Research Article

Some Existence Results for Impulsive Nonlinear Fractional Differential Equations with Closed Boundary Conditions

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We investigate some existence results for the solutions to impulsive fractional differential equations having closed boundary conditions. Our results are based on contracting mapping principle and Burton-Kirk fixed point theorem.

1. Introduction

This paper considers the existence and uniqueness of the solutions to the closed boundary value problem (BVP), for the following impulsive fractional differential equation:

$$^{C}D^{\alpha}x(t) = f(t, x(t)), \quad t \in J := [0, T], \ t \neq t_k, \ 1 < \alpha \leq 2,$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \quad \Delta x'(t_k) = I_k^*(x(t_k^-)), \quad k = 1, 2, \ldots, p, \tag{1.1}$$

$$x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0),$$

where $^{C}D^{\alpha}$ is Caputo fractional derivative, $f \in C(J \times R, R)$, $I_k, I_k^* \in C(R, R)$,

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-), \tag{1.2}$$

with

$$x(t_k^+) = \lim_{h \to 0^+} x(t_k + h), \qquad x(t_k^-) = \lim_{h \to 0^-} x(t_k + h), \tag{1.3}$$

and $\Delta x'(t_k)$ has a similar meaning for $x'(t)$, where

$$0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T, \tag{1.4}$$

a, b, c, and d are real constants with $\Delta := c(1 - b) + (1 - a)(1 - d) \neq 0$.

The boundary value problems for nonlinear fractional differential equations have been addressed by several researchers during last decades. That is why, the fractional derivatives serve an excellent tool for the description of hereditary properties of various materials and processes. Actually, fractional differential equations arise in many engineering and scientific disciplines such as, physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, and porous media (see [1–7]). For some recent development, see, for example, [8–14].

On the other hand, theory of impulsive differential equations for integer order has become important and found its extensive applications in mathematical modeling of phenomena and practical situations in both physical and social sciences in recent years. One can see a noticeable development in impulsive theory. For instance, for the general theory and applications of impulsive differential equations we refer the readers to [15–17].

Moreover, boundary value problems for impulsive fractional differential equations have been studied by some authors (see [18–20] and references therein). However, to the best of our knowledge, there is no study considering closed boundary value problems for impulsive fractional differential equations.

Here, we notice that the closed boundary conditions in (1.1) include quasi-periodic boundary conditions ($b = c = 0$) and interpolate between periodic ($a = d = 1, b = c = 0$) and antiperiodic ($a = d = -1, b = c = 0$) boundary conditions.

Motivated by the mentioned recent work above, in this study, we investigate the existence and uniqueness of solutions to the closed boundary value problem for impulsive fractional differential equation (1.1). Throughout this paper, in Section 2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following sections. In Section 3, we discuss some existence and uniqueness results for solutions of BVP (1.1), that is, the first one is based on Banach's fixed point theorem, the second one is based on the Burton-Kirk fixed point theorem. At the end, we give an illustrative example for our results.

2. Preliminaries

Let us set $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_{k-1} = (t_{k-1}, t_k]$, $J_k = (t_k, t_{k+1}]$, $J' := [0, T] \setminus \{t_1, t_2, \ldots, t_p\}$ and introduce the set of functions:

$PC(J, R) = \{x : J \to R : x \in C((t_k, t_{k+1}], R), k = 0, 1, 2, \ldots, p$ and there exist $x(t_k^+)$ and $x(t_k^-)$, $k = 1, 2, \ldots, p$ with $x(t_k^-) = x(t_k)\}$ and

$PC^1(J, R) = \{x \in PC(J, R), x' \in C((t_k, t_{k+1}], R), k = 0, 1, 2, \ldots, p$ and there exist $x'(t_k^+)$ and $x'(t_k^-)$, $k = 1, 2, \ldots, p$ with $x'(t_k^-) = x'(t_k)\}$ which is a Banach space with the norm $\|x\| = \sup_{t \in J}\{\|x\|_{PC}, \|x'\|_{PC}\}$ where $\|x\|_{PC} := \sup\{|x(t)| : t \in J\}$.

The following definitions and lemmas were given in [4].

Definition 2.1. The fractional (arbitrary) order integral of the function $h \in L^1(J, R_+)$ of order $\alpha \in R_+$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. For a function h given on the interval J, Caputo fractional derivative of order $\alpha > 0$ is defined by

$$^C D_{0^+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds, \quad n = [\alpha] + 1, \tag{2.2}$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.3. *Let $\alpha > 0$, then the differential equation*

$$^C D^\alpha h(t) = 0 \tag{2.3}$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in R, \ i = 0, 1, 2, \ldots, n-1, \ n = [\alpha] + 1. \tag{2.4}$$

The following lemma was given in [4, 10].

Lemma 2.4. *Let $\alpha > 0$, then*

$$I^{\alpha C} D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.5}$$

for some $c_i \in R, i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1$.

The following theorem is known as Burton-Kirk fixed point theorem and proved in [21].

Theorem 2.5. *Let X be a Banach space and $A, D : X \to X$ two operators satisfying:*

(a) *A is a contraction, and*

(b) *D is completely continuous.*

Then either

(i) *the operator equation $x = A(x) + D(x)$ has a solution, or*

(ii) *the set $\varepsilon = \{x \in X : x = \lambda A(x/\lambda) + \lambda D(x)\}$ is unbounded for $\lambda \in (0, 1)$.*

Theorem 2.6 (see [22], Banach's fixed point theorem). *Let S be a nonempty closed subset of a Banach space X, then any contraction mapping T of S into itself has a unique fixed point.*

Next we prove the following lemma.

Lemma 2.7. *Let $1 < \alpha \leq 2$ and let $h : J \rightarrow R$ be continuous. A function $x(t)$ is a solution of the fractional integral equation:*

$$
x(t) = \begin{cases}
\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \Omega_1(t)\Lambda_1 + \Omega_2(t)\Lambda_2, \quad t \in J_0 \\[2ex]
\displaystyle\int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + [1+\Omega_1(t)] \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\[2ex]
\quad + [(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)] \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds \\[2ex]
\quad + [1+\Omega_1(t)] \sum_{i=1}^{k-1} (t_k-t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \Omega_1(t) \int_{t_k}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\[2ex]
\quad + \Omega_2(t) \int_{t_k}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + [1+\Omega_1(t)]\left[\sum_{i=1}^k I_i(x(t_i^-)) + I_k^*(x(t_k^-))\right] \\[2ex]
\quad + \Omega_1(t) \sum_{i=1}^{k-1} (T-t_i)I_i^*(x(t_i^-)) + \Omega_2(t) \sum_{i=1}^{k-1} I_i^*(x(t_i^-)) \\[2ex]
\quad + \sum_{i=1}^{k-1} (t-t_i)I_i^*(x(t_i^-)), \quad t \in J_k, \ k = 1,2,\ldots,p
\end{cases}
$$

$$\tag{2.6}$$

if and only if $x(t)$ is a solution of the fractional BVP

$$^C D^\alpha x(t) = h(t), \quad t \in J',$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta x'(t_k) = I_k^*(x(t_k^-)), \tag{2.7}$$

$$x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0),$$

where

$$
\Lambda_1 := \int_{t_k}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds
$$

$$
+ \sum_{i=1}^k (T-t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k-1} (t_k-t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds
$$

$$
+ \sum_{i=1}^k I_i(x(t_i^-)) + \sum_{i=1}^{k-1} (T-t_i)I_i^*(x(t_i^-)) + I_k^*(x(t_k^-)),
$$

$$\Lambda_2 := \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds + \sum_{i=1}^{k-1} I_i^*(x(t_i^-)),$$

$$\Omega_1(t) := -\frac{(1-d)}{\Delta} - \frac{ct}{T\Delta},$$

$$\Omega_2(t) := \frac{(1-b)T}{\Delta} - \frac{(1-a)t}{\Delta}.$$

(2.8)

Proof. Let x be the solution of (2.7). If $t \in J_0$, then **Lemma 2.4** implies that

$$x(t) = I^\alpha h(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - c_0 - c_1 t,$$

$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds - c_1,$$

(2.9)

for some $c_0, c_1 \in R$.

If $t \in J_1$, then **Lemma 2.4** implies that

$$x(t) = \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - d_0 - d_1(t-t_1),$$

$$x'(t) = \int_{t_1}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds - d_1,$$

(2.10)

for some $d_0, d_1 \in R$. Thus we have

$$x(t_1^-) = \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - c_0 - c_1 t_1, \qquad x(t_1^+) = -d_0,$$

$$x'(t_1^-) = \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds - c_1, \qquad x'(t_1^+) = -d_1.$$

(2.11)

Observing that

$$\Delta x(t_1) = x(t_1^+) - x(t_1^-) = I_1(x(t_1^-)), \qquad \Delta x'(t_1) = x'(t_1^+) - x'(t_1^-) = I_1^*(x(t_1^-)), \qquad (2.12)$$

then we have

$$-d_0 = \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - c_0 - c_1 t_1 + I_1\left(x\left(t_1^-\right)\right),$$

$$-d_1 = \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - c_1 + I_1^*\left(x\left(t_1^-\right)\right),$$

(2.13)

hence, for $t \in (t_1, t_2]$,

$$x(t) = \int_{t_1}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

$$+ (t - t_1) \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1\left(x\left(t_1^-\right)\right) + (t - t_1)I_1^*\left(x\left(t_1^-\right)\right) - c_0 - c_1 t,$$

$$x'(t) = \int_{t_1}^{t} \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1^*\left(x\left(t_1^-\right)\right) - c_1.$$

(2.14)

If $t \in J_2$, then Lemma 2.4 implies that

$$x(t) = \int_{t_2}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - e_0 - e_1(t - t_2),$$

$$x'(t) = \int_{t_2}^{t} \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - e_1,$$

(2.15)

for some $e_0, e_1 \in R$. Thus we have

$$x\left(t_2^-\right) = \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

$$+ (t_2 - t_1) \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1\left(x\left(t_1^-\right)\right) + (t_2 - t_1)I_1^*\left(x\left(t_1^-\right)\right) - c_0 - c_1 t_2,$$

$$x\left(t_2^+\right) = -e_0,$$

$$x'\left(t_2^-\right) = \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1^*\left(x\left(t_1^-\right)\right) - c_1,$$

$$x'\left(t_2^+\right) = -e_1.$$

(2.16)

Similarly we observe that

$$\Delta x(t_2) = x\left(t_2^+\right) - x\left(t_2^-\right) = I_2\left(x\left(t_2^-\right)\right), \qquad \Delta x'(t_2) = x'\left(t_2^+\right) - x'\left(t_2^-\right) = I_2^*\left(x\left(t_2^-\right)\right), \qquad (2.17)$$

thus we have

$$-e_0 = x(t_2^-) + I_2(x(t_2^-)),$$
$$-e_1 = x'(t_2^-) + I_2^*(x(t_2^-)).$$
$$(2.18)$$

Hence, for $t \in (t_2, t_3]$,

$$x(t) = \int_{t_2}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

$$+ (t_2 - t_1) \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds$$

$$+ (t - t_2) \left[\int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha-1)} h(s)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha-1)} h(s)ds \right]$$

$$+ I_1(x(t_1^-)) + I_2(x(t_2^-)) + (t - t_1)I_1^*(x(t_1^-)) + I_2^*(x(t_2^-)) - c_0 - c_1 t.$$
$$(2.19)$$

By a similar process, if $t \in J_k$, then again from Lemma 2.4 we get

$$x(t) = \begin{cases} \displaystyle \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ \displaystyle + \sum_{i=1}^k (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds \\ \displaystyle + \sum_{i=1}^k I_i(x(t_i^-)) + \sum_{i=1}^{k-1} (t - t_i)I_i^*(x(t_i^-)) + I_k^*(x(t_k^-)) - c_0 - c_1 t, \end{cases}$$

$$x'(t) = \left\{ \int_{t_k}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k-1} I_i^*(x(t_i^-)) - c_1. \right.$$
$$(2.20)$$

Now if we apply the conditions:

$$x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0),$$
$$(2.21)$$

we have

$$\Lambda_1 = (1 - a)c_0 + T(1 - b)c_1,$$

$$\Lambda_2 = -\frac{c}{T}c_0 + (1 - d)c_1,$$

$$-c_0 = -\frac{(1-d)\Lambda_1}{\Delta} + \frac{(1-b)T\Lambda_2}{\Delta},$$

$$-c_1 = -\frac{c\Lambda_1}{T\Delta} - \frac{(1-a)\Lambda_2}{\Delta}.$$
$$(2.22)$$

In view of the relations (2.8), when the values of $-c_0$ and $-c_1$ are replaced in (2.9) and (2.20), the integral equation (2.7) is obtained.

Conversely, assume that x satisfies the impulsive fractional integral equation (2.6), then by direct computation, it can be seen that the solution given by (2.6) satisfies (2.7). The proof is complete. ☐

3. Main Results

Definition 3.1. A function $x \in PC^1(J, R)$ with its α-derivative existing on J' is said to be a solution of (1.1), if x satisfies the equation $^C D^\alpha x(t) = f(t, x(t))$ on J' and satisfies the conditions:

$$\Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta x'(t_k) = I_k^*(x(t_k^-)),$$

$$x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0). \tag{3.1}$$

For the sake of convenience, we define

$$\Omega_1^* = \sup_{t \in J}|\Omega_1(t)|, \qquad \Omega_2^* = \sup_{t \in J}|\Omega_2(t)|, \qquad \Omega_1^{**} = \sup_{t \in J}\left|\Omega_1'(t)\right|, \qquad \Omega_2^{**} = \sup_{t \in J}\left|\Omega_2'(t)\right|. \tag{3.2}$$

The followings are main results of this paper.

Theorem 3.2. *Assume that*

(A1) *the function $f : J \times R \to R$ is continuous and there exists a constant $L_1 > 0$ such that $\|f(t, u) - f(t, v)\| \le L_1\|u - v\|$, for all $t \in J$, and $u, v \in R$,*

(A2) *$I_k, I_k^* : R \to R$ are continuous, and there exist constants $L_2 > 0$ and $L_3 > 0$ such that $\|I_k(u) - I_k(v)\| \le L_2\|u - v\|$, $\|I_k^*(u) - I_k^*(v)\| \le L_3\|u - v\|$ for each $u, v \in R$ and $k = 1, 2, \ldots, p$.*

Moreover, consider the following:

$$\left[\frac{L_1 T^\alpha}{\Gamma(\alpha + 1)}(1 + \Omega_1^*)(1 + p + 2p\alpha) + \frac{L_1 T^{\alpha-1}}{\Gamma(\alpha)}\Omega_2^*(1 + p) \right.$$

$$\left. + (1 + \Omega_1^*)(pL_2 + L_3) + (\Omega_1^* T + \Omega_2^* + T)pL_3 \right] < 1. \tag{3.3}$$

Then, BVP (1.1) has a unique solution on J.

Proof. Define an operator $F : PC^1(J, R) \to PC^1(J, R)$ by

$$(Fx)(t) = \int_{t_k}^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)}f(s, x(s))ds + [1 + \Omega_1(t)] \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)}f(s, x(s))ds$$

$$+ [(T - t_k)\Omega_1(t) + \Omega_2(t) + (t - t_k)] \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)}f(s, x(s))ds$$

$$+ [1 + \Omega_1(t)] \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f(s, x(s)) ds$$

$$+ \Omega_1(t) \int_{t_k}^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds$$

$$+ \Omega_2(t) \int_{t_k}^{T} \frac{(T - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f(s, x(s)) ds + [1 + \Omega_1(t)] \left[\sum_{i=1}^{k} I_i(x(t_i^-)) + I_k^*(x(t_k^-)) \right]$$

$$+ \Omega_1(t) \sum_{i=1}^{k-1} (T - t_i) I_i^*(x(t_i^-)) + \Omega_2(t) \sum_{i=1}^{k-1} I_i^*(x(t_i^-)) + \sum_{i=1}^{k-1} (t - t_i) I_i^*(x(t_i^-)).$$

$$(3.4)$$

Now, for $x, y \in PC(J, R)$ and for each $t \in J$, we obtain

$$|(Fx)(t) - (Fy)(t)| \leq \int_{t_k}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds$$

$$+ |1 + \Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds$$

$$+ |(T - t_k)\Omega_1(t) + \Omega_2(t) + (t - t_k)|$$

$$\times \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} |f(s, x(s)) - f(s, y(s))| ds$$

$$+ |1 + \Omega_1(t)| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} |f(s, x(s)) - f(s, y(s))| ds$$

$$+ |\Omega_1(t)| \int_{t_k}^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds$$

$$+ |\Omega_2(t)| \int_{t_k}^{T} \frac{(T - s)^{\alpha-2}}{\Gamma(\alpha - 1)} |f(s, x(s)) - f(s, y(s))| ds$$

$$+ |1 + \Omega_1(t)| \left[\sum_{i=1}^{k} |I_i(x(t_i^-)) - I_i(y(t_i^-))| + |I_k^*(x(t_k^-)) - I_k^*(y(t_k^-))| \right]$$

$$+ |\Omega_1(t)| \sum_{i=1}^{k-1} (T - t_i) |I_i^*(x(t_i^-)) - I_i^*(y(t_i^-))|$$

$$+ |\Omega_2(t)| \sum_{i=1}^{k-1} |I_i^*(x(t_i^-)) - I_i^*(y(t_i^-))|$$

$$+ \sum_{i=1}^{k-1} |t - t_i| |I_i^*(x(t_i^-)) - I_i^*(y(t_i^-))|,$$

$$\|(Fx)(t) - (Fy)(t)\| \le \left[\frac{L_1 T^\alpha}{\Gamma(\alpha + 1)} (1 + \Omega_1^*)(1 + p + 2p\alpha) + \frac{L_1 T^{\alpha-1}}{\Gamma(\alpha)} \Omega_2^*(1 + p) \right.$$

$$\left. + (1 + \Omega_1^*)(pL_2 + L_3) + (\Omega_1^* T + \Omega_2^* + T)pL_3 \right] \|x(s) - y(s)\|. \tag{3.5}$$

Therefore, by (3.3), the operator F is a contraction mapping. In a consequence of Banach's fixed theorem, the BVP (1.1) has a unique solution. Now, our second result relies on the Burton-Kirk fixed point theorem. □

Theorem 3.3. *Assume that (A1)-(A2) hold, and*

(A3) *there exist constants $M_1 > 0$, $M_2 > 0$, $M_3 > 0$ such that $\|f(t, u)\| \le M_1$, $\|I_k(u)\| \le M_2$, $\|I_k^*(u)\| \le M_3$ for each $u, v \in R$ and $k = 1, 2, \ldots, p$.*

Then the BVP (1.1) has at least one solution on J.

Proof. We define the operators $A, D : PC^1(J, R) \to PC^1(J, R)$ by

$$(Ax)(t) = [1 + \Omega_1(t)] \left[\sum_{i=1}^{k} I_i(x(t_i^-)) + I_k^*(x(t_k^-)) \right]$$

$$+ \Omega_1(t) \sum_{i=1}^{k-1} (T - t_i) I_i^*(x(t_i^-)) + \Omega_2(t) \sum_{i=1}^{k-1} I_i^*(x(t_i^-)) + \sum_{i=1}^{k-1} (t - t_i) I_i^*(x(t_i^-)),$$

$$(Dx)(t) = \int_{t_k}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds + [1 + \Omega_1(t)] \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds$$

$$+ [(T - t_k)\Omega_1(t) + \Omega_2(t) + (t - t_k)] \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f(s, x(s)) ds$$

$$+ [1 + \Omega_1(t)] \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f(s, x(s)) ds$$

$$+ \Omega_1(t) \int_{t_k}^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds + \Omega_2(t) \int_{t_k}^{T} \frac{(T - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f(s, x(s)) ds. \tag{3.6}$$

It is obvious that A is contraction mapping for

$$(1 + \Omega_1^*)(pL_2 + L_3) + (\Omega_1^* T + \Omega_2^* + T)pL_3 < 1. \tag{3.7}$$

Now, in order to check that D is completely continuous, let us follow the sequence of the following steps.

Step 1 (*D* is continuous). Let $\{x_n\}$ be a sequence such that $x_n \rightarrow x$ in $PC(J, R)$. Then for $t \in J$, we have

$$
\begin{aligned}
|(Dx_n)(t) - (Dx)(t)| \leq {} & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x_n(s)) - f(s, x(s))| ds \\
& + |1 + \Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x_n(s)) - f(s, x(s))| ds \\
& + |(T - t_k)\Omega_1(t) + \Omega_2(t) + (t - t_k)| \\
& \times \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x_n(s)) - f(s, x(s))| ds \\
& + |1 + \Omega_1(t)| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x_n(s)) - f(s, x(s))| ds \\
& + |\Omega_1(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x_n(s)) - f(s, x(s))| ds \\
& + |\Omega_2(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x_n(s)) - f(s, x(s))| ds.
\end{aligned}
\tag{3.8}
$$

Since f is continuous function, we get

$$
\|(Dx_n)(t) - (Dx)(t)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\tag{3.9}
$$

Step 2 (*D* maps bounded sets into bounded sets in $PC(J, R)$). Indeed, it is enough to show that for any $r > 0$, there exists a positive constant l such that for each $x \in B_r = \{x \in PC(J, R) : \|x\| \leq r\}$, we have $\|D(x)\| \leq l$. By (A3), we have for each $t \in J$,

$$
\begin{aligned}
|(Dx)(t)| \leq {} & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s))| ds \\
& + |1 + \Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s))| ds \\
& + |(T - t_k)\Omega_1(t) + \Omega_2(t) + (t - t_k)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s))| ds \\
& + |1 + \Omega_1(t)| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s))| ds
\end{aligned}
$$

$$+ \left| \Omega_1(t) \right| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s, x(s)) \right| ds + \left| \Omega_2(t) \right| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s, x(s)) \right| ds,$$

$$\|D(x)\| \leq \frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} \left(1 + \Omega_1^* \right) \left(1 + p + 2p\alpha \right) + \frac{M_1 T^{\alpha-1}}{\Gamma(\alpha)} \Omega_2^* (1+p) := l.$$

$$(3.10)$$

Step 3 (D maps bounded sets into equicontinuous sets in $PC^1(J, R)$). Let $\tau_1, \tau_2 \in J_k, 0 \leq k \leq p$ with $\tau_1 < \tau_2$ and let B_r be a bounded set of $PC^1(J, R)$ as in Step 2, and let $x \in B_r$. Then

$$\left| (Dy)(\tau_2) - (D)(\tau_1) \right| \leq \int_{\tau_1}^{\tau_2} \left| (Dy)'(s) \right| ds \leq L(\tau_2 - \tau_1),$$

$$(3.11)$$

where

$$\left| (Dx)'(t) \right| \leq \int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s, x(s)) \right| ds + \left| \Omega_1'(t) \right| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s, x(s)) \right| ds$$

$$+ \left| (T - t_k)\Omega_1'(t) + \Omega_2'(t) + 1 \right| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s, x(s)) \right| ds$$

$$+ \left| \Omega_1'(t) \right| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s, x(s)) \right| ds$$

$$+ \left| \Omega_1'(t) \right| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s, x(s)) \right| ds + \left| \Omega_2'(t) \right| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s, x(s)) \right| ds,$$

$$\leq \frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} \Omega_1^* (1 + p + 2p\alpha) + \frac{M_1 T^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \Omega_2^* \right)(1+p) := L.$$

$$(3.12)$$

This implies that A is equicontinuous on all the subintervals J_k, $k = 0, 1, 2, \ldots, p$. Therefore, by the Arzela-Ascoli Theorem, the operator $D : PC^1(J, R) \rightarrow PC^1(J, R)$ is completely continuous. □

To conclude the existence of a fixed point of the operator $A + D$, it remains to show that the set

$$\varepsilon = \left\{ x \in X : x = \lambda A\left(\frac{x}{\lambda}\right) + \lambda D(x) \text{ for some } \lambda \in (0, 1) \right\}$$

$$(3.13)$$

is bounded.

Let $x \in \varepsilon$, for each $t \in J$,

$$x(t) = \lambda D(x)(t) + \lambda A\left(\frac{x}{\lambda}\right)(t).$$

$$(3.14)$$

Hence, from (A3), we have

$$
\begin{aligned}
|x(t)| \leq{} & \lambda \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s))| ds + \lambda |1+\Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s))| ds \\
& + \lambda |(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,x(s))| ds \\
& + \lambda |1+\Omega_1(t)| \sum_{i=1}^{k-1} (t_k-t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,x(s))| ds \\
& + \lambda |\Omega_1(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s))| ds + \lambda |\Omega_2(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,x(s))| ds \\
& + \lambda |1+\Omega_1(t)| \left[\sum_{i=1}^{k} \left| I_i \left(\frac{x(t_i^-)}{\lambda} \right) \right| + \left| I_k^* \left(\frac{x(t_k^-)}{\lambda} \right) \right| \right] \\
& + \lambda |\Omega_1(t)| \sum_{i=1}^{k-1} (T-t_i) \left| I_i^* \left(\frac{x(t_i^-)}{\lambda} \right) \right| \\
& + \lambda |\Omega_2(t)| \sum_{i=1}^{k-1} \left| I_i^* \left(\frac{x(t_i^-)}{\lambda} \right) \right| + \lambda \sum_{i=1}^{k-1} (t-t_i) \left| I_i^* \left(\frac{x(t_i^-)}{\lambda} \right) \right|,
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
\|x(t)\| \leq{} & \left[\frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} (1+\Omega_1^*)(1+p+2p\alpha) + \frac{M_1 T^{\alpha-1}}{\Gamma(\alpha)} \Omega_2^*(1+p) \right. \\
& \left. + (1+\Omega_1^*)(pM_2 + M_3) + (\Omega_1^* T + \Omega_2^* + T)pM_3 \right].
\end{aligned}
$$

Consequently, we conclude the result of our theorem based on the Burton-Kirk fixed point theorem.

4. An Example

Consider the following impulsive fractional boundary value problem:

$$
\begin{aligned}
{}^{C}D^{3/2} x(t) &= \frac{\sin 2t |x(t)|}{(t+5)^2 (1+|x(t)|)}, \quad t \in [0,1], \ t \neq \frac{1}{3}, \\
\Delta x\left(\frac{1}{3}\right) &= \frac{|x(1^-/3)|}{15+|x(1^-/3)|}, \qquad \Delta x'\left(\frac{1}{3}\right) = \frac{|x'(1^-/3)|}{10+|x'(1^-/3)|}, \\
x(1) &= 5x(0) - 2x'(0), \qquad x'(1) = x(0) + 4x'(0).
\end{aligned}
\tag{4.1}
$$

Here, $a = 5$, $b = -2$, $c = 1$, $d = 4$, $\alpha = 3/2$, $T = 1$, $p = 1$. Obviously, $L_1 = 1/25$, $L_2 = 1/15$, $L_3 = 1/10$, $\Omega_1^* = 4/15$, $\Omega_2^* = 7/15$. Further,

$$
\left[\frac{L_1 T^\alpha}{\Gamma(\alpha+1)} \left(1 + \Omega_1^*\right)\left(1 + p(1 + L_2) + 2p\alpha + L_3\right) + \frac{L_1 T^{\alpha-1}}{\Gamma(\alpha)} \Omega_2^* (1 + p) + \left(\Omega_1^* T + \Omega_2^* + T\right) p L_3 \right] \tag{4.2}
$$
$$
= \frac{464}{1125\sqrt{\pi}} + \frac{173}{450} < 1.
$$

Since the assumptions of Theorem 3.2 are satisfied, the closed boundary value problem (4.1) has a unique solution on $[0, 1]$. Moreover, it is easy to check the conclusion of Theorem 3.3.

Acknowledgments

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