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Research Article

Applications of Measure of Noncompactness in Matrix Operators on Some Sequence Spaces

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We determine the conditions for some matrix transformations from $n(\phi)$, where the sequence space $n(\phi)$, which is related to the ℓ_p spaces, was introduced by Sargent (1960). We also obtain estimates for the norms of the bounded linear operators defined by these matrix transformations and find conditions to obtain the corresponding subclasses of compact matrix operators by using the Hausdorff measure of noncompactness.

1. Introduction and Preliminaries

We shall write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let φ, ℓ_∞, c, and c_0 denote the sets of all finite, bounded, convergent, and null sequences, respectively. We write $\ell_p := \{x \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \le p < \infty$. By e and $e^{(n)} (n \in \mathbb{N})$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section. Moreover, we write bs and cs for the sets of sequences with bounded and convergent partial sums, respectively.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a linear metric space X is called *Schauder basis* if for every $x \in X$, there is a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b^{(n)}$. A sequence space X with a linear topology is called a *K-space* if each of the maps $p_i : X \to C$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called an *FK-space* if X is complete linear metric space; a *BK-space* is a normed FK-space. An FK-space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)} \to x$ as $n \to \infty$.

The spaces c_0, c and $\ell_p (1 \leq p < \infty)$ all have Schauder bases but the space ℓ_∞ has no Schauder basis. Among the other classical sequence spaces, the spaces c_0 and $\ell_p (1 \leq p < \infty)$ have AK.

Let $(X, \| \cdot \|)$ be a normed space. Then the unit sphere and closed unit ball in X are denoted by $S_X := \{x \in X : \|x\| = 1\}$ and $\overline{B}_X := \{x \in X : \|x\| \leq 1\}$. If $X \supset \varphi$ is a BK-space and $a = (a_k) \in w$, then we define

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right| \tag{1.1}$$

provided the expression on the right-hand side exists and is finite.

The α-, β-, and γ-*duals* of a subset X of w are, respectively, defined by

$$\begin{aligned} X^\alpha &= \{a = (a_k) \in w : ax = (a_k x_k) \in \ell_1 \ \forall \ x = (x_k) \in X\}, \\ X^\beta &= \{a = (a_k) \in w : ax = (a_k x_k) \in cs \ \forall \ x = (x_k) \in X\}, \\ X^\gamma &= \{a = (a_k) \in w : ax = (a_k x_k) \in bs \ \forall \ x = (x_k) \in X\}. \end{aligned} \tag{1.2}$$

Throughout this paper, the matrices are infinite matrices of complex numbers. If A is an infinite matrix with complex entries $a_{nk} (n, k \in \mathbb{N})$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the nth row of A, that is, $A_n = (a_{nk})_{n,k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in w$, then we define the A-*transform* of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \quad (n \in \mathbb{N}) \tag{1.3}$$

provided the series on the right converges for each $n \in \mathbb{N}$.

Let X and Y be subsets of w and $A = (a_{nk})$ an infinite matrix. Then, we say that A defines a *matrix mapping* from X into Y, and we denote it by writing $A : X \to Y$ if Ax exists and is in Y for all $x \in X$. By (X, Y), we denote the class of all infinite matrices that map X into Y. Thus $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

Lemma 1.1 (see[1]). *Let $\dagger$ denote any of the symbols α, β, or γ. Then, we have $c_0^\dagger = c^\dagger = \ell_\infty^\dagger = \ell_1$, $\ell_1^\dagger = \ell_\infty$ and $\ell_p^\dagger = \ell_q$, where $1 < p < \infty$ and $q = p/(p-1)$.*

Lemma 1.2 (see[1, 2]). *Let X be any of the spaces c_0, c, ℓ_∞, or $\ell_p (1 \leq p < \infty)$. Then, we have $\| \cdot \|_X^* = \| \cdot \|_{X^\beta}$ on X^β, where $\| \cdot \|_{X^\beta}$ denotes the natural norm on the dual space X^β.*

Lemma 1.3 (see[1, 2]). *Let $X \supset \varphi$ and Y be BK-spaces. Then, we have*

(a) *$(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$;*

(b) *if X has AK, then $\mathcal{B}(X, Y) \subset (X, Y)$, that is, for every operator $L \in \mathcal{B}(X, Y)$ there exists a matrix $A \in (X, Y)$ such that $L(x) = Ax$ for all $x \in X$.*

Furthermore, we have the following results on the operator norms.

Lemma 1.4 (see[2]). *Let $X \supset \varphi$ be a BK-space and Y any of the spaces c_0, c, or ℓ_∞. If $A \in (X, Y)$, then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty, \tag{1.4}$$

where $\|A\|_{(X,\ell_\infty)}$ denotes the operator norm for the matrix $A \in (X, \ell_\infty)$.

Sargent [3] defined the following sequence spaces.

Let C denote the space whose elements are finite sets of distinct positive integers. Given any element σ of C, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ such that $c_n(\sigma) = 1$ for $n \in \sigma$ and $c_n(\sigma) = 0$ otherwise. Further

$$C_s = \left\{\sigma \in C : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\right\}, \tag{1.5}$$

that is, C_s is the set of those σ whose support has cardinality at most s, and we get

$$\Phi = \left\{\phi = (\phi_k) \in w : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1} \text{ and } (n+1)\phi_n \geq n\phi_{n+1}\right\}. \tag{1.6}$$

For $\phi \in \Phi$, the following sequence spaces were introduced by Sargent [3] and further studied in [4]

$$\begin{aligned} m(\phi) &= \left\{x = (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s\geq 1}\sup_{\sigma \in C_s}\left(\frac{1}{\phi_s}\sum_{k\in\sigma}|x_k|\right) < \infty\right\}, \\ n(\phi) &= \left\{x = (x_k) \in w : \|x\|_{n(\phi)} = \sup_{u \in S(x)}\left(\sum_{k=1}^{\infty}|u_k|\Delta\phi_k\right) < \infty\right\}, \end{aligned} \tag{1.7}$$

where $\Delta\phi_k = \phi_k - \phi_{k-1}$ and $S(x)$ denotes the set of all sequences that are rearrangements of x.

Remark 1.5 ([3]). (i) The spaces $m(\phi)$ and $n(\phi)$ are BK spaces with their respective norms. (ii) If $\phi_n = 1$ for all $n \in \mathbb{N}$, then $m(\phi) = \ell_1$, $n(\phi) = \ell_\infty$; if $\phi_n = n$ for all $n \in \mathbb{N}$, then $m(\phi) = \ell_\infty$, $n(\phi) = \ell_1$. (iii) $\ell_1 \subseteq m(\phi) \subseteq \ell_\infty[\ell_\infty \supseteq n(\phi) \supseteq \ell_1]$ for all ϕ of Φ.(iv) $(m(\phi))^\dagger = n(\phi)$ and $(n(\phi))^\dagger = m(\phi)$, where $\dagger$ is any of the symbols α, β, or γ.

Recently, Makowsky and Mursaleen [5] have characterized the classes of compact operators on some BK-spaces, namely, $\mathcal{C}(m(\phi), \ell_p)$ $(1 \leq p \leq \infty)$, $\mathcal{C}(n(\phi), \ell_p)(1 \leq p < \infty)$, $\mathcal{C}(\ell_p, m(\phi))(1 < p \leq \infty)$, and $\mathcal{C}(\ell_p, n(\phi))(1 \leq p \leq \infty)$. In this paper, we determine the conditions for the classes of matrix transformations $(n(\phi), c_0)$, $(n(\phi), c)$, and $(n(\phi), \ell_\infty)$, and establish estimates for the norms of the bounded linear operators defined by these matrix transformations. Further, we obtain the necessary and sufficient (or only sufficient) conditions for the corresponding subclasses of compact matrix operators $\mathcal{C}(n(\phi), c_0)$, $\mathcal{C}(n(\phi), c)$, and $\mathcal{C}(n(\phi), \ell_\infty)$ by using the Hausdorff measure of noncompactness.

2. The Hausdorff Measure of Noncompactness

Let $(X, \|\cdot\|)$ be a normed space. Then the unit sphere and closed unit ball in X are denoted by $S_X := \{x \in X : \|x\| = 1\}$ and $\overline{B}_X := \{x \in X : \|x\| \leq 1\}$. If X and Y are Banach spaces then $B(X,Y)$ is the set of all bounded linear operators $L : X \to Y$; $B(X,Y)$ is a Banach space with the operator norm given by $\|L\| = \sup_{x \in S_X} \|L(x)\|$ for all $L \in \mathcal{B}(X,Y)$. A linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X, and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a subsequence which converges in Y. We denote the class of all compact operators in $\mathcal{B}(X,Y)$ by $\mathcal{C}(X,Y)$. An operator $L \in \mathcal{B}(X,Y)$ is said to be *of finite rank* if $\dim R(L) < \infty$, where $R(L)$ is the range space of L. An operator of finite rank is clearly compact. In particular, if $Y = \mathbb{C}$ then we write X^* for the set of all continuous linear functionals on X with the norm $\|f\| = \sup_{x \in S_X} |f(x)|$.

The Hausdorff measure of noncompactness was defined by *Goldenštein* et al. in 1957 [6].

Let S and M be subsets of a metric space (X,d) and $\varepsilon > 0$. Then, S is called an *ε-net* of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x,s) < \varepsilon$. Further, if the set S is finite, then the ε-net S of M is called a *finite ε-net* of M, and we say that M has a finite ε-net in X. A subset of a metric space is said to be *totally bounded* if it has a finite ε-net for every $\varepsilon > 0$.

By $\mathcal{M}_X$, we denote the collection of all bounded subsets of a metric space (X,d). If $Q \in \mathcal{M}_X$, then the *Hausdorff measure of noncompactness* of the set Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) := \inf\left\{\epsilon > 0 : Q \subset \cup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon (i = 1, 2, \ldots), n \in \mathbb{N}\right\}. \tag{2.1}$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the *Hausdorff measure of noncompactness*.

The basic properties of the Hausdorff measure of noncompactness can be found in [2, 7–9] and for recent developments, see [10–18]. If Q, Q_1, and Q_2 are bounded subsets of a metric space (X,d), then

$$\begin{aligned} \chi(Q) = 0 &\quad \text{iff } Q \text{ is totally bounded}, \\ Q_1 \subset Q_2 &\quad \text{implies } \chi(Q_1) \leq \chi(Q_2). \end{aligned} \tag{2.2}$$

Further, if X is a normed space, then the function χ has some additional properties connected with the linear structure, for example,

$$\begin{aligned} \chi(Q_1 + Q_2) &\leq \chi(Q_1) + \chi(Q_2), \\ \chi(\alpha Q) &= |\alpha| \chi(Q) \quad \forall \alpha \in \mathbb{C}. \end{aligned} \tag{2.3}$$

Let X and Y be Banach spaces and χ_1 and χ_2 the Hausdorff measures of noncompactness on X and Y, respectively. An operator $L : X \to Y$ is said to be (χ_1,χ_2)-bounded if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$ and there exist a constant $C \geq 0$ such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in \mathcal{M}_X$. If an operator L is (χ_1,χ_2)-*bounded*, then the number $\|L\|_{(\chi_1,\chi_2)} := \inf\{C \geq 0 : \ \chi_2(L(Q)) \leq C\chi_1(Q) \text{ for all } Q \in \mathcal{M}_X\}$ is called the (χ_1, χ_2)-*measure* of *noncompactness* $of L$. If $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi_1,\chi_2)} = \|L\|_{\chi}$.

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows: Let X and Y be Banach spaces and $L \in B(X,Y)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_{\chi}$, can be determined by

$$\|L\|_{\chi} = \chi(L(S_X)), \tag{2.4}$$

and we have that L is compact if and only if

$$\|L\|_{\chi} = 0. \tag{2.5}$$

Now, the following result gives an estimate for the Hausdorff measure of noncompactness in Banach spaces with Schauder bases. It is known that if $(b_k)_{k=0}^{\infty}$ is a Schauder basis for a Banach space X, then every element $x \in X$ has a unique representation $x = \sum_{k=0}^{\infty} \alpha_k(x)\ b_k$, where $\alpha_k (k \in \mathbb{N})$ are called the basis functionals. Moreover, for each $n \in \mathbb{N}$, the operator $P_n : X \to X$ defined by $P_n(x) = \sum_{k=0}^{n} \alpha_k(x)\ b_k (x \in X)$ is called the projector onto the linear span of $\{b_0, b_1, \ldots, b_n\}$. Besides, all operators P_n and $I - P_n$ are equibounded, where I denotes the identity operator on X.

Theorem 2.1 (see[7]). *Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $Q \in \mathcal{M}_X$, and $P_n : X \to X (n \in \mathbb{N})$ the projector onto the linear span of $\{b_0, b_1, \ldots, b_n\}$. Then, we have*

$$\frac{1}{a} \cdot \limsup_{n \to \infty} \left(\sup_{x \in Q} \|(I - P_n)(x)\| \right) \leq \chi(Q) \leq \limsup_{n \to \infty} \left(\sup_{x \in Q} \|(I - P_n)(x)\| \right), \tag{2.6}$$

where $a = \limsup_{n \to \infty} \|I - P_n\|$.

In particular, the following result shows how to compute the Hausdorff measure of noncompactness in the spaces c_0 and $\ell_p (1 \leq p < \infty)$, which are BK-spaces with AK.

Theorem 2.2 (see[7]). *Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \leq p < \infty$ or c_0. If $P_n : X \to X (n \in \mathbb{N})$ is the operator defined by $P_n(x) = x^{[n]} = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ for all $x = (x_k)_{k=0}^{\infty} \in X$, then we have*

$$\chi(Q) = \lim_{n \to \infty} \left(\sup_{x \in Q} \|(I - P_n)(x)\| \right). \tag{2.7}$$

The Haudorff measure of noncompactness for $n(\phi)$ has recently been determined in [19] as follows.

Theorem 2.3. *Let Q be a bounded subset of $n(\phi)$. Then*

$$\chi(Q) = \lim_{k \to \infty} \sup_{x \in Q} \left(\sup_{u \in S(x)} \left(\sum_{n=k}^{\infty} |u_n| \Delta\phi_n \right) \right). \tag{2.8}$$

3. Main Results

First we prove the following basic lemma.

Lemma 3.1. *If $A \in (n(\phi), c)$, then the following hold*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \quad \text{exists for every } k \in \mathbb{N}, \tag{3.1}$$

$$\alpha = (\alpha_k) \in m(\phi), \tag{3.2}$$

$$\sup_n \|A_n - \alpha\|^*_{n(\phi)} < \infty, \tag{3.3}$$

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \quad \forall x = (x_k) \in n(\phi). \tag{3.4}$$

Proof. We write $\|x\| = \|x\|_{n(\phi)}$, for short. Since $A \in (n(\phi), c)$, we have

$$\|L_A\| = \sup_n \|A_n\|^*_{n(\phi)} < \infty. \tag{$*$}$$

Further, since $e^{(k)} \in n(\phi)$ and hence $Ae^{(k)} \in c$ for all $k \in \mathbb{N}$. Consequently, the limits α_k in (3.1) exist for all $k \in \mathbb{N}$.

Now, let $x \in n(\phi)$ be given. Then there is a positive constant K such that $\|x^{[p]}\| \le K \|x\|$ for all $p \in \mathbb{N}$. Thus we have

$$\left|\sum_{k=0}^{p} a_{nk} x_k\right| = \left|A_n\left(x^{[p]}\right)\right| \le \left\|Ax^{[p]}\right\|_{l_\infty} = \left\|L_A(x^{[p]})\right\|_{l_\infty} \le K\|L_A\|\|x\| \tag{3.5}$$

for all $p,\ n \in \mathbb{N}$. Hence, we obtain from (3.1) that

$$\left|\sum_{k=0}^{p} \alpha_k x_k\right| = \lim_{n\to\infty}\left|\sum_{k=0}^{p} a_{nk} x_k\right| \le K\ \|L_A\|\|x\|; \quad (p \in \mathbb{N}). \tag{3.6}$$

This implies that $\alpha x = (\alpha_k x_k) \in bs$, and since $x \in X$ was arbitrary, we deduce that $\alpha \in n(\phi)^\gamma$. But $n(\phi)^\gamma = n(\phi)^\beta$ and hence (3.2) holds. Moreover, since $n(\phi) \supset \varphi$ is a BK space, (3.2) implies $\|\alpha\|^*_{n(\phi)} < \infty$ by (Wilansky [19, Theorem 7.2.9]). Therefore, we get (3.3) from (15) by using (1.3).

Now, define the matrix $B = (b_{nk})$ by $b_{nk} = a_{nk} - \alpha_k$ for all $n, k \in \mathbb{N}$. Then, it is obvious that $B_n \in n(\phi)^\beta = m(\phi)$ for all $n \in \mathbb{N}$. Also, it follows by (3.3) that

$$\sup_n \|B_n\|^*_{n(\phi)} = \sup_n \|A_n - \alpha\|^*_{n(\phi)} < \infty. \tag{3.7}$$

Furthermore, we have from (3.1) that

$$\lim_{n\to\infty} B_n\left(e^{(k)}\right) = \lim_{n\to\infty} b_{nk} = 0 \quad (k \in \mathbb{N}), \tag{3.8}$$

that is, $Be^{(k)} \in c_0$ for all $k \in \mathbb{N}$. This leads us to the consequence that $B \in (n(\phi), c_0)$ by (Malkowsky-Rakocevic [2, Theorem 1.23(c)]). Hence, $\lim_{n\to\infty} B_n(x) = 0$ for all $x \in n(\phi)$, which yields (3.4). □

This completes the proof of the lemma.

Theorem 3.2. *(a) If $A \in (n(\phi), c_0)$, then*

$$\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n>r}\|A_n\|^*_{n(\phi)}\right) \tag{3.9}$$

(b) If $A \in (n(\phi), c)$, then

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n-\alpha\|^*_{n(\phi)}\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n-\alpha\|^*_{n(\phi)}\right), \tag{3.10}$$

where $\alpha = (\alpha_k)$ with $\alpha_k = \lim_{r\to\infty} a_{nk}$ for all $k \in \mathbb{N}$.

(c) If $A \in (n(\phi), \ell_\infty)$, then

$$0 \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n\|^*_{n(\phi)}\right). \tag{3.11}$$

Proof. We write $S = S_{n(\phi)}$, for short. Then, we have by (2.4) and Lemma 1.3 (a) that

$$\|L_A\|_\chi = \chi(AS). \tag{3.12}$$

For (a), we have $AS \in \mathcal{M}_{c_0}$. Thus, it follows by Theorem 2.2 that

$$\|L_A\|_\chi = \chi(AS) = \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right), \tag{3.13}$$

where $P_r : c_0 \to c_0 (r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. This yields that $\|(I-P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r}|A_n(x)|$ for all $x \in n(\phi)$ and every $r \in \mathbb{N}$. Thus, by combining (1.1) and (1.3), we have for every $r \in \mathbb{N}$ that

$$\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r}\|A_n\|^*_{n(\phi)}. \tag{3.14}$$

Hence, by (3.13) we get (3.9).

To prove (b), we have $AS \in \mathcal{M}_c$. Thus, we are going to apply Theorem 2.1 to get an estimate for the value of $\chi(AS)$ in (3.12). For this, we know that every $z = (z_n) \in c$ has a unique representation $z = \bar{z}e + \sum_{n=0}^{\infty}(z_n - \bar{z})e^{(n)}$, where $\bar{z} = \lim_{n\to\infty} z_n$. Thus, we define the projectors $P_r : c \to c(r \in \mathbb{N})$ by $P_0(z) = \overline{z}e$ and $P_r(z) = \overline{z}e + \sum_{n=0}^{r-1}(z_n - \overline{z})\, e^{(n)}$ for $r \geq 1$. Then, we have for every $r \in \mathbb{N}$ that $(I - P_r)(z) = \sum_{n=r}^{\infty}(z_n - \overline{z})e^{(n)}$ and hence

$$\|(I - P_r)(z)\|_{\ell_\infty} = \sup_{n>r}|z_n - \overline{z}| \tag{3.15}$$

for all $z \in c$ and every $r \in \mathbb{N}$. Obviously $\|(I - P_r)(z)\|_{\ell_\infty} \leq 2\|z\|_{\ell_\infty}$, hence $\|I - P_r\| \leq 2$ for all $r \in \mathbb{N}$. Further, for each $r \in \mathbb{N}$, we define the sequence $z^{(r)} = (z_n^{(r)}) \in c$ by $z_r^{(r)} = -1$ and $z_n^{(r)} = 1$ for $n \neq r$. Then $\|z^{(r)}\|_{\ell_\infty} = 1$ and $\lim_{n\to\infty} z_n^{(r)} = 1$. Therefore, $\|I - P_r\| \geq \|(I - P_r)(z^{(r)})\|_{\ell_\infty} = 2$ by (3.15). Consequently, we have $\|I - P_r\| = 2$ for all $r \in \mathbb{N}$. Hence, from (3.12) we obtain by applying Theorem 2.1 that

$$\frac{1}{2} \cdot \mu(A) \leq \|L_A\|_\chi \leq \mu(A), \tag{3.16}$$

where

$$\mu(A) = \lim_{r\to\infty} \sup\left(\sup_{x\in S}\|(I - P_r)(Ax)\|_{\ell_\infty}\right). \tag{3.17}$$

Now, it is given that $A \in (n(\phi), c)$. Thus, it follows from Lemma 3.1 that the limits $\alpha_k = \lim_{n\to\infty} a_{nk}$ exist for all k, $\alpha = (\alpha_k) \in n(\phi)^\beta = m(\phi)$ and

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty}\alpha_k x_k \tag{3.18}$$

for all $x = (x_k) \in n(\phi)$. Therefore, we derive from (3.15) that

$$\begin{aligned}\|(I - P_r)(Ax)\|_{\ell_\infty} &= \sup_{n\geq r}\left|A_n(x) - \sum_{k=0}^{\infty}\alpha_k x_k\right| \\ &= \sup_{n\geq r}\left|\sum_{k=0}^{\infty}(a_{nk} - \alpha_k)x_k\right|\end{aligned} \tag{3.19}$$

for all $x = (x_k) \in n(\phi)$ and every $r \in \mathbb{N}$. Consequently, we obtain by (1.3) that

$$\sup_{x\in S}\|(I - P_r)(Ax)\|_{\ell_\infty} = \sup_{n\geq r}\|A_n - \alpha\|^*_{n(\phi)} \quad (r \in \mathbb{N}). \tag{3.20}$$

Hence, we get (3.10) from (3.16).

Finally, to prove (c) we define $P_r : \ell_\infty \to \ell_\infty(r \in \mathbb{N})$ as in the proof of part (a) for all $x = (x_k) \in \ell_\infty$.

Then, it is clear that

$$AS \subset P_r(AS) + (I - P_r)(AS) \quad (r \in \mathbb{N}). \tag{3.21}$$

Thus, it follows by the elementary properties of the function χ that

$$\begin{aligned} 0 \le \ \chi(AS) &\le \chi(P_r(AS)) + \chi((I-P_r)(AS)) \\ &= \chi((I-P_r)(AS)) \\ &\le \sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} \\ &= \sup_{n>r}\|A_n\|^*_{n(\phi)} \end{aligned} \tag{3.22}$$

for all $r \in \mathbb{N}$. This and (3.12) together imply (3.11). This completes the proof of the theorem. □

As an immediate consequence of Theorem 3.2 and (2.5), we have the following.

Corollary 3.3. *(a) If $A \in (n(\phi), c_0)$, then*

$$L_A \text{ is compact iff } \lim_{r\to\infty}\left(\sup_{n>r}\|A_n\|^*_{n(\phi)}\right) = 0. \tag{3.23}$$

(b) If $A \in (n(\phi), c)$, then

$$L_A \text{ is compact iff } \lim_{r\to\infty}\left(\sup_{n\ge r}\|A_n - \alpha\|^*_{n(\phi)}\right) = 0, \tag{3.24}$$

where $\alpha = (\alpha_k)$ with $\alpha_k = \lim_{n\to\infty} a_{nk}$for all $k \in \mathbb{N}$.

(c) If $A \in (n(\phi), \ell_\infty)$, then

$$L_A \text{ is compact if } \lim_{r\to\infty}\left(\sup_{n>r}\|A_n\|^*_{n(\phi)}\right) = 0. \tag{3.25}$$

Remark 3.4. It is worth mentioning that the condition in (3.25) is only a sufficient condition for the operator L_A to be compact, where $A \in (n(\phi), \ell_\infty)$. In the following example, we show that it is possible for L_A to be compact while $\lim_{r\to\infty}(\sup_{n>r}\|A_n\|^*_{n(\phi)}) \neq 0$.

Choose a fixed $m \in \mathbb{N}$ such that $x'_m \neq 0$ for some $x' = (x'_k) \in S_{n(\phi)}$. Now, we define the matrix $A = (a_{nk})$ by $a_{nm} = 1$ and $a_{nk} = 0$ for all $k \neq m$ $(n \in \mathbb{N})$. Then, we have $Ax = x_m e$ for all $x = (x_k) \in w$, hence $A \in (w, \ell_\infty) \subset (n(\phi), \ell_\infty)$. Also, since L_A is of finite rank, L_A is compact. On the other hand, we have $A_n = e^{(m)}$ and hence $\|A_n\|^*_{n(\phi)} = \sup_{x\in S_{n(\phi)}}|x_m|$ for all $n \in \mathbb{N}$ by (1.3). This implies that

$$\lim_{r\to\infty}\left(\sup_{n>r}\|A_n\|^*_{n(\phi)}\right) = \operatorname*{syp}_{x\in S_{n(\phi)}} |x_m| \ge \left|x'_m\right| > 0. \tag{3.26}$$

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