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Research Article

Positive Solutions of Nonlinear Fractional Differential Equations with Integral Boundary Value Conditions

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We investigate the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary value conditions ${}^{C}D^{\alpha}u(t) + f(t, u(t)) = 0$, $0 < t < 1$, $u(0) = u''(0) = 0$, $u(1) = \lambda \int_0^1 u(s)ds$, where $2 < \alpha < 3$, $0 < \lambda < 2$ and ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function. Our analysis relies on a fixed point theorem in partially ordered sets. Moreover, we compare our results with others that appear in the literature.

1. Introduction

Many papers and books on fractional differential equations have appeared recently (see, for example, [1–22]). The interest of the study of fractional-order differential equations lies in the fact that fractional-order models are more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models.

Integral boundary conditions have various applications in chemical engineering, thermo-elasticity, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers (see, [23–30]) and the references therein. Recently, Cabada and Wang in [31] investigated the existence of positive solutions for the fractional boundary value problem

$$\begin{aligned} &{}^{C}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u''(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s)ds, \end{aligned} \tag{1.1}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

The main tool used in [31] is the well-known Guo-Krasnoselskii fixed point theorem and the question of uniqueness of solutions is not treated. We consider our paper as an alternative answer to the results of [31]. The fixed point theorem in partially ordered sets is the main tool used in our results. The existence of fixed points in partially ordered sets has been considered recently (see, e.g. [32–34]).

2. Preliminaries and Basic Facts

For the convenience of the reader, we present in this section some notations and lemmas which will be used in the proofs of our results. For details, see [35, 36].

Definition 2.1. The Caputo derivative of fractional order $\alpha > 0$ of a function $f : [0,\infty) \to \mathbb{R}$ is defined by

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \tag{2.1}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.2. The Riemman-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{2.2}$$

provided that such integral exists.

Definition 2.3. The Riemman-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$ D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds, \tag{2.3}$$

where $n = [\alpha] + 1$, provided that the right hand side is pointwise defined on $(0,\infty)$.

Lemma 2.4. *Let $\alpha > 0$ then the fractional differential equation*

$$ {}^{C}D^{\alpha} u(t) = 0, \tag{2.4}$$

has

$$ u(t) = \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^{j}, \tag{2.5}$$

as unique solution.

Lemma 2.5. *Let $\alpha > 0$ then*

$$I^{\alpha}\ {}^{C}D^{\alpha}u(t) = u(t) - \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^{j}. \tag{2.6}$$

In [31], the authors obtain the Green's function associated with Problem (1.1). More precisely, they proved the following result.

Theorem 2.6 (see [31]). *Let $2 < \alpha < 3$ and $\lambda \neq 2$. Suppose that $f \in C[0,1]$ then the unique solution of*

$$\begin{gathered} {}^{C}D^{\alpha}u(t) + f(t) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s)ds \end{gathered} \tag{2.7}$$

is $u(t) = \int_0^1 G(t,s) f(s) ds$, where

$$G(t,s) = \frac{1}{(2-\lambda)\Gamma(\alpha+1)} \begin{cases} 2t(1-s)^{\alpha-1}(\alpha - \lambda + \lambda s) - (2-\lambda)\alpha(t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ 2t(1-s)^{\alpha-1}(\alpha - \lambda + \lambda s), & 0 \le t \le s \le 1. \end{cases} \tag{2.8}$$

In [31], the following lemma is proved.

Lemma 2.7. *Let $G(t,s)$ be the Green's function associated to Problem* (2.7)*, which has the expression* (2.8)*. Then:*

(i) *$G(t,s) > 0$ for all $t,s \in (0,1)$ if and only if $\lambda \in [0,2)$.*

(ii) *$G(t,s) \le 2/(2-\lambda)\Gamma(\alpha)$ for all $t,s \in [0,1]$ and $\lambda \in [0,2)$.*

(iii) *For $2 < \alpha < 3$ and $\lambda \neq 2 G(t,s)$ is a continuous function on $[0,1] \times [0,1]$.*

In the sequel, we present the fixed point theorem which we will be use later. This result appears in [32].

Theorem 2.8 (see [32]). *Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \le Tx_0$.*

Suppose that

$$d(Tx,Ty) \le d(x,y) - \psi(d(x,y)) \quad \text{for } x,y \in X \text{ with } x \ge y, \tag{2.9}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function, ψ is positive on $(0,\infty)$ and $\psi(0) = 0$.

Assume that either T is continuous or X is such that

$$\text{if } \{x_n\} \text{ is a nondecreasing sequencein } X \text{ such that } x_n \to x \text{ then } x_n \le x,\ \forall n \in \mathbb{N}. \tag{2.10}$$

Besides, if

$$\text{for each } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \tag{2.11}$$

then T has a unique fixed point.

Remark 2.9. Notice that the condition $\lim_{t\to\infty}\psi(t) = \infty$ is superfluous in Theorem 2 of [32].

Remark 2.10. If we look at the proof of Theorem 2.2 in [32] we notice that the condition about the continuity of ψ is redundant.

In fact, from $x_0 \le Tx_0$ the authors generate the sequence $\{T^n x_0\}$ and if we put $x_{n+1} = T^n x_0$ it is proved that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le d(x_n, x_{n-1}) - \psi(d(x_n, x_{n-1})) \le d(x_n, x_{n-1}). \tag{2.12}$$

Consequently, $\{d(x_{n+1}, x_n)\}$ is a nonnegative decreasing sequence of real numbers and hence $\{d(x_{n+1}, x_n)\}$ posseses a limit ρ^*.

Taking limit when $n \to \infty$ in the last inequality, we obtain

$$\rho^* \le \rho^* - \lim_{n\to\infty}\psi(d(x_n, x_{n-1})) \le \rho^*, \tag{2.13}$$

and, therefore,

$$\lim_{n\to\infty}\psi(d(x_n, x_{n-1})) = 0. \tag{2.14}$$

Suppose that $\rho^* > 0$, since $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence $\rho^* \le d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$, and, since ψ is a nondecreasing function, we have $\psi(\rho^*) \le \psi(d(x_{n+1}, x_n))$ for all $n \in \mathbb{N}$.

As ψ is positive on $(0, \infty)$, $0 < \psi(\rho^*) \le \psi(d(x_{n+1}, x_n))$ for all $n \in \mathbb{N}$ and, therefore,

$$0 < \psi(\rho^*) \le \lim \psi(d(x_{n+1}, x_n)). \tag{2.15}$$

This contradicts to (2.14). Consequently, $\rho^* = 0$.

The rest of the proof works well and the condition about the continuity of ψ is not used.

Theorem 2.11. *Theorem 2.8 is valid without the assumption $\psi : [0, \infty) \to [0, \infty)$ is continuous.*

In our considerations, we will work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbb{R}, \text{continuous}\}$ with the classical metric given by $d(x, y) = \sup_{0\le t\le 1}\{|x(t) - y(t)|\}$.

Notice that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \le y \Longleftrightarrow x(t) \le y(t) \quad \text{for } t \in [0,1]. \tag{2.16}$$

In [33] it is proved that $(C[0,1], \le)$ satisfies condition (2.10) of Theorem 2.8. Moreover, for $x, y \in C[0,1]$, as the function $\max\{x, y\} \in C[0,1]$, $(C[0,1], \le)$ satisfies condition (2.11) of Theorem 2.8.

3. Main Result

Our starting point in this section is to present the class of function $\mathcal{A}$ which we will use later. By $\mathcal{A}$ we will denote the class of those functions $\phi : [0,\infty) \to [0,\infty)$ which are nondecreasing and such that if $\varphi(x) = x - \phi(x)$ then the following conditions are satisfied:

(a) $\varphi : [0,\infty) \to [0,\infty)$ and φ is nondecreasing.

(b) $\varphi(0) = 0$.

(c) φ is positive on $(0,\infty)$.

Examples of such functions are $\phi(x) = \arctan x$ and $\phi(x) = x/(1+x)$.

In what follows, we formulate our main result.

Theorem 3.1. *Suppose that* $2 < \alpha < 3$, $0 < \lambda < 2$ *and* $f : [0,1] \times [0,\infty) \to [0,\infty)$ *satisfies the following assumptions:*

(i) f *is continuous.*

(ii) $f(t,x)$ *is nondecreasing respect to the second argument for each* $t \in [0,1]$.

(iii) *There exist* $0 < \rho \leq (2-\lambda)\Gamma(\alpha)/2$ *and* $\phi \in \mathcal{A}$ *such that*

$$f(t,y) - f(t,x) \leq \rho\phi(y-x), \tag{3.1}$$

for $x, y \in [0,\infty)$ *with* $y \geq x$ *and* $t \in [0,1]$.

Then Problem (1.1) *has a unique nonnegative solution.*

Proof. Consider the cone

$$P = \{u \in C[0,1] : u \geq 0\}. \tag{3.2}$$

Notice that, as P is a closed set of $C[0,1]$, P is a complete metric with the distance given by $d(x,y) = \sup_{0\leq t\leq 1}\{|x(t) - y(t)|\}$ satisfying conditions (2.10) and (2.11) of Theorem 2.8.

Now, for $u \in P$ we define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))ds, \tag{3.3}$$

where $G(t,s)$ is the Green's function defined by (2.8).

By Lemma 2.7 and assumption (i), it is clear that T applies P into itself.

In the sequel, we will check that the assumptions of Theorem 2.11 are satisfied.

Firstly, the operator T is nondecreasing. In fact, by (ii), for $u \geq v$ we have

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))ds \geq \int_0^1 G(t,s) f(s,v(s))ds = (Tv)(t). \tag{3.4}$$

Besides, for $u \geq v$ and taking into account our assumptions, we can obtain

$$\begin{aligned} d(Tu,Tv) &= \max_{0\leq t\leq 1}\{|(Tu)(t)-(Tv)(t)|\} \\ &= \max_{0\leq t\leq 1}\{Tu(t)-Tv(t)\} \\ &= \max_{0\leq t\leq 1}\left[\int_0^1 G(t,s)\left(f(s,u(s))-f(s,v(s))\right)ds\right] \\ &\leq \max_{0\leq t\leq 1}\left[\int_0^1 G(t,s)\rho\phi(u(s)-v(s))ds\right]. \end{aligned} \tag{3.5}$$

Since ϕ is nondecreasing and $u \geq v$, we have

$$\phi(u(s)-v(s)) \leq \phi(d(u,v)), \tag{3.6}$$

and, from the last inequality it follows

$$d(Tu,Tv) \leq \rho\phi(d(u,v))\max_{0\leq t\leq 1}\int_0^1 G(t,s)ds. \tag{3.7}$$

By Lemma 2.7(ii) and since $\rho \leq (2-\lambda)\Gamma(\alpha)/2$, we can obtain

$$d(Tu,Tv) \leq \phi(d(u,v)) = d(u,v) - \left[d(u,v)-\phi(d(u,v))\right]. \tag{3.8}$$

Since $\phi \in \mathcal{A}$, $\psi(x) = x - \phi(x)$ satifies the properties (a), (b), and (c) mentioned at the beginning of this section, for $u \geq v$ we have

$$d(Tu,Tv) \leq d(u,v) - \psi(d(u,v)). \tag{3.9}$$

Finally, taking into account that the zero function, $0 \leq T0$ (where $T0 = Tu$ with $u(t) = 0$ for all $t \in [0,1]$), by Theorem 2.11, Problem (1.1) has a unique nonnegative solution. □

Now, we present a sufficient condition for the existence and uniqueness of a positive solution for Problem (1.1) (positive solution means a solution satisfying $x(t) > 0$ for $t \in (0,1)$).

Theorem 3.2. *Under assumptions of Theorem 3.1 and adding the following condition*

$$f(t_0,0) \neq 0 \quad \text{for certain } t_0 \in [0,1], \tag{3.10}$$

we obtain existence and uniqueness of a positive solution for Problem (1.1).

Proof. Consider the nonnegative solution $x(t)$ for Problem (1.1) whose existence is guaranteed by Theorem 3.1.

Notice that $x(t)$ satisfies

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) ds. \tag{3.11}$$

In what follows, we will prove that $x(t) > 0$ for $t \in (0,1)$.

In fact, in contrary case we can find $0 < t^* < 1$ such that $x(t^*) = 0$, and therefore,

$$x(t^*) = \int_0^1 G(t^*,s) f(s, x(s)) ds = 0. \tag{3.12}$$

Since $x \geq 0$ and $G(t,s) \geq 0$ (see, Lemma 2.7) and, taking into account the nondecreasing character with respect to the second argument of the function f, from the last inequality it follows

$$0 = x(t^*) = \int_0^1 G(t^*,s) f(s, x(s)) ds \geq \int_0^1 G(t^*,s) f(s, 0) ds \geq 0. \tag{3.13}$$

Thus, $\int_0^1 G(t^*,s) f(s,0) ds = 0$. This fact and the nonnegative character of the functions $G(t^*,s)$ and $f(s,0)$ give us

$$G(t^*,s) f(s,0) = 0 \quad \text{a.e. } (s). \tag{3.14}$$

Since $G(t^*,s) > 0$ for $s \in (0,1)$, we get

$$f(s,0) = 0 \quad \text{a.e. } (s). \tag{3.15}$$

By (3.10), since $f(t_0, 0) \neq 0$ for certain $t_0 \in [0,1]$, this means that $f(t_0, 0) > 0$, and taking into account the continuity of f, we can find a set $\Omega \subset [0,1]$ with $t_0 \in \Omega$ and $\mu(\Omega) > 0$ (where μ is the Lebesgue measure) such that $f(t,0) > 0$ for any $t \in \Omega$. This contradicts to (3.15).

Therefore, $x(t) > 0$. □

Remark 3.3. In Theorem 3.2, the condition $f(t_0, 0) \neq 0$ for certain $t_0 \in [0,1]$ seems to be a strong condition in order to obtain a positive solution for Problem (1.1), but when the solution is unique we will see that the condition is very adjusted one. In fact, under the assumption that Problem (1.1) has a unique nonnegative solution $x(t)$ we have that

$$f(t,0) = 0 \quad \text{for any } t \in [0,1] \text{ iff } x(t) \equiv 0. \tag{3.16}$$

Indeed, if $f(t,0) = 0$ for any $t \in [0,1]$ then it is easily seen that the zero function is a solution for Problem (1.1) and the uniqueness of solution gives us $x(t) \equiv 0$.

The reverse implication is obvious.

Remark 3.4. Notice that assumptions in Theorem 3.1 are invariant by nonnegative and continuous perturbations. More precisely, if $f(t,0) = 0$ for any $t \in [0,1]$ and f satisfies conditions (i), (ii), and (iii) of Theorem 3.1 then $g(t,x) = f(t,x) + a(t)$, where $a : [0,1] \to [0,\infty)$ continuous and $a \neq 0$ satisfies assumptions of Theorem 3.2 and, consequently, the boundary value problem

$$\begin{gathered} {}^{C}D^{\alpha}u(t) + g(t,u(t)) = 0, \quad 0 < t < 1, 2 < \alpha < 3 \\ u(0) = u''(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s)ds, \end{gathered} \tag{3.17}$$

with $0 < \lambda < 2$, has a unique positive solution.

Example 3.5. Now, consider the following boundary value problem

$$\begin{gathered} {}^{C}D^{\alpha}u(t) + t + \frac{\gamma u(t)}{1+u(t)} = 0, \quad 0 < t < 1, 2 < \alpha < 3, \gamma > 0 \\ u(0) = u''(0) = 0, \quad u(1) = \frac{\sin 1}{1-\cos 1} \int_0^1 u(s)ds. \end{gathered} \tag{3.18}$$

In this case, $f(t,u) = t + (\gamma u/(1+u))$. Obviously, $f : [0,1] \times [0,\infty) \to [0,\infty)$ and f is continuous. Since $\partial f/\partial u = \gamma/(1+u)^2 > 0$, f satisfies condition (ii) of Theorem 3.1.

Moreover, for $u \geq v$ and $t \in [0,1]$ we have

$$\begin{aligned} f(t,u) - f(t,v) &= t + \frac{\gamma u}{1+u} - t - \frac{\gamma v}{1+v} \\ &= \gamma\left(\frac{u}{1+u} - \frac{v}{1+v}\right) = \gamma\left(\frac{u-v}{(1+u)(1+v)}\right) \\ &\leq \gamma\frac{u-v}{1+u-v} = \gamma\phi(u-v), \end{aligned} \tag{3.19}$$

where $\phi(x) = x/(1+x)$. It is easily seen that ϕ belongs to the class $\mathcal{A}$. In this case, $\lambda = \sin 1/(1-\cos 1) \approx 1,83048$, consequently $0 < \lambda < 2$, and for $0 < \gamma \leq (2-\lambda)/2 \cdot \Gamma(\alpha) \approx 0,08476\Gamma(\alpha)$, Problem (3.18) satisfies (iii) of Theorem 3.1. Since $f(t,0) = t \neq 0$ for $t \neq 0$, Theorem 3.2 says that Problem (3.18) has a unique positive solution for $0 < \gamma \leq 0,08476\Gamma(\alpha)$, where $2 < \alpha < 3$.

4. Some Remarks and Examples

In [31] the authors consider Problem (1.1).

In order to present the main result of [31] we need the following notation. Denote by f_0 and f_∞ the following limits:

$$f_0 = \lim_{u \to 0^+}\left\{\min_{t\in[0,1]}\frac{f(t,u)}{u}\right\}, \qquad f_\infty = \lim_{u\to\infty}\left\{\max_{t\in[0,1]}\frac{f(t,u)}{u}\right\}. \tag{4.1}$$

The main result of [31] is the following theorem.

Theorem 4.1. *Assume that one of the two following conditions is fulfilled:*

(i) *(sublinear case)* $f_0 = \infty$ *and* $f_\infty = 0$,

(ii) *(superlinear case)* $f_0 = 0$, $f_\infty = \infty$ *and there exist* $\mu > 0$ *and* $\theta > 0$ *for which* $f(t, \rho x) \geq \mu\rho^\theta f(t, x)$ *for all* $\rho \in (0, 1]$.

Then, Problem (1.1) *has at least one positive solution that belongs to*

$$P = \left\{ u \in C[0,1] : \frac{u}{id_{[0,1]}} \in C[0,1], u(t) \geq \frac{t\lambda(\alpha-2)}{2\alpha} \|u\| \ \forall t \in [0,1] \right\}, \tag{4.2}$$

where $id_{[0,1]}$ *is the identity mapping on* $[0,1]$.

Notice that in Example 3.5, $f(t,u) = t + (\gamma u/(1+u))$ and in this case we have

$$\min_{t\in[0,1]} \frac{f(t,u)}{u} = \frac{\gamma}{1+u}. \tag{4.3}$$

Consequently,

$$f_0 = \lim_{u\to 0^+} \left\{ \min_{t\in[0,1]} \frac{f(t,u)}{u} \right\} = \lim_{u\to 0^+} \frac{\gamma}{1+u} = \gamma. \tag{4.4}$$

Therefore, for $0 < \gamma \leq 0,08476\Gamma(\alpha)$ Example 3.5 cannot be treated by Theorem 4.1.

Example 4.2. Consider the following boundary value problem

$$\begin{gathered} {}^C D^\alpha u(t) + c + \gamma \ \arctan \ u(t) = 0, \quad 0 < t < 1, c > 0, \gamma > 0, 2 < \alpha < 3, \\ u(0) = u''(0) = 0, \qquad u(1) = \frac{1}{2}\int_0^1 u(s)ds, \end{gathered} \tag{4.5}$$

In this case, $0 < \lambda = (1/2) < 2$, $f(t,u) = c + \gamma \arctan u$.

It is easily proved that f satisfies condition (i) and (ii) of Theorem 3.1. In [37], it is proved that if $u \geq v \geq 0$

$$\arctan\ u - \arctan\ v \leq \arctan(u - v). \tag{4.6}$$

Using this fact, for $u \geq v \geq 0$ and $t \in [0,1]$, we have

$$f(t,u) - f(t,v) = \gamma(\arctan\ u - \arctan\ v) \leq \gamma\ \arctan(u - v) = \gamma\phi(u - v), \tag{4.7}$$

where $\phi(x) = \arctan x$. It is easily proved that $\phi \in \mathcal{A}$.

Then, for $0 < \gamma \leq (3/4)\Gamma(\alpha)$, the function f satisfies condition (iii) of Theorem 3.1. Moreover, since $f(t,0) = c > 0$, Theorem 3.2 gives us the existence and uniqueness of a positive solution for Problem (4.5) when $0 < \gamma \leq (3/4)\Gamma(\alpha)$.

On the other hand, we have

$$\max_{t\in[0,1]} \frac{f(t,u)}{u} = \min_{t\in[0,1]} \frac{f(t,u)}{u} = \frac{c+\gamma \ \arctan \ u}{u},$$
$$f_0 = \lim_{u\to 0^+} \frac{c+\gamma \arctan \ u}{u} = \infty, \qquad f_\infty = \lim_{u\to\infty} \frac{c+\gamma \arctan \ u}{u} = 0. \tag{4.8}$$

Consequently, this example corresponds to the sublinear case of Theorem 4.1. Therefore, Theorem 4.1 gives us the existence of at least one positive solution for $0 < \gamma \leq (3/4)\Gamma(\alpha)$. The question of uniqueness of solutions is not treated in [31].

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