Research Article

Attracting Periodic Cycles for an Optimal Fourth-Order Nonlinear Solver

Mi Young Lee and Changbum Chun

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea

Correspondence should be addressed to Changbum Chun, cbchun@skku.edu

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We consider an optimal fourth-order method for solving nonlinear equations and construct polynomials such that the rational map arising from the method applied to these polynomials has an attracting periodic orbit of any prescribed period.

1. Introduction

Recently, an optimal fourth-order iterative method to find a simple root ρ, that is, $f(\rho) = 0$ and $f'(\rho) \neq 0$, of a nonlinear equation $f(x) = 0$, which is given by

$$x_{n+1} = x_n - \frac{16f(x_n)f'(x_n)}{-5f'^2(x_n) + 30f'(x_n)f'(y_n) - 9f'^2(y_n)}, \tag{1.1}$$

where $y_n = x_n - (2/3)(f(x_n)/f'(x_n))$ was proposed in [1] and its dynamics behavior was investigated and analyzed in detail. By an optimal method, we mean a multipoint method without memory which requires $n + 1$ functional evaluations per iteration, but achieves the order of convergence 2^n [2].

The method (1.1) may be used to approximate both real and complex roots of the nonlinear equation with $x_0 \in \mathbb{C}$. If the initial guess x_0 is chosen sufficiently near a zero of f, then an iterative method is expected to converge. This is, however, not true in general if there exists an attracting periodic orbit or cycle of period $k \geq 2$ (whose definition will be introduced below). If the initial guess happens to be chosen from the basin of attraction of an attractive periodic cycle, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the attractive cycle, not to a zero of f since any root of $f(x) = 0$ may be considered as a period orbit of period 1. Thus the existence of attracting periodic cycles of period greater than or equal to 2 could interfere with an iterative method searching for a root of the nonlinear equation. As a result,

it has been an important concern from practical aspect in iteration theory to construct specific polynomials for a given method such that the map arising from the iterative method applied to the polynomials has an attractive periodic orbit. In this direction there was some result in [3] where attractive periodic orbits of any prescribed period were constructed for some classical third-order methods. Motivated by this, in this paper we extend the construction of attractive periodic cycles of any prescribed period to higher-order iterative methods. To this end, we will recall some preliminaries, see for example Milnor [4], Amat et al. [5], and Chun et al. [1]. Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map on the Riemann sphere.

Definition 1.1. For $z \in \widehat{\mathbb{C}}$ we define its orbit as the set

$$\text{orb}(z) = \left\{ z, R(z), R^2(z), \dots, R^n(z), \dots \right\}. \tag{1.2}$$

Definition 1.2. A point z_0 is a fixed point of R if $R(z_0) = z_0$.

Definition 1.3. A periodic point z_0 of period m is such that $R^m(z_0) = z_0$ where m is the smallest such integer. The set of the m distinct points $\{z, R(z), R^2(z), \dots, R^{m-1}(z)\}$ is called a periodic cycle.

Remark 1.4. If z_0 is periodic of period m then it is a fixed point for R^m.

Definition 1.5. If z_0 is a periodic point of period m, then the derivative $(R^m)'(z_0)$ is called the eigenvalue of the periodic point z_0.

Remark 1.6. By the chain rule, if z_0 is a periodic point of period m, then its eigenvalue is the product of the derivatives of R at each point on the orbit of z_0, and we have

$$(R^m)'(z_0) = (R^m)'(z_1) = \cdots = (R^m)'(z_{n-1}), \tag{1.3}$$

that is, all the points of a cycle have the same eigenvalue.

We classify the fixed points of a map based on the magnitude of the derivative.

Definition 1.7. A point z_0 is called attracting if $|R'(z_0)| < 1$, repelling if $|R'(z_0)| > 1$, and neutral if $|R'(z_0)| = 1$. If the derivative is zero then the point is called superattracting.

Definition 1.8. The Julia set of a nonlinear map $R(z)$, denoted $J(R)$, is the closure of the set of its repelling periodic points. The complement of $J(R)$ is the Fatou set $\mathbb{F}(R)$.

By its definition, $J(R)$ is a closed subset of $\widehat{\mathbb{C}}$. A point z_0 belongs to the Julia set if and only if dynamics in a neighborhood of z_0 displays sensitive dependence on the initial conditions, so that nearby initial conditions lead to wildly different behavior after a number of iterations. As a simple example, consider the map $R(z) = z^2$ on $\widehat{\mathbb{C}}$. The entire open disk is contained in $\mathbb{F}(R)$, since successive iterates on any compact subset converge uniformly to zero. Similarly the exterior is contained in $\mathbb{F}(R)$. On the other hand if z_0 is on the unit circle than in any neighborhood of z_0 any limit of the iterates would necessarily have a jump discontinuity as we cross the unit circle. Therefore $J(R)$ is the unit circle. Such smooth Julia sets are exceptional.

Lemma 1.9 (Invariance lemma (Milnor [4])). *The Julia set $J(R)$ of a holomorphic map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is fully invariant under R. That is, z belongs to J if and only if $R(z)$ belongs to J.*

Lemma 1.10 (Iteration Lemma). *For any $k > 0$, the Julia set $J(R^k)$ of the k-fold iterate coincides with $J(R)$.*

Definition 1.11. If O is an attracting periodic orbit of period m, we define the basin of attraction to be the open set $A \in \widehat{\mathbb{C}}$ consisting of all points $z \in \widehat{\mathbb{C}}$ for which the successive iterates $R^m(z), R^{2m}(z), \ldots$ converge towards some point of O.

   The basin of attraction of a periodic orbit may have infinitely many components.

Definition 1.12. The immediate basin of attraction of a periodic orbit is the connected component containing the periodic orbit.

Lemma 1.13. *Every attracting periodic orbit is contained in the Fatou set of R. In fact the entire basin of attraction A for an attracting periodic orbit is contained in the Fatou set. However, every repelling periodic orbit is contained in the Julia set.*

2. Attractive Cycles Results

Let us consider the map R_f associated to f

$$R_f(z) = z + \frac{16f(z)f'(z)}{5f'^2(z) - 30f'(z)f'(y) + 9f'^2(y)}, \tag{2.1}$$

where

$$y = z - \frac{2}{3}\frac{f(z)}{f'(z)}, \tag{2.2}$$

for which the optimal method (1.1) is written as

$$x_{n+1} = R_f(x_n), \quad n \ge 0. \tag{2.3}$$

   Toward the aim to construct periodic orbits of any prescribed period for the method (1.1), we have the following characterization.

Proposition 2.1. *Let $\Omega = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct complex numbers, and let f be a complex analytic function. Then Ω is periodic orbit of period n of iteration function R_f if and only if*

$$\frac{16f'(x_i)f(x_i)}{x_{i+1} - x_i} = 5f'^2(x_i) - 30f'(x_i)f'(y_i) + 9f'^2(y_i), \quad i = 1, 2, \ldots, n-1,$$

$$\frac{16f'(x_n)f(x_n)}{x_1 - x_n} = 5f'^2(x_n) - 30f'(x_n)f'(y_n) + 9f'^2(y_n), \tag{2.4}$$

where $y_i = x_i - (2/3)(f(x_i)/f'(x_i))$, $i = 1, 2, \ldots, n$.

Proof. Assume that Ω is a periodic orbit of R_f. Then $R_f(x_i) = x_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $R_f(x_n) = x_1$. We have

$$
\begin{aligned}
x_{i+1} &= x_i + \frac{16 f'(x_i) f(x_i)}{5 f'^2(x_i) - 30 f'(x_i) f'(y_i) + 9 f'^2(y_i)}, \quad i = 1, 2, \ldots, n-1, \\[2mm]
x_1 &= x_n + \frac{16 f'(x_n) f(x_n)}{5 f'^2(x_n) - 30 f'(x_n) f'(y_n) + 9 f'^2(y_n)}.
\end{aligned}
\tag{2.5}
$$

Therefore, we obtain

$$
\begin{aligned}
x_{i+1} - x_i &= \frac{16 f'(x_i) f(x_i)}{5 f'^2(x_i) - 30 f'(x_i) f'(y_i) + 9 f'^2(y_i)}, \quad i = 1, 2, \ldots, n-1, \\[2mm]
x_1 - x_n &= \frac{16 f'(x_n) f(x_n)}{5 f'^2(x_n) - 30 f'(x_n) f'(y_n) + 9 f'^2(y_n)}.
\end{aligned}
\tag{2.6}
$$

Conversely, suppose that f satisfies condition (2.4). Then we easily have

$$
\begin{aligned}
R_f(x_i) &= x_i + \frac{16 f'(x_i) f(x_i)}{(16 f'(x_i) f(x_i))/(x_{i+1} - x_i)} = x_{i+1}, \quad i = 1, 2, \ldots, n-1, \\[2mm]
R_f(x_n) &= x_n + \frac{16 f'(x_n) f(x_n)}{(16 f'(x_n) f(x_n))/(x_1 - x_n)} = x_1.
\end{aligned}
\tag{2.7}
$$

$\square$

Proposition 2.2. *For any positive integer $n \geq 2$, there exists a polynomial $f(x)$ of degree less than or equal to $3n - 1$ for which R_f has a periodic orbit of period n.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a given a set of n distinct complex numbers, let $z_1, z_2, \ldots, z_n$ be a set of n nonzero complex numbers, and let $s_1, s_2, \ldots, s_n$ be another set of nonzero complex numbers such that

$$
\begin{aligned}
y_i &\neq y_j \quad \forall i \neq j, \\[1mm]
y_i &\neq x_j \quad \forall i, j = 1, 2, \ldots, n, \\[1mm]
15^2 - 9\left(5 s_i^2 - \frac{16 s_i z_i}{x_{i+1} - x_i}\right) &\geq 0, \quad i = 1, 2, \ldots, n-1, \\[2mm]
15^2 - 9\left(5 s_n^2 - \frac{16 s_n z_n}{x_1 - x_n}\right) &\geq 0,
\end{aligned}
\tag{2.8}
$$

where $y_i = x_i - (2/3)(z_i s_i)$.

Assume that there exists a polynomial $f(x)$ such that

$$f(x_i) = z_i, \quad i = 1, 2, \ldots, n,$$

$$f'(x_i) = s_i, \quad i = 1, 2, \ldots, n,$$

$$9f'^2(y_i) - 30s_i f'(y_i) + 5s_i^2 = \frac{16s_i z_i}{x_{i+1} - x_i}, \quad i = 1, 2, \ldots, n-1, \qquad (2.9)$$

$$9f'^2(y_n) - 30s_n f'(y_n) + 5s_n^2 = \frac{16s_n z_n}{x_1 - x_n}.$$

Then by Proposition 2.1, $\Omega = \{x_1, x_2, \ldots, x_n\}$ is a periodic orbit of period n of R_f. Using (2.9) and condition (2.8) yields

$$f'(y_i) = \frac{15 \pm \sqrt{15^2 - 9\left(5s_i^2 - \left(16s_i z_i / (x_{i+1} - x_i)\right)\right)}}{9} \quad i = 1, 2, \ldots, n-1,$$

$$(2.10)$$

$$f'(y_n) = \frac{15 \pm \sqrt{15^2 - 9\left(5s_n^2 - \left(16s_n z_n / (x_1 - x_n)\right)\right)}}{9}.$$

We now show that such a polynomial exists. For this, we use the Hermite interpolation procedure. We begin the construction by writing $f(x)$ as

$$f(x) = \sum_{i=1}^{3n} a_i f_i(x), \qquad (2.11)$$

where the functions $f_i(x)$ are polynomials defined as follows:

$$f_1(x) = 1,$$

$$f_2(x) = (x - x_1),$$

$$f_3(x) = (x - x_1)^2,$$

$$f_4(x) = (x - x_1)^2 (x - y_1)^2,$$

$$f_5(x) = (x - x_1)^2 (x - y_1)^2 (x - x_2), \qquad (2.12)$$

$$f_6(x) = (x - x_1)^2 (x - y_1)^2 (x - x_2)^2,$$

$$f_7(x) = (x - x_1)^2 (x - y_1)^2 (x - x_2)^2 (x - y_2)^2,$$

$$\vdots$$

$$f_{3n}(x) = (x - x_1)^2 (x - y_1)^2 (x - x_2)^2 (x - y_2)^2 \cdots (x - x_n)^2.$$

To determine the polynomial $f(x)$ we must find suitable coefficients a_i for $i = 1, 2, \ldots, 3n$. For this we must solve a linear system of $3n$ equations with $3n$ unknown of the form $AX = B$. The matrix A associated with the linear system of equations is given by

$$
A = \begin{pmatrix}
f_1(x_1) & f_2(x_1) & f_3(x_1) & \cdots & f_{3n}(x_1) \\
f_1'(x_1) & f_2'(x_1) & f_3'(x_1) & \cdots & f_{3n}'(x_1) \\
f_1'(y_1) & f_2'(y_1) & f_3'(y_1) & \cdots & f_{3n}'(y_1) \\
f_1(x_2) & f_2(x_2) & f_3(x_2) & \cdots & f_{3n}(x_2) \\
f_1'(x_2) & f_2'(x_2) & f_3'(x_2) & \cdots & f_{3n}'(x_2) \\
f_1'(y_2) & f_2'(y_2) & f_3'(y_2) & \cdots & f_{3n}'(y_2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
f_1'(y_n) & f_2'(y_n) & f_3'(y_n) & \cdots & f_{3n}'(y_n)
\end{pmatrix}
\tag{2.13}
$$

and the column vectors $X, B \in \mathbb{R}^n$ are given by $X = (a_1, a_2, \ldots, a_{3n})^T$ and $B = (f(x_1), f'(x_1), f'(y_1), \ldots, f(x_n), f'(x_n), f'(y_n))^T$. The components of the upper triangle of the matrix A are zero, and the components of the diagonal of A are nonzero. Therefore, the linear system $AX = B$ has a unique solution. □

Example 2.3. Let us construct a polynomial $f(x)$ for which the iterative method R_f has a periodic orbit of period 2. For this, we let $x_1 = 0, x_2 = 1, z_1 = f(x_1) = 2, z_2 = f(x_2) = 2, s_1 = f'(x_1) = 2$ and $s_2 = f'(x_2) = 2$. Then we have $y_1 = x_1 - (2/3)(z_1/s_1) = -2/3$, $y_2 = x_2 - (2/3)(z_2/s_2) = 1/3$, $f'(y_1) = (15 \pm \sqrt{15^2 - 9(5s_1^2 - 16s_1z_1/(x_2 - x_1))})/9 = 22/3, -2/3$, and $f'(y_2) = (15 \pm \sqrt{15^2 - 9(5s_2^2 - 16s_2z_2/(x_1 - x_2))})/9 = 14/3, 2$. We construct a polynomial $f(x) = \sum_{i=1}^{6} a_i f_i(x)$, where $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$, $f_4(x) = x^2(x + (2/3))^2$, $f_5(x) = x^2(x + (2/3))^2(x - 1)$, $f_6(x) = x^2(x + (2/3))^2(x - 1)^2$. Then matrix A is given by

$$
A = \begin{pmatrix}
f_1(x_1) & 0 & 0 & 0 & 0 & 0 \\
f_1'(x_1) & f_2'(x_1) & 0 & 0 & 0 & 0 \\
f_1'(y_1) & f_2'(y_1) & f_3'(y_1) & 0 & 0 & 0 \\
f_1(x_2) & f_2(x_2) & f_3(x_2) & f_4(x_2) & 0 & 0 \\
f_1'(x_2) & f_2'(x_2) & f_3'(x_2) & f_4'(x_2) & f_5'(x_2) & 0 \\
f_1'(y_2) & f_2'(y_2) & f_3'(y_2) & f_4'(y_2) & f_5'(y_2) & f_6'(y_2)
\end{pmatrix}.
\tag{2.14}
$$

By evaluating A we obtain

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\dfrac{4}{3} & 0 & 0 & 0 \\ 1 & 1 & 1 & \dfrac{25}{9} & 0 & 0 \\ 0 & 1 & 2 & \dfrac{80}{9} & \dfrac{25}{9} & 0 \\ 0 & 1 & \dfrac{2}{3} & \dfrac{8}{9} & -\dfrac{13}{27} & \dfrac{20}{81} \end{pmatrix}. \tag{2.15}$$

Now we consider the system of linear equations $AX = B$, where $X = (a_1, a_2, a_3, a_4, a_5, a_6)^T$ and $B = (f(x_1), f'(x_1), f'(y_1), f(x_2), f'(x_2), f'(y_2))^T$. Here, we can take $B = (2, 2, 22/3, 2, 2, 14/3)^T$. Hence we have the following system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \dfrac{-4}{3} & 0 & 0 & 0 \\ 1 & 1 & 1 & \dfrac{25}{9} & 0 & 0 \\ 0 & 1 & 2 & \dfrac{80}{9} & \dfrac{25}{9} & 0 \\ 0 & 1 & \dfrac{2}{3} & \dfrac{8}{9} & \dfrac{-13}{27} & \dfrac{20}{81} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ \dfrac{22}{3} \\ 2 \\ 2 \\ \dfrac{14}{3} \end{pmatrix} \tag{2.16}$$

whose solution is $X = (2, 2, -4, 18/25, 72/125, 12582/625)^T$. Therefore, the polynomial f is given by $f(x) = 2 + 2x - 4x^2 + (18/25)x^2(x + (2/3))^2 + (72/125)x^2(x + 2/3)^2(x - 1) + (12582/625)x^2(x + 2/3)^2(x - 1)^2 = 2 + 2x + (3132/625)x^2 - (14808/625)x^4 + (5872/625)x^3 - (8028/625)x^5 + (12582/625)x^6$. From (2.1), we obtain $R_f(0) = 1$ and $R_f(1) = 0$. So, $\Omega = \{0, 1\}$ is a periodic orbit of period 2 for the method $R_f(x)$.

Proposition 2.4. *Let $f(x)$ be a polynomial for which R_f has a periodic orbit of period n, say $\Omega = \{x_1, x_2, \ldots, x_n\}$. If $f''(x_i) = 0 = f''(y_i)$ and $f'(x_i) = f'(y_i)$ or $f''(x_i) = 0 = f''(y_i)$ and $7f'(x_i) = 3f'(y_i)$ for some $i = 1, 2, \ldots, n$, then Ω is an superattracting periodic orbit of period n for R_f.*

Proof. Without loss of generality, we may assume that $f''(x_1) = 0 = f''(y_1)$ and $f'(x_1) = f'(y_1)$. By the chain rule, we have

$$\left(R_f^n\right)'(x_1) = R_f'(x_n)R_f'(x_{n-1})R_f'(x_{n-2}) \cdots R_{f'}(x_1). \tag{2.17}$$

Note that on differentiating (2.1) we have

$$
\begin{aligned}
R_{f'}(x) = {} & \left[f'(x) \left(5f'^2(x) - 30f'(x)f'(y) + 9f'^2(y) \right)^2 \right]^{-1} \\
& \times \Big[f'(x) \left(5f'^2(x) - 30f'(x)f'(y) + 9f'^2(y) \right) \left(21f'(x)^2 - 30f'(x)f'(y) \right) \\
& \quad + 9f'^2(y) + 16f(x)f''(x) \\
& \quad - 16f(x) \Big[10f'^3(x)f''(x) - 30f'^2(x)f''(x)f'(y) \\
& \qquad\qquad -2 \left(f'^2(x) + 2f(x)f''(x) \right) \left(5f'(x)f''(y) - 3f'(y)f''(y) \right) \Big] \Big].
\end{aligned}
\tag{2.18}
$$

We therefore obtain $R'_f(x_1) = 0$. By (2.17) we conclude that $(R_f^n)'(x_1) = 0$.   □

Acknowledgments

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