inverse are valued in $\mathbb{R}$. Hence, the function $g(s) = h(\lambda h^{-1}(s))$ is also a real function and Theorem 3.1 implies its invertible analyticity.

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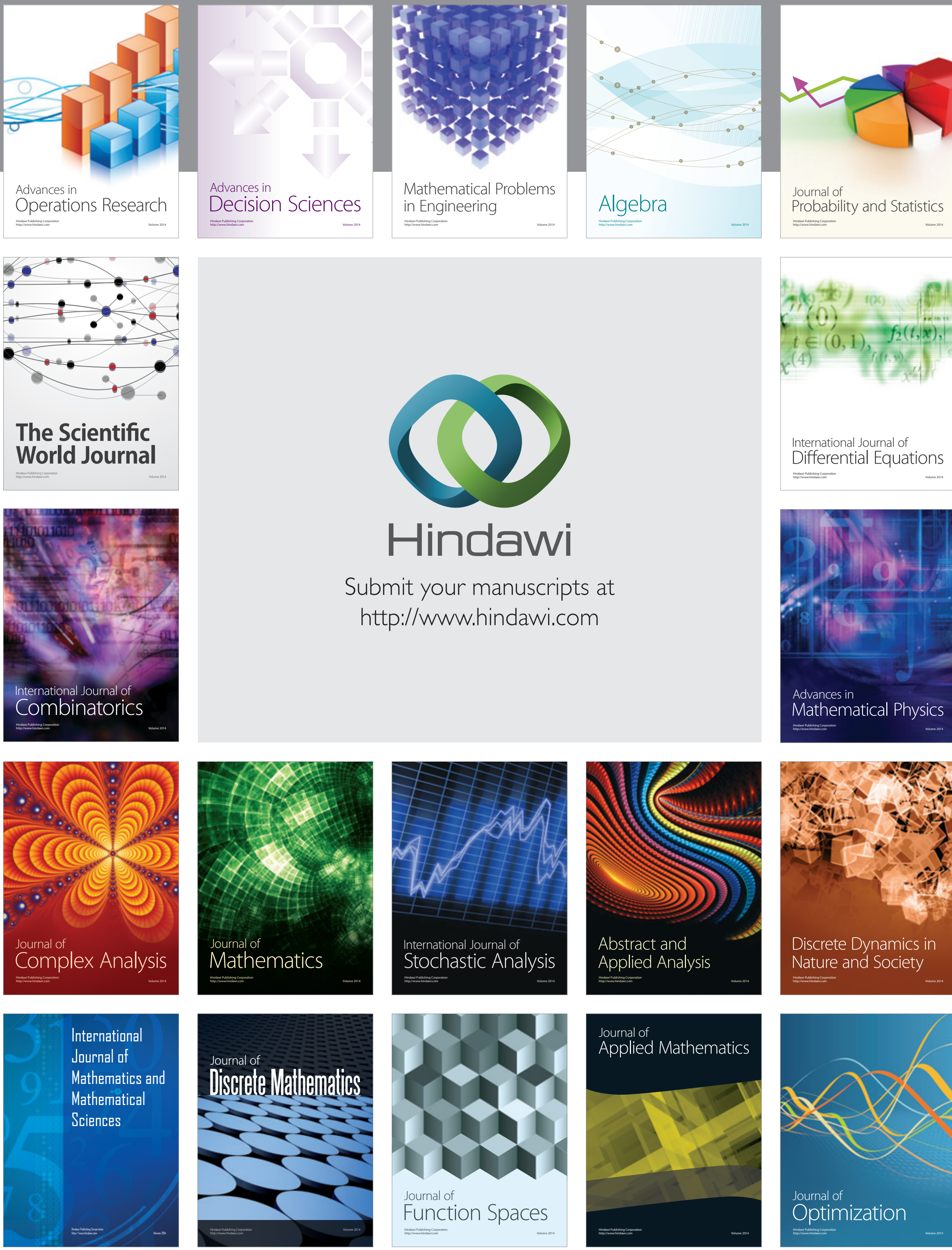
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