Research Article

Optimal Bounds for Neuman-Sándor Mean in Terms of the Convex Combinations of Harmonic, Geometric, Quadratic, and Contraharmonic Means

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We present the best possible lower and upper bounds for the Neuman-Sándor mean in terms of the convex combinations of either the harmonic and quadratic means or the geometric and quadratic means or the harmonic and contraharmonic means.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $M(a, b)$ [1] is defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the theory of bivariate means have been the subject of intensive research [2–17]. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a, b)$ can be found in the literature [1, 18–20].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric,

logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contraharmonic means of a and b, respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a,b) < G(a,b) < L(a,b) < P(a,b) < A(a,b) \\ < M(a,b) < T(a,b) < Q(a,b) < C(a,b) \end{aligned} \tag{1.2}$$

hold for all $a, b > 0$ with $a \neq b$.

In [1, 18], Neuman and Sándor proved that the double inequalities

$$\begin{aligned} A(a,b) < M(a,b) < T(a,b), \\ P(a,b)M(a,b) < A^2(a,b), \\ A(a,b)T(a,b) < M^2(a,b) < \frac{A^2(a,b) + T^2(a,b)}{2} \end{aligned} \tag{1.3}$$

hold for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1 - a$ and $b' = 1 - b$. Then the following Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \tag{1.4}$$

were presented in [1].

The double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ for all $a, b > 0$ with $a \neq b$ was established by Li et al. in [19], where $L_p(a,b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$ and $L_{-1}(a,b) = (b-a)/(\log b - \log a)$ is the pth generalized logarithmic mean of a and b, and $p_0 = 1.843\ldots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

Neuman [20] proved that the double inequalities

$$\begin{aligned} \alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b), \\ \lambda Q(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu Q(a,b) + (1-\mu)A(a,b) \end{aligned} \tag{1.5}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 1 - \log(\sqrt{2}+1)/[(\sqrt{2}-1)\log(\sqrt{2}+1)] = 0.3249\ldots$, $\beta \geq 1/3$, $\lambda \leq 1 - \log(\sqrt{2}+1)/\log(\sqrt{2}+1) = 0.1345\ldots$ and $\mu \geq 1/6$.

The main purpose of this paper is to find the least values α_1, α_2, α_3, and the greatest values β_1, β_2, β_3, such that the double inequalities

$$\begin{aligned} \alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b), \\ \alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b), \\ \alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b) \end{aligned} \tag{1.6}$$

hold true for all $a, b > 0$ with $a \neq b$.

Our main results are presented in Theorems 1.1–1.3.

Theorem 1.1. *The double inequality*

$$\alpha_1 H(a,b) + (1 - \alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1 - \beta_1)Q(a,b) \tag{1.7}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \geq 2/9 = 0.2222\ldots$ and $\beta_1 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$.

Theorem 1.2. *The double inequality*

$$\alpha_2 G(a,b) + (1 - \alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1 - \beta_2)Q(a,b) \tag{1.8}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \geq 1/3 = 0.3333\ldots$ and $\beta_2 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$.

Theorem 1.3. *The double inequality*

$$\alpha_3 H(a,b) + (1 - \alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1 - \beta_3)C(a,b) \tag{1.9}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \geq 1 - 1/[2\log(1 + \sqrt{2})] = 0.4327\ldots$ and $\beta_3 \leq 5/12 = 0.4166\ldots$.

2. Lemmas

In order to prove our main results we need two Lemmas, which we present in this section.

Lemma 2.1 (see [21, Lemma 1.1]). *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and $b_n > 0$ for all $n \in \{0, 1, 2, \ldots\}$. Let $h(x) = f(x)/g(x)$, then the following statements are true.*

(1) *If the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0, r)$.*

(2) *If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0 < n \leq n_0$ and (strictly) decreasing (increasing) for $n > n_0$, then there exists $x_0 \in (0, r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0, x_0)$ and (strictly) decreasing (increasing) on (x_0, r).*

Lemma 2.2. *Let $p \in (0, 1)$, $\lambda_0 = 1 - 1/[\sqrt{2}\log(1 + \sqrt{2})] = 0.1977\ldots$ and*

$$f_p(x) = \sinh^{-1}(x) - \frac{x}{\sqrt{1 + x^2} - p\left(\sqrt{1 + x^2} - \sqrt{1 - x^2}\right)}. \tag{2.1}$$

Then $f_{1/3}(x) < 0$ and $f_{\lambda_0}(x) > 0$ for all $x \in (0, 1)$.

Proof. From (2.1), one has

$$f_p(0) = 0, \tag{2.2}$$

$$f_p(1) = \log\left(1 + \sqrt{2}\right) - \frac{1}{\sqrt{2}(1 - p)}, \tag{2.3}$$

$$f_p'(x) = \frac{g_p(x)}{\sqrt{1 - x^4}\left(\sqrt{1 + x^2} + p\left(\sqrt{1 - x^2} - \sqrt{1 + x^2}\right)\right)^2}, \tag{2.4}$$

where

$$g_p(x) = \sqrt{1-x^2}\left(\sqrt{1+x^2} + p\left(\sqrt{1-x^2} - \sqrt{1+x^2}\right)\right)^2 - \sqrt{1-x^2} - p\left(\sqrt{1+x^2} - \sqrt{1-x^2}\right).$$
(2.5)

We divide the proof into two cases.

Case 1. ($p = 1/3$). Then (2.5) leads to

$$g_{1/3}(0) = 0, \qquad g_{1/3}(1) = -\frac{\sqrt{2}}{3} < 0,$$
(2.6)

$$g'_{1/3}(x) = \frac{x^3}{\sqrt{1-x^4}} h_{1/3}(x),$$
(2.7)

where

$$h_{1/3}(x) = \frac{14}{9\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} - \left(\sqrt{1+x^2} + \sqrt{1-x^2}\right) - \frac{\sqrt{1-x^2}}{3}.$$
(2.8)

We clearly see that the function $\sqrt{1+x^2} + \sqrt{1-x^2}$ is strictly decreasing in $(0,1)$. Then from (2.8), we get

$$h_{1/3}(x) < h_{1/3}(1) = -\frac{2\sqrt{2}}{9} < 0$$
(2.9)

for $x \in (0,1)$.

Therefore, $f_{1/3}(x) < 0$ for all $x \in (0,1)$ follows easily from (2.2), (2.4), (2.6), (2.7), and (2.9).

Case 2. ($p = \lambda_0$). Then (2.3) and (2.5) yield

$$f_{\lambda_0}(1) = g_{\lambda_0}(0) = 0, \qquad g_{\lambda_0}(1) = -\sqrt{2}\lambda_0 < 0,$$
(2.10)

$$g'_{\lambda_0}(x) = \frac{x}{\sqrt{1-x^4}} h_{\lambda_0}(x),$$
(2.11)

where

$$h_{\lambda_0}(x) = \left[\left(2 - 3\lambda_0 - 2\lambda_0^2\right) - (3 - 6\lambda_0)x^2\right]\sqrt{1+x^2} - \left[\left(3\lambda_0 - 2\lambda_0^2\right) + \left(6\lambda_0 - 6\lambda_0^2\right)x^2\right]\sqrt{1-x^2}.$$
(2.12)

We divide the discussion of this case into two subcases, and all computations are carried out using MATHEMATICA software.

Subcase A. $x \in (0.9, 1)$. Then from (2.12) and the fact that

$$\left(2 - 3\lambda_0 - 2\lambda_0^2\right) - (3 - 6\lambda_0)x^2 < \left(2 - 3\lambda_0 - 2\lambda_0^2\right) - (3 - 6\lambda_0) \times (0.9)^2 = -0.1404\cdots < 0,$$
(2.13)

we know that

$$h_{\lambda_0}(x) < 0 \tag{2.14}$$

for $x \in (0.9, 1)$.

Subcase B. $x \in (0, 0.9]$. Then from (2.12), one has

$$h_{\lambda_0}(0) = 0.8137\cdots > 0, \qquad h_{\lambda_0}(0.9) = -0.7494\cdots < 0,$$
$$h'_{\lambda_0}(x) = \frac{x}{\sqrt{1 - x^4}}\mu(x), \tag{2.15}$$

where

$$\mu(x) = \left[\left(18\lambda_0 - 18\lambda_0^2\right)x^2 - \left(9\lambda_0 - 10\lambda_0^2\right)\right]\sqrt{1 + x^2} - \left[(9 - 18\lambda_0)x^2 + \left(4 - 9\lambda_0 + 2\lambda_0^2\right)\right]\sqrt{1 - x^2}. \tag{2.16}$$

We conclude that

$$\mu(x) < 0 \tag{2.17}$$

for all $x \in (0, 0.9]$. Indeed, if $x \in (0, 1/2)$, then (2.17) follows from (2.16) and the inequality

$$\left(18\lambda_0 - 18\lambda_0^2\right)x^2 - \left(9\lambda_0 - 10\lambda_0^2\right) < 5.5\lambda_0^2 - 4.5\lambda_0 = -0.6747\cdots < 0. \tag{2.18}$$

If $x \in [1/2, 0.9]$, then (2.17) follows from (2.16) and the inequalities

$$\begin{aligned} \left(18\lambda_0 - 18\lambda_0^2\right)x^2 &- \left(9\lambda_0 - 10\lambda_0^2\right) \\ &\leq \left(18\lambda_0 - 18\lambda_0^2\right) \times (0.9)^2 - \left(9\lambda_0 - 10\lambda_0^2\right) \\ &= 5.58\lambda_0 - 4.58\lambda_0^2 = 0.9242\ldots, \\ (9 - 18\lambda_0)x^2 + \left(4 - 9\lambda_0 + 2\lambda_0^2\right) &\geq \frac{1}{4}(9 - 18\lambda_0) + \left(4 - 9\lambda_0 + 2\lambda_0^2\right) \\ &= 6.25 - 13.5\lambda_0 + 2\lambda_0^2 = 3.6589\ldots, \\ \left[\left(18\lambda_0 - 18\lambda_0^2\right)x^2 - \left(9\lambda_0 - 10\lambda_0^2\right)\right]&\sqrt{1 + x^2} - \left[(9 - 18\lambda_0)x^2 + \left(4 - 9\lambda_0 + 2\lambda_0^2\right)\right]\sqrt{1 - x^2} \\ &\leq \left(5.58\lambda_0 - 4.58\lambda_0^2\right)\sqrt{1 + (0.9)^2} - \left(6.25 - 13.5\lambda_0 + 2\lambda_0^2\right)\sqrt{1 - (0.9)^2} \\ &= -0.3514\cdots < 0. \end{aligned} \tag{2.19}$$

From (2.15) together with (2.17) we clearly see that there exists $x_0 \in (0, 0.9)$ such that $h_{\lambda_0}(x) > 0$ for $x \in [0, x_0)$ and $h_{\lambda_0}(x) < 0$ for $(x_0, 0.9]$.

Subcases A and B lead to the conclusion that $h_{\lambda_0}(x) > 0$ for $x \in [0, x_0)$ and $h_{\lambda_0}(x) < 0$ for $x \in (x_0, 1)$. Thus from (2.11), we know that $g_{\lambda_0}(x)$ is strictly increasing in $(0, x_0]$ and strictly decreasing in $[x_0, 1)$.

It follows from (2.4) and (2.10) together with the piecewise monotonicity of $g_{\lambda_0}(x)$ that there exists $x_1 \in (0, 1)$ such that $f_{\lambda_0}(x)$ is strictly increasing in $[0, x_1)$ and strictly decreasing in $[x_1, 1)$.

Therefore, $f_{\lambda_0}(x) > 0$ for $x \in (0, 1)$ follows from (2.2) and (2.10) together with the piecewise monotonicity of $f_{\lambda_0}(x)$. ☐

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. Since $H(a, b)$, $M(a, b)$ and $Q(a, b)$ are symmetric and homogeneous of degree 1. Hence, without loss of generality, we assume that $a > b$. Let $x = (a - b)/(a + b)$ and $t = \sinh^{-1}(x)$. Then $x \in (0, 1)$, $t \in (0, \log(1 + \sqrt{2}))$, $M(a, b)/A(a, b) = x/\sinh^{-1}(x) = \sinh(t)/t$, $H(a, b)/A(a, b) = 1 - x^2 = 1 - \sinh^2(t) = [3 - \cosh(2t)]/2$, $Q(a, b)/A(a, b) = \sqrt{1 + x^2} = \cosh(t)$ and

$$
\begin{aligned}
\frac{Q(a, b) - M(a, b)}{Q(a, b) - H(a, b)} &= \frac{\sqrt{1 + x^2}\sinh^{-1}(x) - x}{\left[\sqrt{1 + x^2} - (1 - x^2)\right]\sinh^{-1}(x)} \\
&= \frac{t\cosh(t) - \sinh(t)}{t[(1/2)\cosh(2t) + \cosh(t) - (3/2)]} := \varphi(t).
\end{aligned}
\tag{3.1}
$$

Making use of power series $\sinh(t) = \sum_{n=0}^{\infty} t^{2n+1}/(2n+1)!$ and $\cosh(t) = \sum_{n=0}^{\infty} t^{2n}/(2n)!$, we can express (3.1) as follows:

$$
\varphi(t) = \frac{\sum_{n=1}^{\infty}[2n/((2n+1)(2n)!)]t^{2n+1}}{\sum_{n=1}^{\infty}\left[(2^{2n-1}+1)/(2n)!\right]t^{2n+1}}.
\tag{3.2}
$$

Let $a_n = 2n/((2n+1)(2n)!)$ and $b_n = (2^{2n-1}+1)/(2n)!$ Then $a_n/b_n = 2n/[(2n+1)(2^{2n-1}+1)]$. Moreover, by a simple calculation, we see that

$$
\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{2 + (2 - 18n - 12n^2)2^{2n-1}}{(2n+1)(2n+3)(2^{2n-1}+1)(2^{2n+1}+1)} < 0
\tag{3.3}
$$

for $n \geq 1$.

Equations (3.1) and (3.2) together with inequality (3.3) and Lemma 2.1 lead to the conclusion that $\varphi(t)$ is strictly decreasing in $(0, \log(1 + \sqrt{2}))$. This in turn implies that

$$
\lim_{t \to 0^+} \varphi(t) = \frac{2}{9}, \qquad \lim_{t \to \log(1+\sqrt{2})} \varphi(t) = 1 - \frac{1}{\sqrt{2}\log\left(1 + \sqrt{2}\right)}.
\tag{3.4}
$$

Therefore, Theorem 1.1 follows from (3.1) and (3.4) together with the monotonicity of $\varphi(t)$. ☐

Proof of Theorem 1.2. Since $G(a,b)$, $M(a,b)$ and $Q(a,b)$ are symmetric and homogeneous of degree 1. Hence, without loss of generality, we assume that $a > b$. Let $x = (a - b)/(a + b)$, $p \in (0,1)$ and $\lambda_0 = 1 - 1/[\sqrt{2}\log(1 + \sqrt{2})]$. Then making use of $G(a,b)/A(a,b) = \sqrt{1 - x^2}$ gives

$$\frac{Q(a,b) - M(a,b)}{Q(a,b) - G(a,b)} = \frac{\sqrt{1 + x^2}\sinh^{-1}(x) - x}{\left(\sqrt{1 + x^2} - \sqrt{1 - x^2}\right)\sinh^{-1}(x)}. \tag{3.5}$$

Moreover, we obtain

$$\lim_{x \to 0^+} \frac{\sqrt{1 + x^2}\sinh^{-1}(x) - x}{\left(\sqrt{1 + x^2} - \sqrt{1 - x^2}\right)\sinh^{-1}(x)} = \frac{1}{3}, \tag{3.6}$$

$$\lim_{x \to 1^-} \frac{\sqrt{1 + x^2}\sinh^{-1}(x) - x}{\left(\sqrt{1 + x^2} - \sqrt{1 - x^2}\right)\sinh^{-1}(x)} = 1 - \frac{1}{\sqrt{2}\log\left(1 + \sqrt{2}\right)} = \lambda_0. \tag{3.7}$$

We take the difference between the additive convex combination of $G(a,b)$, $Q(a,b)$, and $M(a,b)$ as follows:

$$pG(a,b) + (1 - p)Q(a,b) - M(a,b)$$

$$= A(a,b)\left[p\sqrt{1 - x^2} + (1 - p)\sqrt{1 + x^2} - \frac{x}{\sinh^{-1}(x)}\right] \tag{3.8}$$

$$= \frac{A(a,b)\left[p\sqrt{1 - x^2} + (1 - p)\sqrt{1 + x^2}\right]}{\sinh^{-1}(x)}f_p(x),$$

where $f_p(x)$ is defined as in Lemma 2.2.

Therefore, $(1/3)G(a,b) + (2/3)Q(a,b) < M(a,b) < \lambda_0 G(a,b) + (1 - \lambda_0)Q(a,b)$ for all $a, b > 0$ with $a \neq b$ follows from (3.8) and Lemma 2.2. This conjunction with the following statement gives the asserted result.

(i) If $p < 1/3$, then (3.5) and (3.6) imply that there exists $0 < \delta_1 < 1$ such that $M(a,b) < pG(a,b) + (1 - p)Q(a,b)$ for all $a, b > 0$ with $(a - b)/(a + b) \in (0, \delta_1)$.

(ii) If $p > \lambda_0$, then (3.5) and (3.7) imply that there exists $0 < \delta_2 < 1$ such that $M(a,b) > pG(a,b) + (1 - p)Q(a,b)$ for all $a, b > 0$ with $(a - b)/(a + b) \in (1 - \delta_2, 1)$. $\qquad\square$

Proof of Theorem 1.3. We will follow, to some extent, lines in the proof of Theorem 1.1. First we rearrange terms of (1.9) to obtain

$$\beta_3 < \frac{C(a,b) - M(a,b)}{C(a,b) - H(a,b)} < \alpha_3. \tag{3.9}$$

Use of $C(a,b)/A(a,b) = 1 + x^2$ followed by a substitution $x = \sinh(t)$ gives

$$\beta_3 < \phi(t) < \alpha_3, \tag{3.10}$$

where

$$\phi(t) = \frac{t[\cosh(2t) + 1] - 2\sinh(t)}{2t[\cosh(2t) - 1]}, \quad |t| < \log\left(1 + \sqrt{2}\right). \tag{3.11}$$

Since the function $\phi(t)$ is an even function, it suffices to investigate its behavior on the interval $(0, \log(1 + \sqrt{2}))$.

Using power series of $\sinh(t)$ and $\cosh(t)$, then (3.11) can be rewritten as

$$\phi(t) = \frac{\sum_{n=1}^{\infty}\left[2^{2n}/(2n)! - 2/(2n+1)!\right]t^{2n+1}}{\sum_{n=1}^{\infty}\left[2^{2n+1}/(2n)!\right]t^{2n+1}}. \tag{3.12}$$

Let $c_n = 2^{2n}/(2n)! - 2/(2n+1)!$ and $d_n = 2^{2n+1}/(2n)!$. Then

$$\frac{c_n}{d_n} = \frac{1}{2} - \frac{1}{(2n+1)2^{2n}}. \tag{3.13}$$

It follows from (3.13) that the sequence $\{c_n/d_n\}$ is strictly increasing for $n \geq 1$.

Equations (3.12) and (3.13) together with Lemma 2.1 and the monotonicity of $\{c_n/d_n\}$ lead to the conclusion that $\phi(t)$ is strictly increasing in $(0, \log(1 + \sqrt{2}))$. Moreover,

$$\lim_{t \to 0^+} \phi(t) = \frac{c_1}{d_1} = \frac{5}{12}, \qquad \lim_{t \to \log(1+\sqrt{2})} \phi(t) = 1 - \frac{1}{2\log\left(1 + \sqrt{2}\right)}. \tag{3.14}$$

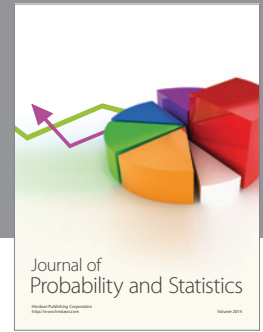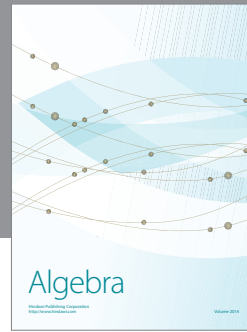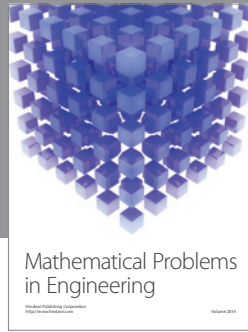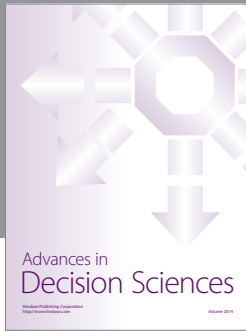Making use of (3.14) and (3.10) together with the monotonicity of $\phi(t)$ gives the asserted result. $\qquad\square$

Acknowledgments

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