Research Article

Approximate Solutions to Three-Point Boundary Value Problems with Two-Space Integral Condition for Parabolic Equations

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We construct a novel reproducing kernel space and give the expression of reproducing kernel skillfully. Based on the orthogonal basis of the reproducing kernel space, an efficient algorithm is provided firstly to solve a three-point boundary value problem of parabolic equations with two-space integral condition. The exact solution of this problem can be expressed by the series form. The numerical method is supported by strong theories. The numerical experiment shows that the algorithm is simple and easy to implement by the common computer and software.

1. Introduction

Nonclassical boundary value problems with nonlocal boundary conditions arise naturally in various engineering models and physical phenomena, for example, chemical engineering, thermoelasticity, underground water flow, and population dynamics [1–4]. The importance of boundary value problems with integral boundary conditions has been pointed out by Samarskiĭ [5].

Boundary value problems for parabolic equations with an integral boundary condition are investigated in the literature for the development, analysis, and implementation of accurate methods [6–11]. Integral boundary conditions of models emerged in previous literatures can be summed up as

$$\int_a^b u(x,t)dx = h(t), \quad u(x,t) \in [a,b] \times [0,T]. \tag{1.1}$$

However, Marhoune [12] studied the parabolic equation with a generalized integral boundary condition (1.2)–(1.4). This model is more universal, and it extended usual integral boundary conditions. The form is as follows:

$$\frac{\partial u(x,t)}{\partial t} - a(t)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in (0,1) \times (0,T), \tag{1.2}$$

subject to the initial-boundary value conditions

$$\begin{aligned} u(x,0) &= \varphi(x), & x \in (0,1), \\ u(0,t) &= u(1,t), & t \in (0,T), \end{aligned} \tag{1.3}$$

and the integral condition

$$\int_0^\alpha u(x,t)dx + \int_\beta^1 u(x,t)dx = 0, \quad 0 < \alpha < \beta < 1, \ \alpha + \beta = 1, \tag{1.4}$$

with the function $a(t)$, and its derivatives are bounded on the interval $[0,T]$: $0 < a_0 < a(t) < a_1$, $0 < a_2 < a'(t) < a_3$. In the following, we may assume $\varphi(x) = 0$ because it can be got from homogeneous boundary conditions. The existence and uniqueness of the solution for (1.2)–(1.4) have been proved in [12].

A practical model is typed by various definite conditions under different environment. Investigation about the definite conditions is the key problem to the model. Due to condition (1.4), it is difficult to construct reproducing kernel space, so nobody gives the algorithm for the above problem by applying reproducing kernel theory. In this paper, the author successfully constructs a novel reproducing kernel space which includes boundary conditions (1.3)-(1.4) and gains the expression of the reproducing kernel skillfully. Meanwhile, we provide a simple algorithm for solving (1.2)–(1.4). Based on the orthogonal basis in the reproducing kernel space, the exact solution is given by the form of series. Meanwhile, using a similar process, it is possible to solve other linear ordinary differential equations, partial differential equations with the same boundary value conditions.

2. Constructive Method for the Reproducing Kernel Space $H_0(\Omega)$

$H(\Omega)$ and $H_0(\Omega)(\Omega = [0,1] \times [0,T])$ are inner product spaces and $H_0(\Omega) \subset H(\Omega)$, and they are defined in the following.

Definition 2.1. The inner product space $H(\Omega)$ is defined by

$$\begin{aligned} H(\Omega) = \Big\{ u(x,t) \mid \partial_{x^2t}^3 u(x,t) \text{ is absolutely continuous, } u(x,0) = 0, \\ u(0,t) = u(1,t), \ \partial_{x^3t^2}^5 u(x,t) \in L^2(\Omega) \Big\}. \end{aligned} \tag{2.1}$$

The inner product of $H(\Omega)$ is defined by

$$\langle u(x,t), v(x,t) \rangle_{\mathrm{H}} = \sum_{j=0}^{1} \left[\sum_{i=0}^{1} \partial_{t^j x^i}^{j+i} u(0,0) \partial_{t^j x^i}^{j+i} v(0,0) + \int_0^1 \partial_{t^j x^3}^{j+3} u(x,0) \partial_{t^j x^3}^{j+3} v(x,0) dx \right]$$
$$+ \sum_{i=0}^{1} \int_0^T \partial_{t^2 x^i}^{2+i} u(0,t) \partial_{t^2 x^i}^{2+i} v(0,t) dt + \int_0^T \int_0^1 \partial_{t^2 x^3}^{5} u(x,t) \partial_{t^2 x^3}^{5} v(x,t) dx \, dt. \tag{2.2}$$

And it possesses associated norm $\| \cdot \|$.

Lemma 2.2. *Inner space $H(\Omega)$ is a Hilbert reproducing kernel space. Its reproducing kernel function*

$$K(x,y,t,s) = R(x,y)G(t,s), \tag{2.3}$$

and for any $u(x,t) \in H(\Omega)$,

$$u(x,t) = (u(y,s), K(x,y,t,s))_H, \tag{2.4}$$

where

$$R(x,y) = \begin{cases} r_y(x), & x \le y, \\ r_x(y), & y < x, \end{cases} \qquad G(t,s) = \begin{cases} \dfrac{st(2+t)}{2} - \dfrac{t^3}{6}, & t \le s, \\ \dfrac{ts(2+s)}{2} - \dfrac{s^3}{6}, & t > s, \end{cases} \tag{2.5}$$

where

$$r_y(x) = \frac{1}{18720} \Big(18720 + x(y-1) \big(\big((5-x)x^3(2+y) - 120 \big) y(18 + (y-6)y) \tag{2.6}$$
$$-156x^4 - 10xy(3+x)(y(6+(y-4)y) - 120) \big) \Big).$$

This proof can be found in [13–17].
Clearly, one has

$$\begin{align} &(1)\ G(t,s) = G(s,t), \qquad R(x,y) = R(y,x), \\ &(2)\ \forall s \in [0,T], \qquad G(0,s) = 0, \tag{2.7} \\ &(3)\ \forall y \in [0,1], \qquad R(0,y) = R(1,y). \end{align}$$

Lemma 2.3. *Fix an $s \in [0,T]$, $G(t,s)(\int_0^\alpha R(x,y)dy + \int_\beta^1 R(x,y)dy) \in H(\Omega)$.*

Definition 2.4. The subspace $H_0(\Omega)$ is defined by

$$H_0(\Omega) = \left\{ u(x,t) \in H(\Omega), \int_0^\alpha u(x,t)dx + \int_\beta^1 u(x,t)dx = 0 \right\}. \tag{2.8}$$

Obviously, $H_0(\Omega)$ is a closed subspace of the reproducing kernel $H(\Omega)$. It is very important to obtain the representation of reproducing kernel in $H_0(\Omega)$, which is the base of our algorithm. Therefore, our work begins with some lemmas to provide constructive method for reproducing kernel in $H_0(\Omega)$.

Lemma 2.5. *Fix an* $s \in [0,T]$, $G(t,s)(\int_0^\alpha R(x,y)dy + \int_\beta^1 R(x,y)dy) \not\equiv 0$.

Proof. Otherwise, for all $u(x,t) \in H(\Omega)$, by (2.4),

$$
\int_0^\alpha u(x,t)dx + \int_\beta^1 u(x,t)dx = \int_0^\alpha \langle u(y,s), R(x,y)G(t,s) \rangle_H dx + \int_\beta^1 \langle u(y,s), R(x,y)G(t,s) \rangle_H dx
$$
$$
= \left\langle u(y,s), G(t,s) \left(\int_0^\alpha R(x,y)dx + \int_\beta^1 R(x,y)dx \right) \right\rangle_H \equiv 0,
$$
(2.9)

then $u(x,t) \in H_0(\Omega)$, which is contradictory.   □

Lemma 2.6. *Fix an* $s \in [0,T]$, $G(t,s)(\int_0^\alpha R(x,y)dy + \int_\beta^1 R(x,y)dy) \notin H_0(\Omega)$, *namely,* $\int_0^\alpha [\int_0^\alpha Rdy + \int_\beta^1 Rdy]dx + \int_\beta^1 [\int_0^\alpha Rdy + \int_\beta^1 Rdy]dx \neq 0$.

Proof. Otherwise, for all $u(x,t) \in H_0(\Omega) \subset H(\Omega)$, then

$$
0 = \int_0^\alpha u(x,t)dx + \int_\beta^1 u(x,t)dx
$$
$$
= \int_0^\alpha \langle u(y,s), R(x,y)G(t,s) \rangle_H dx + \int_\beta^1 \langle u(y,s), R(x,y)G(t,s) \rangle_H dx \qquad (2.10)
$$
$$
= \left\langle u(y,s), G(t,s) \left(\int_0^\alpha R(x,y)dx + \int_\beta^1 R(x,y)dx \right) \right\rangle_H,
$$

which implies that $G(t,s)(\int_0^\alpha R(x,y) + \int_\beta^1 R(x,y)) \equiv 0$, and this contradicts with **Lemma 2.5**.
                                                                                    □

Consider a function

$$
K_0(x,y,t,s) = K(x,y,t,s) + G(t,s) \frac{\left(\int_0^\alpha Rdy + \int_\beta^1 Rdy \right) \left(\int_\beta^1 Rdx - \int_0^\alpha Rdx \right)}{\int_0^\alpha \left[\int_0^\alpha Rdy + \int_\beta^1 Rdy \right] dx + \int_\beta^1 \left[\int_0^\alpha Rdy + \int_\beta^1 Rdy \right] dx}, \quad (2.11)
$$

and one can check carefully that $K_0(x,y,t,s) \in H_0(\Omega)$ is the reproducing kernel of $H_0(\Omega)$.

3. An Orthogonal Basis of $H_0(\Omega)$

Let an operator $\mathbb{L} : H_0(\Omega) \to L^2(\Omega)$, putting

$$
(\mathbb{L}u)(x,t) = \partial_t u(x,t) - a(t)\partial_{x^2}^2 u(x,t), \qquad (3.1)
$$

rewriting (1.2)–(1.4) as

$$(\mathbb{L}u)(x,t) = f(x,t),$$
$$u(x,0) = 0, \qquad u(0,t) = u(1,t),$$
$$\int_0^\alpha u(x,t)dx + \int_\beta^1 u(x,t)dx = 0. \tag{3.2}$$

Lemma 3.1. $\mathbb{L}$ *is a bounded linear operator.*

Proof. Noting that

$$|\partial_t u(x,t)| = \left|\langle u(y,s), \partial_t K_0(x,y,t,s)\rangle_{H_0}\right| \le M_1\|u\|_{H_0},$$
$$\left|\partial_{x^2}^2 u(x,t)\right| = \left|\langle u(y,s), \partial_{x^2}^2 K_0(x,y,t,s)\rangle_{H_0}\right| \le M_2\|u\|_{H_0}, \tag{3.3}$$

where M_1, M_2 are positive real numbers,

$$\|\mathbb{L}u\|_{L^2}^2 = \int_0^T \int_0^1 ((\mathbb{L}u)(x,t))^2 dx\, dt = \int_0^T \int_0^1 \left(\partial_t u(x,t) - a(t)\partial_{x^2}^2 u(x,t)\right)^2 dx\, dt$$
$$\le \int_0^T \int_0^1 \left(|\partial_t u|^2 + a^2(t)\left|\partial_{x^2}^2 u\right|^2 + 2|a(t)||\partial_t u|\left|\partial_{x^2}^2 u\right|\right) dx\, dt. \tag{3.4}$$

Combining with the bounded function $a(t)$, it holds that $\mathbb{L}$ is a bounded linear operator. $\square$

We will choose and fix a countable dense subset $S = \{(x_1,t_1),(x_2,t_2),\ldots\} \subset \Omega$ and define $\psi_i(x,t)$ by

$$\psi_i(x,t) \overset{\text{def}}{=} (\mathbb{L}_{(y,s)}K_0(x,y,t,s))(x_i,t_i). \tag{3.5}$$

Lemma 3.2. *Consider the following:* $\psi_i(x,t) \in H_0(\Omega)$, $i = 1,\ 2,\ldots$, *(see [11]).*

Lemma 3.3. *The function system* $\{\psi_i(x,t)\}_{i=1}^\infty$ *is a complete system of the space* $H_0(\Omega)$.

Proof. For every i, we have

$$0 = \langle u(x,t), \psi_i(x,t)\rangle_{H_0} = \langle u(x,t), (\mathbb{L}_{(y,s)}K_0(x,y,t,s))(x_i,t_i)\rangle_{H_0}$$
$$= \mathbb{L}_{(y,s)}\left(\langle u(x,t), K_0(x,y,t,s)\rangle_{H_0}\right)(x_i,t_i) = \mathbb{L}_{(y,s)}(u(y,s))(x_i,t_i)$$
$$= (\mathbb{L}u)(x_i,t_i), \tag{3.6}$$

which shows that $(\mathbb{L}u)(x,t) = 0$ due to the denseness of S. It follows that $u(x,t) \equiv 0$ from the existence of $\mathbb{L}^{-1}$. $\square$

Applying Gram-Schmidt process, we obtain an orthogonal basis $\{\widetilde{\psi}_i(x,t)\}_{i=1}^{\infty}$ of $H_0(\Omega)$, such that

$$\widetilde{\psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x,t), \tag{3.7}$$

where β_{ik} are orthogonal coefficients.

4. Numerical Algorithm

In this section, it is explained how to deduce the exact solution from the orthogonal basis $\{\widetilde{\psi}_i(x,t)\}_{i=1}^{\infty}$ of $H_0(\Omega)$.

Theorem 4.1. *The exact solution of* (3.2) *can be expressed by*

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} u(x_k,t_k) \widetilde{\psi}_i(x,t). \tag{4.1}$$

Proof. The exact solution $u(x,t)$ can be expanded to a Fourier series in terms of normal orthogonal basis $\widetilde{\psi}_i(x,t)$ in $H_0(\Omega)$,

$$
\begin{aligned}
u(x,t) &= \sum_{i=1}^{\infty} \langle u(x,t), \widetilde{\psi}_i(x,t) \rangle_{H_0} \widetilde{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x,t), \psi_k(x,t) \rangle_{H_0} \widetilde{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x,t), (\mathbb{L}_{(y,s)} K_0(x,y,t,s))(x_k,t_k) \rangle_{H_0} \widetilde{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \mathbb{L}_{(y,s)} \left(\langle u(x,t), K_0(x,y,t,s) \rangle_{H_0} \right)(x_k,t_k) \widetilde{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (\mathbb{L}u)(x_k,t_k) \widetilde{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k,t_k) \widetilde{\psi}_i(x,t).
\end{aligned}
\tag{4.2}
$$

$\square$

We obtain the n-truncation approximate solution of (3.2),

$$u_n(x,t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k,t_k) \widetilde{\psi}_i(x,t), \tag{4.3}$$

which is n-truncation Fourier series of the exact solution $u(x,t)$ in (3.2), so $u_n(x,t) \overset{\|\cdot\|_{H_0}}{\to} u(x,t)$, as $n \to \infty$.

Table 1: Numerical results.

(x,t)	$u(x,t)$	$|(u-u_{25})/u|$	$|(u-u_{100})/u|$	$|(u-u_{400})/u|$
(0.00,0.0001)	0.000700035	0.000000000	$2.45579E-10$	$1.12923E-9$
(0.95,0.05)	0.187934	0.000230442	0.0000112112	$1.94207E-6$
(0.85,0.15)	-0.241376	0.0000224685	$6.90183\ E-6$	$1.36152E-6$
(0.75,0.25)	-1.36687	0.00086302	0.0000715374	$7.60769E-8$
(0.65,0.35)	-2.7296	0.00322583	0.000250607	$5.17148E-7$
(0.55,0.45)	-3.87049	0.00287117	0.000571766	$1.33238E-6$
(0.45,0.55)	-4.34049	0.00517334	0.000894053	$2.81685E-6$
(0.35,0.65)	-3.71389	0.00142632	0.000913571	$5.62867E-6$
(0.25,0.75)	-1.60569	0.00213525	0.000480841	$1.37775E-5$
(0.15,0.85)	2.3062	0.00450865	0.000593452	$2.81487E-6$
(0.05,0.95)	8.25283	0.0020922	0.00048678	$5.00647E-6$

Theorem 4.2. *Consider the following:* $|u_n(x,t) - u(x,t)| \to 0$, *as* $n \to \infty$.

Proof. Since $K_0(x,x,t,t)$ is continuous on Ω, it follows that $K_0(x,x,t,t) \leq C$, for all $(x,t) \in \Omega$, where C is a constant. When $\|u_n - u\|_{H_0} \to 0$,

$$
\begin{aligned}
|u_n(x,t) - u(x,t)| &= \left| \langle u_n(x,t) - u(x,t), K_0(x,y,t,s) \rangle_{H_0} \right| \\
&\leq \|u_n - u\|_{H_0} \|K_0(x,y,t,s)\|_{H_0} \\
&= \sqrt{K_0(x,x,t,t)} \ \|u_n - u\|_{H_0} \longrightarrow 0.
\end{aligned}
\tag{4.4}
$$

$\square$

5. Numerical Example

In this section, a numerical example is studied to demonstrate the accuracy of the present algorithm. The example is computed by Mathematica 5.0. Results obtained by the algorithm are compared with the analytical solution and are found to be in good agreement.

Example 5.1. Consider a three-point boundary value problem of parabolic equations with two-space integral condition

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} - \sin t \frac{\partial^2 u(x,t)}{\partial x^2} &= e^t \left(7 - 36x + 36x^3\right) - 216 \left(e^t - 1\right) x \sin(t), \\
u(x,0) &= 0, \quad x \in (0,1), \\
u(0,t) = u(1,t), \quad \int_0^{1/3} u(x,t)dx + \int_{2/3}^1 u(x,t)dx &= 0, \quad t \in (0,1).
\end{aligned}
\tag{5.1}
$$

The exact solution is $u(x,t) = (e^t - 1)(7 - 36x + 36x^3)$. The numerical results are collected in Table 1.

6. Conclusion

In this paper, we construct a reproducing kernel space by new method, in which each function satisfies boundary value conditions of considered problems. In this space, a numerical algorithm is presented for solving a class of parabolic equations with two-space integral boundary condition. Exact solution with series form is given. Approximate solution obtained by present algorithm converges to exact solution uniformly.

Acknowledgments

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