

Research Article

Existence of Almost Periodic Solutions to N th-Order Neutral Differential Equations with Piecewise Constant Arguments

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We present some conditions for the existence and uniqueness of almost periodic solutions of N th-order neutral differential equations with piecewise constant arguments of the form $(x(t) + px(t-1))^{(N)} = qx([t]) + f(t)$, here $[\cdot]$ is the greatest integer function, p and q are nonzero constants, N is a positive integer, and $f(t)$ is almost periodic.

1. Introduction

In this paper we study certain functional differential equations of neutral delay type with piecewise constant arguments of the form

$$(x(t) + px(t-1))^{(N)} = qx([t]) + f(t), \quad (1.1)$$

here $[\cdot]$ is the greatest integer function, p and q are nonzero constants, N is a positive integer, and $f(t)$ is almost periodic. Throughout this paper, we use the following notations: \mathbb{R} is the set of reals; \mathbb{R}^+ the set of positive reals; \mathbb{Z} the set of integers; that is, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$; \mathbb{Z}^+ the set of positive integers; \mathbb{C} denotes the set of complex numbers. A function $x : \mathbb{R} \rightarrow \mathbb{R}$ is called a solution of (1.1) if the following conditions are satisfied:

- (i) x is continuous on \mathbb{R} ;
- (ii) the N th-order derivative of $x(t) + p(t)x(t-1)$ exists on \mathbb{R} except possibly at the points $t = n$, $n \in \mathbb{Z}$, where one-sided N th-order derivatives of $x(t) + p(t)x(t-1)$ exist;
- (iii) x satisfies (1.1) on each interval $(n, n+1)$ with integer $n \in \mathbb{Z}$.

Differential equations with piecewise constant arguments are usually referred to as a hybrid system, and could model certain harmonic oscillators with almost periodic forcing. For some excellent works in this field we refer the reader to [1–5] and references therein, and for a survey of work on differential equations with piecewise constant arguments we refer the reader to [6].

In paper [1, 2], Yuan and Li and He, respectively, studied the existence of almost periodic solutions for second-order equations involving the argument $2[(t + 1)/2]$ in the unknown function. In paper [3], Seifert intensively studied the special case of (1.1) for $N = 2$ and $|p| < 1$ by using different methods. However, to the best of our knowledge, there are no results regarding the existence of almost periodic solutions for N th-order neutral differential equations with piecewise constant arguments as (1.1) up to now.

Motivated by the ideas of Yuan [1] and Seifert [3], in this paper we will investigate the existence of almost periodic solutions to (1.1). Both the cases when $|p| < 1$ and $|p| > 1$ are considered.

2. The Main Results

We begin with some definitions, which can be found (or simply deduced from the theory) in any book, say [7], on almost periodic functions.

Definition 2.1. A set $K \subset \mathbb{R}$ is said to be relatively dense if there exists $L > 0$ such that $[a, a + L] \cap K \neq \emptyset$ for all $a \in \mathbb{R}$.

Definition 2.2. A bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ (resp., \mathbb{C}) is said to be almost periodic if the ε -translation set of f

$$T(f, \varepsilon) = \{ \tau \in \mathbb{R} : |f(t + \tau) - f(t)| < \varepsilon \forall t \in \mathbb{R} \} \quad (2.1)$$

is relatively dense for each $\varepsilon > 0$. We denote the set of all such function f by $AP(\mathbb{R}, \mathbb{R})$ (resp., $AP(\mathbb{R}, \mathbb{C})$).

Definition 2.3. A sequence $x : \mathbb{Z} \rightarrow \mathbb{R}^k$ (resp., \mathbb{C}^k), $k \in \mathbb{Z}, k > 0$, denoted by $\{x_n\}$, is called an almost periodic sequence if the ε -translation set of $\{x_n\}$

$$T(\{x_n\}, \varepsilon) = \{ \tau \in \mathbb{Z} : |x_{n+\tau} - x_n| < \varepsilon \forall n \in \mathbb{Z} \} \quad (2.2)$$

is relatively dense for each $\varepsilon > 0$, here $|\cdot|$ is any convenient norm in \mathbb{R}^k (resp., \mathbb{C}^k). We denote the set of all such sequences $\{x_n\}$ by $APS(\mathbb{Z}, \mathbb{R}^k)$ (resp., $APS(\mathbb{Z}, \mathbb{C}^k)$).

Proposition 2.4. $\{x_n\} = \{(x_{n1}, x_{n2}, \dots, x_{nk})\} \in APS(\mathbb{Z}, \mathbb{R}^k)$ (resp., $APS(\mathbb{Z}, \mathbb{C}^k)$) if and only if $\{x_{ni}\} \in APS(\mathbb{Z}, \mathbb{R})$ (resp., $APS(\mathbb{Z}, \mathbb{C})$), $i = 1, 2, \dots, k$.

Proposition 2.5. Suppose that $\{x_n\} \in APS(\mathbb{Z}, \mathbb{R})$, $f \in AP(\mathbb{R}, \mathbb{R})$. Then the sets $T(f, \varepsilon) \cap \mathbb{Z}$ and $T(\{x_n\}, \varepsilon) \cap T(f, \varepsilon)$ are relatively dense.

Now one rewrites (1.1) as the following equivalent system

$$\begin{aligned} (x(t) + px(t-1))' &= y_1(t), & (2.3_1) \\ y_1'(t) &= y_2(t), & (2.3_2) \\ &\vdots & \\ y_{N-2}'(t) &= y_{N-1}(t), & (2.3_{N-1}) \\ y_{N-1}'(t) &= qx([t]) + f(t). & (2.3_N) \end{aligned} \tag{2.3}$$

Let $(x(t), y_1(t), \dots, y_{N-1}(t))$ be solutions of system (2.3) on \mathbb{R} , for $n \leq t < n + 1, n \in \mathbb{Z}$, using (2.3_N) we obtain

$$y_{N-1}(t) = y_{N-1}(n) + qx(n)(t - n) + \int_n^t f(t_1) dt_1, \tag{2.4}$$

and using this with (2.3_{N-1}) we obtain

$$y_{N-2}(t) = y_{N-2}(n) + y_{N-1}(n)(t - n) + \frac{1}{2}qx(n)(t - n)^2 + \int_n^t \int_n^{t_2} f(t_1) dt_1 dt_2. \tag{2.5}$$

Continuing this way, and, at last, we get

$$\begin{aligned} x(t) + px(t-1) &= x(n) + px(n-1) + y_1(n)(t - n) + \frac{1}{2}y_2(n)(t - n)^2 + \dots \\ &+ \frac{1}{(N-1)!}y_{N-1}(n)(t - n)^{N-1} + \frac{1}{N!}qx(n)(t - n)^N \\ &+ \int_n^t \int_n^{t_N} \dots \int_n^{t_2} f(t_1) dt_1 dt_2 \dots dt_N. \end{aligned} \tag{2.6}$$

Since $x(t)$ must be continuous at $n + 1$, using these equations we get for $n \in \mathbb{Z}$,

$$x(n+1) = \left(1 - p + \frac{q}{N!}\right)x(n) + y_1(n) + \frac{1}{2!}y_2(n) + \dots + \frac{1}{(N-1)!}y_{N-1}(n) + px(n-1) + f_n^{(1)}, \tag{2.7_1}$$

$$y_1(n+1) = \frac{q}{(N-1)!}x(n) + y_1(n) + y_2(n) + \frac{1}{2!}y_3(n) + \dots + \frac{1}{(N-2)!}y_{N-1}(n) + f_n^{(2)}, \tag{2.7_2}$$

$$\vdots \tag{2.7_{N-1}}$$

$$y_{N-2}(n+1) = \frac{q}{2}x(n) + y_{N-2}(n) + y_{N-1}(n) + f_n^{(N-1)}, \tag{2.7_{N-1}}$$

$$y_{N-1}(n+1) = qx(n) + y_{N-1}(n) + f_n^{(N)}, \tag{2.7_N}$$

(2.7)

where

$$\begin{aligned} f_n^{(1)} &= \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} f(t_1) dt_1 dt_2 \cdots dt_N, \dots, f_n^{(N-1)} = \int_n^{n+1} \int_n^{t_2} f(t_1) dt_1 dt_2, \\ f_n^{(N)} &= \int_n^{n+1} f(t_1) dt_1. \end{aligned} \quad (2.8)$$

Lemma 2.6. *If $f \in AP(\mathbb{R}, \mathbb{R})$, then sequences $\{f_n^{(i)}\} \in APS(\mathbb{Z}, \mathbb{R}), i = 1, 2, \dots, N$.*

Proof. We typically consider $\{f_n^{(1)}\}$ for all $\varepsilon > 0$ and $\tau \in T(f, \varepsilon) \cap \mathbb{Z}$, we have

$$\begin{aligned} \left| f_{n+\tau}^{(1)} - f_n^{(1)} \right| &= \left| \int_{n+\tau}^{n+\tau+1} \int_{n+\tau}^{t_N} \cdots \int_{n+\tau}^{t_2} f(t_1) dt_1 dt_2 \cdots dt_N - \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} f(t_1) dt_1 dt_2 \cdots dt_N \right| \\ &\leq \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} |f(t_1 + \tau) - f(t_1)| dt_1 dt_2 \cdots dt_N \\ &\leq \frac{\varepsilon}{N!}. \end{aligned} \quad (2.9)$$

From Definition 2.3, it follows that $\{f_n^{(1)}\}$ is an almost periodic sequence. In a manner similar to the proof just completed, we know that $\{f_n^{(2)}\}, \{f_n^{(3)}\}, \dots, \{f_n^{(N)}\}$ are also almost periodic sequences. This completes the proof of the lemma. \square

Lemma 2.7. *The system of difference equations*

$$c_{n+1} = \left(1 - p + \frac{q}{N!}\right) c_n + d_n^{(1)} + \frac{1}{2!} d_n^{(2)} + \cdots + \frac{1}{(N-1)!} d_n^{(N-1)} + p c_{n-1} + f_n^{(1)}, \quad (2.10_1)$$

$$d_{n+1}^{(1)} = \frac{q}{(N-1)!} c_n + d_n^{(1)} + d_n^{(2)} + \frac{1}{2!} d_n^{(3)} + \cdots + \frac{1}{(N-2)!} d_n^{(N-1)} + f_n^{(2)}, \quad (2.10_2)$$

\vdots

\vdots

$$d_{n+1}^{(N-2)} = \frac{q}{2} c_n + d_n^{(N-2)} + d_n^{(N-1)} + f_n^{(N-1)}, \quad (2.10_{N-1})$$

$$d_{n+1}^{(N-1)} = q c_n + d_n^{(N-1)} + f_n^{(N)}, \quad (2.10_N) \quad (2.10)$$

has solutions on \mathbb{Z} ; these are in fact uniquely determined by $c_0, c_{-1}, d_0^{(1)}, \dots, d_0^{(N-1)}$.

Proof. It is easy to check that $c_n, d_n^{(i)}, i = 1, 2, \dots, N-1$ are uniquely determined in term of $c_0, c_{-1}, d_0^{(1)}, d_0^{(2)}, \dots, d_0^{(N-1)}$ for $n \in \mathbb{Z}^+$. For $n = -1$, (2.10_N) uniquely determines $d_{-1}^{(N-1)}$, (2.10_{N-1}) uniquely determines $d_{-1}^{(N-2)}, \dots$, (2.10₂) uniquely determines $d_{-1}^{(1)}$, and thus since $p \neq 0$, (2.10₁) uniquely determines c_{-2} . So $c_{-1}, c_{-2}, d_{-1}^{(1)}, d_{-1}^{(2)}, \dots, d_{-1}^{(N-1)}$ are determined. Continuing in this way, we establish the lemma. \square

Lemma 2.8. For any solution $(c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$, $n \in \mathbb{Z}$, of system (2.10), there exists a solution $(x(t), y_1(t), y_2(t), \dots, y_{N-1}(t))$, $t \in \mathbb{R}$, of (2.3) such that $x(n) = c_n$, $y_1(n) = d_n^{(1)}, \dots$, $y_{N-1}(n) = d_n^{(N-1)}$, $n \in \mathbb{Z}$.

Proof. Define

$$\begin{aligned} w(t) = & c_n + pc_{n-1} + d_n^{(1)}(t-n) + \frac{1}{2!}d_n^{(2)}(t-n)^2 + \dots \\ & + \frac{1}{(N-1)!}d_n^{(N-1)}(t-n)^{N-1} + \frac{1}{N!}qc_n(t-n)^N + \int_n^t \int_n^{t_1} \dots \int_n^{t_{N-1}} f(t_1)dt_1 dt_1 \dots dt_N, \end{aligned} \quad (2.11)$$

for $n \leq t < n+1$, $n \in \mathbb{Z}$. It can easily be verified that $w(t)$ is continuous on \mathbb{R} ; we omit the details.

Define $x(t) = \varphi(t)$, $-1 \leq t \leq 0$, where $\varphi(t)$ is continuous, and $\varphi(0) = c_0$, $\varphi(-1) = c_{-1}$;

$$\begin{aligned} x(t) = & \frac{[w(t+1) - \varphi(t+1)]}{p}, \quad -2 \leq t < -1, \\ x(t) = & \frac{[w(t+1) - x(t+1)]}{p}, \quad -3 \leq t < -2. \end{aligned} \quad (2.12)$$

Continuing this way, we can define $x(t)$ for $t < 0$. Similarly, define

$$\begin{aligned} x(t) = & -p\varphi(t-1) + w(t), \quad 0 \leq t < 1, \\ x(t) = & -px(t-1) + w(t), \quad 1 \leq t < 2, \end{aligned} \quad (2.13)$$

continuing in this way $x(t)$ is defined for $t \geq 0$, and so $x(t)$ is defined for all $t \in \mathbb{R}$.

Next, define $y_1(t) = w'(t), y_2(t) = w''(t), \dots, y_{N-1}(t) = w^{(N-1)}(t)$, $t \neq n \in \mathbb{Z}$, and by the appropriate one-sided derivative of $w'(t), w''(t), \dots, w^{(N-1)}(t)$ at $n \in \mathbb{Z}$. It is easy to see that $y_1(t), y_2(t), \dots, y_{N-1}(t)$ are continuous on \mathbb{R} , and $(x(n), y_1(n), y_2(n), \dots, y_{N-1}(n)) = (c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$ for $n \in \mathbb{Z}$; we omit the details. \square

Next we express system (2.7) in terms of an equivalent system in \mathbb{R}^{N+1} give by

$$v_{n+1} = Av_n + h_n, \quad (2.14)$$

where

$$A = \begin{pmatrix} 1 - p + \frac{q}{N!} & 1 & \frac{1}{2!} & \cdots & \frac{1}{(N-1)!} & p \\ \frac{q}{(N-1)!} & 1 & 1 & \cdots & \frac{1}{(N-2)!} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{q}{2!} & 0 & 0 & \cdots & 1 & 0 \\ \frac{q}{1!} & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (2.15)$$

$$v_n = (x(n), y_1(n), y_2(n), \dots, y_{N-1}(n), x(n-1))^T, \quad h_n = (f_n^{(1)}, f_n^{(2)}, \dots, f_n^{(N)}, 0)^T.$$

Lemma 2.9. *Suppose that all eigenvalues of A are simple (denoted by $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$) and $|\lambda_i| \neq 1$, $1 \leq i \leq N+1$. Then system (2.14) has a unique almost periodic solution.*

Proof. From our hypotheses, there exists a $(N+1) \times (N+1)$ nonsingular matrix P such that $PAP^{-1} = \Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N+1})$ and $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$ are the distinct eigenvalues of A . Define $\bar{v}_n = Pv_n$, then (2.14) becomes

$$\bar{v}_{n+1} = \Lambda \bar{v}_n + \bar{h}_n, \quad (2.16)$$

where $\bar{h}_n = Ph_n$.

For the sake of simplicity, we consider first the case $|\lambda_1| < 1$. Define

$$\bar{v}_{n1} = \sum_{m \leq n} \lambda_1^{n-m} \bar{h}_{(m-1)1}, \quad (2.17)$$

where $\bar{h}_n = (\bar{h}_{n1}, \bar{h}_{n2}, \dots, \bar{h}_{n(N+1)})^T$, $n \in \mathbb{Z}$. Clearly $\{\bar{h}_{n1}\}$ is almost periodic, since $\bar{h}_n = Ph_n$, and $\{h_n\}$ is. For $\tau \in T(\{\bar{h}_{n1}\}, \varepsilon)$, we have

$$\begin{aligned} |\bar{v}_{(n+\tau)1} - \bar{v}_{n1}| &= \left| \sum_{m \leq n+\tau} \lambda_1^{n+\tau-m} \bar{h}_{(m-1)1} - \sum_{m \leq n} \lambda_1^{n-m} \bar{h}_{(m-1)1} \right| \\ &\quad \text{(letting } m = m' + \tau, \text{ then replacing } m' \text{ by } m) \\ &= \left| \sum_{m \leq n} \lambda_1^{n-m} \bar{h}_{(m+\tau-1)1} - \sum_{m \leq n} \lambda_1^{n-m} \bar{h}_{(m-1)1} \right| \\ &= \left| \sum_{m \leq n} \lambda_1^{n-m} (\bar{h}_{(m+\tau-1)1} - \bar{h}_{(m-1)1}) \right| \\ &\leq \frac{\varepsilon}{1 - |\lambda_1|}, \end{aligned} \quad (2.18)$$

this shows that $\{\bar{v}_{n1}\} \in \text{APS}(\mathbb{Z}, \mathbb{C})$.

If $|\lambda_i| < 1, 2 \leq i \leq N + 1$, in a manner similar to the proof just completed for λ_1 , we know that $\{\bar{v}_{ni}\} \in PAS(\mathbb{Z}, \mathbb{C}), 2 \leq i \leq N + 1$, and so $\{\bar{v}_n\} \in APS(\mathbb{Z}, \mathbb{C}^{N+1})$. It follows easily that then $\{P^{-1}\bar{v}_n\} = \{v_n\} \in APS(\mathbb{Z}, \mathbb{R}^{N+1})$ and our lemma follows.

Assume now $|\lambda_1| > 1$. Now define

$$\bar{v}_{n1} = \sum_{m \leq n} \lambda_1^{m-n} \bar{h}_{(m-1)1}, \quad n \in \mathbb{Z}. \tag{2.19}$$

As before, the fact that $\{\bar{v}_{n1}\} \in APS(\mathbb{Z}, \mathbb{C})$ follows easily from the fact that $\{\bar{h}_{n1}\} \in APS(\mathbb{Z}, \mathbb{C})$. So in every possible case, we see that each component $v_{ni}, i = 1, 2, \dots, N + 1$, of v_n is almost periodic and so $\{v_n\} \in APS(\mathbb{Z}, \mathbb{R}^{N+1})$.

The uniqueness of this almost periodic solution $\{v_n\}$ of (2.14) follows from the uniqueness of the solution \bar{v}_n of (2.16) since $P^{-1}\bar{v}_n = v_n$, and the uniqueness of \bar{v}_n of (2.16) follows, since if \tilde{v}_n were a solution of (2.16) distinct from $\bar{v}_n, u_n = \bar{v}_n - \tilde{v}_n$ would also be almost periodic and solve $u_{n+1} = \Lambda u_n, n \in \mathbb{Z}$. But by our condition on Λ , it follows that each component of u_n must become unbounded either as $n \rightarrow \infty$ or as $n \rightarrow -\infty$, and that is impossible, since it must be almost periodic. This proves the lemma. \square

Lemma 2.10. *Suppose that conditions of Lemma 2.9 hold, $w(t)$ is as defined in the proof of Lemma 2.8 with $(c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$ the unique first N components of the almost periodic solution of (2.14) given by Lemma 2.9, then $w(t)$ is almost periodic.*

Proof. For $\tau \in T(\{c_n\}, \varepsilon) \cap T(\{d_n^{(1)}\}, \varepsilon) \cap T(\{d_n^{(2)}\}, \varepsilon) \cap \dots \cap T(\{d_n^{(N-1)}\}, \varepsilon) \cap T(f, \varepsilon)$,

$$\begin{aligned} & |w(t + \tau) - w(t)| \\ &= \left| (c_{n+\tau} - c_n) + p(c_{n+\tau-1} - c_{n-1}) + (d_{n+\tau}^{(1)} - d_n^{(1)})(t - n) + \frac{1}{2!}(d_{n+\tau}^{(2)} - d_n^{(2)})(t - n)^2 + \dots \right. \\ &\quad \left. + \frac{1}{(N-1)!}(d_{n+\tau}^{(N-1)} - d_n^{(N-1)})(t - n)^{N-1} + \frac{q}{N!}(c_{n+\tau} - c_n)(t - n)^N \right. \\ &\quad \left. + \int_{n+\tau}^{t+\tau} \int_{n+\tau}^{t_N} \dots \int_{n+\tau}^{t_2} f(t_1) dt_1 dt_2 \dots dt_N - \int_n^t \int_n^{t_N} \dots \int_n^{t_2} f(t_1) dt_1 dt_2 \dots dt_N \right| \\ &\leq \left(1 + |p| + \frac{|q|}{N!} + \sum_{i=0}^{N-1} \frac{1}{i!} \right) \varepsilon. \end{aligned} \tag{2.20}$$

It follows from definition that $w(t)$ is almost periodic. \square

Theorem 2.11. *Suppose that $|p| \neq 1$ and all eigenvalues of A in (2.14) are simple (denoted by $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$) and satisfy $|\lambda_i| \neq 1, 1 \leq i \leq N + 1$. Then (1.1) has a unique almost periodic solution $\bar{x}(t)$, which can, in fact be determined explicitly in terms of $w(t)$ as defined in the proof of Lemma 2.8.*

Proof. Consider the following.

Case 1 ($|p| < 1$). For each $m \in \mathbb{Z}^+$ define $x_m(t)$ as follows:

$$x_m(t) = w(t) - px_m(t-1), \quad t > -m, \quad (2.21)$$

$$x_m(t) = \phi(t), \quad t \leq -m, \quad (2.22)$$

here $w(t)$ is as defined in the proof of Lemma 2.8, and

$$\phi(t) = c_n + (c_{n+1} - c_n)(t - n), \quad n \leq t < n + 1, \quad n \in \mathbb{Z}, \quad (2.23)$$

where c_n is the first component of the solution v_n of (2.14) given by Lemma 2.9. Let $l \in \mathbb{Z}^+$, then from (2.21) we get

$$(-p)^l x_m(t-l) = (-p)^l w(t-l) + (-p)^{l+1} x_m(t-l-1), \quad t > -m. \quad (2.24)$$

It follows that

$$x_m(t) = \sum_{j=0}^{l-1} (-p)^j w(t-j) + (-p)^l x_m(t-l), \quad t > -m. \quad (2.25)$$

If $l > t + m$, $x_m(t-l) = \phi(t-l)$, and so for such l ,

$$\left| x_m(t) - \sum_{j=0}^{l-1} (-p)^j w(t-j) \right| \leq |p|^l |\phi(t-l)|. \quad (2.26)$$

Let $l \rightarrow \infty$, we get

$$x_m(t) = \begin{cases} \sum_{j=0}^{\infty} (-p)^j w(t-j), & t > -m, \\ \phi(t), & t \leq -m. \end{cases} \quad (2.27)$$

Since $w(t)$ and $\phi(t)$ are uniformly continuous on \mathbb{R} , it follows that $\{x_m(t) : m \in \mathbb{Z}^+\}$ is equicontinuous on each interval $[-L, L]$, $L \in \mathbb{Z}^+$, and by the Ascoli-Arzelá Theorem, there exists a subsequence, which we again denote by $x_m(t)$, and a function $\bar{x}(t)$ such that $x_m(t) \rightarrow \bar{x}(t)$ uniformly on $[-L, L]$, and by a familiar diagonalization procedure, can find a subsequence, again denoted by $x_m(t)$ which is such that $x_m(t) \rightarrow \bar{x}(t)$ for each $t \in \mathbb{R}$. From (2.27) it follows that

$$x_m(t) = \sum_{j=0}^{\infty} (-p)^j w(t-j), \quad (2.28)$$

and so $\bar{x}(t)$ is almost periodic since $w(t - j)$ is almost periodic in t for each $j \geq 0$, and $|p| < 1$. From (2.21), letting $m \rightarrow \infty$, we get $\bar{x}(t) + p\bar{x}(t - 1) = w(t)$, $t \in \mathbb{R}$, and since $w(t)$ solves (1.1), $\bar{x}(t)$ does also. The uniqueness of $\bar{x}(t)$ as an almost periodic solution of (1.1) follows from the uniqueness of the almost periodic solution $v_n : \mathbb{Z} \rightarrow \mathbb{R}^{N+1}$ of (2.14) given by Lemma 2.9, which determines the uniqueness of $w(t)$, and therefore from (2.21) the uniqueness of $\bar{x}(t)$.

Case 2 ($|p| > 1$). Rewriting (2.24) as

$$\left(\frac{-1}{p}\right)^l x_m(t-l) = \left(\frac{-1}{p}\right)^l w(t-l) + \left(\frac{-1}{p}\right)^{l+1} x_m(t-l-1), \quad t > -m, \quad (2.29)$$

we deduce in a similar manner that

$$x_m(t) = \begin{cases} \sum_{j=0}^{\infty} \left(\frac{-1}{p}\right)^j w(t-j), & t > -m, \\ \phi(t), & t \leq -m. \end{cases} \quad (2.30)$$

The remainder of the proof is similar to that of Case 1, we omit the details. □

If $p = 0$, the system of difference equations (2.10) of Lemma 2.7 now becomes

$$\begin{aligned} c_{n+1} &= \left(1 + \frac{1}{N!}q\right)c_n + d_n^{(1)} + \frac{1}{2!}d_n^{(2)} + \dots + \frac{1}{(N-1)!}d_n^{(N-1)} + f_n^{(1)}, \\ d_{n+1}^{(1)} &= \frac{1}{(N-1)!}qc_n + d_n^{(1)} + d_n^{(2)} + \frac{1}{2!}d_n^{(3)} + \dots + \frac{1}{(N-2)!}d_n^{(N-1)} + f_n^{(2)}, \\ &\vdots \\ d_{n+1}^{(N-2)} &= \frac{q}{2}c_n + d_n^{(N-1)} + d_n^{(N-2)} + f_n^{(N-1)}, \\ d_{n+1}^{(N-1)} &= qc_n + d_n^{(N-1)} + f_n^{(N)}, \end{aligned} \quad (2.31)$$

and system (2.14) reduces to

$$v_{n+1}^* = A^*v_n^* + h_n^*, \quad (2.32)$$

where

$$A^* = \begin{pmatrix} 1 + \frac{q}{N!} & 1 & \frac{1}{2!} & \dots & \frac{1}{(N-2)!} & \frac{1}{(N-1)!} \\ \frac{q}{(N-1)!} & 1 & 1 & \dots & \frac{1}{(N-3)!} & \frac{1}{(N-2)!} \\ \frac{q}{(N-2)!} & 0 & 1 & \dots & \frac{1}{(N-4)!} & \frac{1}{(N-3)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{q}{2!} & 0 & 0 & \dots & 1 & 1 \\ q & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad (2.33)$$

and $v_n^* = (x(n), y_1(n), y_2(n), \dots, y_{N-1}(n))^T$, $h_n^* = (f_n^{(1)}, f_n^{(2)}, \dots, f_n^{(N)})^T$. Then we have the following theorem.

Theorem 2.12. *Let $p = 0$ and $q \neq (-1)^N N!$, if all eigenvalues of A^* in (2.32) are simple (denoted by $\lambda_1, \lambda_2, \dots, \lambda_N$) and satisfy $|\lambda_i| \neq 1, 1 \leq i \leq N$, then (1.1) has a unique almost periodic solution $\bar{x}(t)$.*

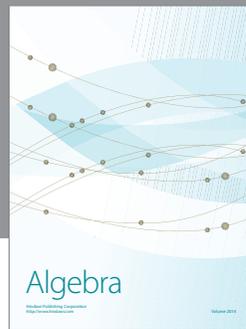
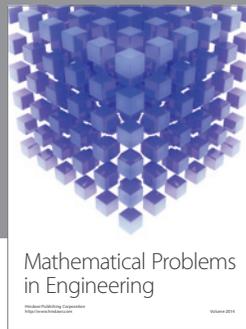
Proof. System (2.32) has a solution on \mathbb{Z} since A^* is nonsingular because $q \neq (-1)^N N!$. The rest of the proof follows in the same way as the proof of Theorem 2.11 and is omitted. \square

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References

- [1] R. Yuan, "The existence of almost periodic solutions to two-dimensional neutral differential equations with piecewise constant argument," *Scientia Sinica A*, vol. 27, no. 10, pp. 873–881, 1997.
- [2] Z. Li and M. He, "The existence of almost periodic solutions of second order neutral differential equations with piecewise constant argument," *Northeastern Mathematical Journal*, vol. 15, no. 3, pp. 369–378, 1999.
- [3] G. Seifert, "Second-order neutral delay-differential equations with piecewise constant time dependence," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 1, pp. 1–9, 2003.
- [4] R. Yuan, "A new almost periodic type of solutions of second order neutral delay differential equations with piecewise constant argument," *Science in China. Series A*, vol. 43, no. 4, pp. 371–383, 2000.
- [5] R. Yuan, "Pseudo-almost periodic solutions of second-order neutral delay differential equations with piecewise constant argument," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 41, pp. 871–890, 2000.
- [6] K. L. Cooke and J. Wiener, "A survey of differential equations with piecewise continuous arguments," in *Delay Differential Equations and Dynamical Systems*, vol. 1475, pp. 1–15, Springer, Berlin, Germany, 1991.
- [7] A. M. Fink, *Almost Periodic Differential Equations*, vol. 377 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1974.



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