Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 432501, 9 pages
doi:10.1155/2012/432501

Research Article

An Explicit Method for the Split Feasibility Problem with Self-Adaptive Step Sizes

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Received 12 September 2012; Accepted 24 October 2012

Academic Editor: Wenming Zou

An explicit iterative method with self-adaptive step-sizes for solving the split feasibility problem is presented. Strong convergence theorem is provided.

1. Introduction

Since its publication in 1994, the split feasibility problem has been studied by many authors. For some related works, please consult [1–18]. Among them, a more popular algorithm that solves the split feasibility problems is Byrne's CQ method [2]:

$$x_{n+1} = P_C\left(x_n - \tau A^*\left(I - P_Q\right)Ax_n\right), \tag{1.1}$$

where C and Q are two closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. The CQ algorithm only involves the computations of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions.

Note that CQ algorithm can be obtained from optimization. If we set

$$f(x) := \frac{1}{2}\left\|Ax - P_QAx\right\|^2, \tag{1.2}$$

then the convex objective f is differentiable and has a Lipschitz gradient given by

$$\nabla f(x) = A^*\left(I - P_Q\right)A. \tag{1.3}$$

Thus, the CQ algorithm can be obtained by minimizing the following convex minimization problem

$$\min_{x\in C} f(x). \tag{1.4}$$

We can use a gradient projection algorithm below to solve the split feasibility problem:

$$x_{n+1} = P_C\left(x_n - \tau_n \nabla f(x_n)\right), \tag{1.5}$$

where τ_n, the step size at iteration n, is chosen in the interval $(0, 2/L)$, where L is the Lipschitz constant of ∇f.

However, we observe that the determination of the step size τ_n depends on the operator (matrix) norm $\|A\|$ (or the largest eigenvalue of A^*A). This means that in order to implement the CQ algorithm, one has first to compute (or, at least, estimate) the matrix norm of A, which is in general not an easy work in practice. To overcome the above difficulty, the so-called self-adaptive method which permits step size τ_n being selected self-adaptively was developed. See, for example, [10, 14, 15, 19–23].

Inspired by the above results and the self-adaptive method, in this paper, we present an explicit iterative method with self-adaptive step sizes for solving the split feasibility problem. Convergence analysis result is given.

2. Preliminaries

Let H_1 and H_2 be two real Hilbert spaces and C and Q two closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem is to find a point x^* such that

$$x^* \in C, \quad Ax^* \in Q. \tag{2.1}$$

Next, we use Γ to denote the solution set of the split feasibility problem, that is, $\Gamma = \{x \in C : Ax \in Q\}$.

We know that a point $x^* \in C$ is a stationary point of problem (1.4) if it satisfies

$$\left\langle \nabla f(x^*), x - x^* \right\rangle \geq 0, \quad \forall x \in C. \tag{2.2}$$

Given $x^* \in H_1$. x^* solves the split feasibility problem if and only if x^* solves the fixed point equation

$$x^* = P_C\left(x^* - \gamma A^*(I - P_Q)Ax^*\right). \tag{2.3}$$

Next we adopt the following notation:

(i) $x_n \to x$ means that x_n converges strongly to x;

(ii) $x_n \rightharpoonup x$ means that x_n converges weakly to x;

(iii) $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

Recall that a function $f : H \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{2.4}$$

for all $\lambda \in (0,1)$ and $\forall x, y \in H$. It is known that a differentiable function f is convex if and only if there holds the relation:

$$f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle, \tag{2.5}$$

for all $z \in H$. Recall that an element $g \in H$ is said to be a subgradient of $f : H \to \mathbb{R}$ at x if

$$f(z) \geq f(x) + \langle g, z - x \rangle, \tag{2.6}$$

for all $z \in H$. If the function $f : H \to \mathbb{R}$ has at least one subgradient at x is said to be subdifferentiable at x. The set of subgradients of f at the point x is called the subdifferential of f at x, and is denoted by $\partial f(x)$. A function f is called sub-differentiable if it is subdifferentiable at all $x \in H$. If f is convex and differentiable, then its gradient and subgradient coincide. A function $f : H \to \mathbb{R}$ is said to be weakly lower semi continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n \to \infty} f(x_n). \tag{2.7}$$

f is said to be w-lsc on H if it is w-lsc at every point $x \in H$.

A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{2.8}$$

for all $x, y \in C$.

Recall that the (nearest point or metric) projection from H onto C, denoted P_C, assigns, to each $x \in H$, the unique point $P_C(x) \in C$ with the property

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}. \tag{2.9}$$

It is well known that the metric projection P_C of H onto C has the following basic properties:

(a) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ for all $x, y \in H$;

(b) $\langle x - y, P_C(x) - P_C(y) \rangle \geq \|P_C(x) - P_C(y)\|^2$ for every $x, y \in H$;

(c) $\langle x - P_C(x), y - P_C(x) \rangle \leq 0$ for all $x \in H$, $y \in C$.

Lemma 2.1 (see [24]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \tag{2.10}$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (see [25]). *Let* (s_n) *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* (s_{n_i}) *of* (s_n) *such that* $s_{n_i} \leq s_{n_i+1}$ *for all* $i \geq 0$. *For every* $n \geq n_0$, *define an integer sequence* $(\tau(n))$ *as*

$$\tau(n) = \max\{k \leq n : s_{n_i} < s_{n_i+1}\}. \tag{2.11}$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$ *and for all* $n \geq n_0$

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}. \tag{2.12}$$

3. Main Results

In this section, we will introduce our algorithm and prove our main results.

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. In the sequel, we assume that the split feasibility problem is consistent, that is $\Gamma \neq \emptyset$.

Algorithm 3.1. For $u \in C$ and given $x_0 \in C$, let the sequence $\{x_{n+1}\}$ defined by

$$\begin{aligned} y_n &= \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} &= P_C\left(y_n - \tau_n \frac{f(y_n)\nabla f(y_n)}{\|\nabla f(y_n)\|^2}\right), \quad n \geq 0, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\tau_n\} \subset (0,2)$.

Remark 3.2. In the sequel, we may assume that $\nabla f(y_n) \neq 0$ for all n. Note that this fact can be guaranteed if the sequence $\{y_n\}$ is infinite; that is, Algorithm 3.1 does not terminate in a finite number of iterations.

Theorem 3.3. *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\inf_n \tau_n(2-\tau_n) > 0$.

Then $\{x_n\}$ *defined by* (3.1) *converges strongly to* $P_\Gamma(u)$.

Proof. Let $\nu \in \Gamma$. It follows that $\nabla f(\nu) = 0$ for all $\nu \in \Gamma$. From (2.5), we deduce that

$$f(y_n) = f(y_n) - f(\nu) \leq \langle \nabla f(y_n), y_n - \nu \rangle. \tag{3.2}$$

Thus, by (3.1) and (3.2), we have

$$\begin{aligned}
\|x_{n+1} - \nu\|^2 &= \left\| P_C\left(y_n - \tau_n \frac{f(y_n)\nabla f(y_n)}{\|\nabla f(y_n)\|^2} \right) - \nu \right\|^2 \\
&\leq \left\| y_n - \tau_n \frac{f(y_n)\nabla f(y_n)}{\|\nabla f(y_n)\|^2} - \nu \right\|^2 \\
&= \|y_n - \nu\|^2 - 2\tau_n \frac{f(y_n)}{\|\nabla f(y_n)\|^2} \langle \nabla f(y_n), y_n - \nu \rangle + \tau_n^2 \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \\
&\leq \|y_n - \nu\|^2 - 2\tau_n \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} + \tau_n^2 \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \\
&= \|y_n - \nu\|^2 - \tau_n(2 - \tau_n) \frac{f^2(y_n)}{\|\nabla f(y_n)\|^2}, \\
\|y_n - \nu\|^2 &= \|\alpha_n(u - \nu) + (1 - \alpha_n)(x_n - \nu)\|^2 \leq \alpha_n\|u - \nu\|^2 + (1 - \alpha_n)\|x_n - \nu\|^2.
\end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - \nu\|^2 &\leq \alpha_n\|u - \nu\|^2 + (1 - \alpha_n)\|x_n - \nu\|^2 - \tau_n(2 - \tau_n)\frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \\
&\leq \alpha_n\|u - \nu\|^2 + (1 - \alpha_n)\|x_n - \nu\|^2 \\
&\leq \max\left\{ \|u - \nu\|^2, \|x_n - \nu\|^2 \right\}.
\end{aligned} \tag{3.4}$$

By induction, we deduce

$$\|x_{n+1} - \nu\| \leq \max\{\|u - \nu\|, \|x_0 - \nu\|\}. \tag{3.5}$$

Hence, $\{x_n\}$ is bounded.

At the same time, we note that

$$\|y_n - \nu\|^2 = \|\alpha_n(u - \nu) + (1 - \alpha_n)(x_n - \nu)\|^2 \leq (1 - \alpha_n)\|x_n - \nu\|^2 + 2\alpha_n\langle u - \nu, y_n - \nu \rangle. \tag{3.6}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\leq (1-\alpha_n)\|x_n - v\|^2 + 2\alpha_n\langle u - v, y_n - v\rangle - \tau_n(2-\tau_n)\frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \\
&= (1-\alpha_n)\|x_n - v\|^2 + 2\alpha_n\langle u - v, \alpha_n(u-v) + (1-\alpha_n)(x_n - v)\rangle \\
&\quad - \tau_n(2-\tau_n)\frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \\
&= (1-\alpha_n)\|x_n - v\|^2 + 2\alpha_n^2\|u-v\|^2 + 2\alpha_n(1-\alpha_n)\langle u - v, x_n - v\rangle \\
&\quad - \tau_n(2-\tau_n)\frac{f^2(y_n)}{\|\nabla f(y_n)\|^2}.
\end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - v\|^2 - \|x_n - v\|^2 + \alpha_n\Big(\|x_n - v\|^2 - 2\alpha_n\|u-v\|^2 + 2\alpha_n\langle u - v, x_n - v\rangle\Big) \\
&\quad + \tau_n(2-\tau_n)\frac{f^2(y_n)}{\|\nabla f(y_n)\|^2} \leq 2\alpha_n\langle u - v, x_n - v\rangle.
\end{aligned} \tag{3.8}$$

Next, we will prove that $x_n \to v$. Set $\omega_n = \|x_n - v\|^2$ for all $n \geq 0$. Since $\alpha_n \to 0$ and $\inf_n \tau_n(2-\tau_n) > 0$, we may assume without loss of generality that $\tau_n(2-\tau_n) \geq \sigma$ for some $\sigma > 0$. Thus, we can rewrite (3.8) as

$$\omega_{n+1} - \omega_n + \alpha_n U_n + \frac{\sigma f^2(y_n)}{\|\nabla f(y_n)\|^2} \leq 2\alpha_n\langle u - v, x_n - v\rangle, \tag{3.9}$$

where $U_n = \|x_n - v\|^2 - 2\alpha_n\|u-v\|^2 + 2\alpha_n\langle u - v, x_n - v\rangle$.

Now, we consider two possible cases.

Case 1. Assume that $\{\omega_n\}$ is eventually decreasing; that is, there exists $N > 0$ such that $\{\omega_n\}$ is decreasing for $n \geq N$. In this case, $\{\omega_n\}$ must be convergent and from (3.9) it follows that

$$\begin{aligned}
0 \leq \frac{\sigma f^2(y_n)}{\|\nabla f(y_n)\|^2} &\leq \omega_n - \omega_{n+1} - \alpha_n U_n + 2\alpha_n\|u-v\|\|x_n - v\| \\
&\leq \omega_n - \omega_{n+1} + M\alpha_n,
\end{aligned} \tag{3.10}$$

where $M > 0$ is a constant such that $\sup_n\{2\|u-v\|\|x_n - v\| + \|U_n\|\} \leq M$. Letting $n \to \infty$ in (3.10), we get

$$\lim_{n\to\infty} f(y_n) = 0. \tag{3.11}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging weakly to $\tilde{x} \in C$. Since, $x_n - y_n \to 0$, we also have $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to $\tilde{x} \in C$. From the weak lower semicontinuity of f, we have

$$0 \leq f(\tilde{x}) \leq \liminf_{k \to \infty} f(y_{n_k}) = \lim_{n \to \infty} f(y_n) = 0. \tag{3.12}$$

Hence, $f(\tilde{x}) = 0$; that is, $A\tilde{x} \in Q$. This indicates that

$$\omega_w(y_n) = \omega_w(x_n) \subset \Gamma. \tag{3.13}$$

Furthermore, by using the property of the projection (c), we deduce

$$\limsup_{n \to \infty} \langle u - v, x_n - v \rangle = \max_{\tilde{x} \in \omega_w(x_n)} \langle u - P_\Gamma(u), \tilde{x} - P_\Gamma(u) \rangle \leq 0. \tag{3.14}$$

From (3.8), we obtain

$$\omega_{n+1} \leq (1 - \alpha_n)\omega_n + \alpha_n \Big(2\alpha_n \|u - v\|^2 + 2(1 - \alpha_n)\langle u - v, x_n - v \rangle \Big). \tag{3.15}$$

This together with Lemma 2.1 imply that $\omega_n \to 0$.

Case 2. Assume that $\{\omega_n\}$ is not eventually decreasing. That is, there exists an integer n_0 such that $\omega_{n_0} \leq \omega_{n_0+1}$. Thus, we can define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = \max\{k \in \mathbb{N} \mid n_0 \leq k \leq n, \omega_k \leq \omega_{k+1}\}. \tag{3.16}$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to +\infty$ as $n \to \infty$ and

$$\omega_{\tau(n)} \leq \omega_{\tau(n)+1}, \tag{3.17}$$

for all $n \geq n_0$. In this case, we derive from (3.10) that

$$\frac{\sigma f^2(y_{\tau(n)})}{\|\nabla f(y_{\tau(n)})\|^2} \leq M\alpha_{\tau(n)} \longrightarrow 0. \tag{3.18}$$

It follows that

$$\lim_{n \to \infty} f(y_{\tau(n)}) = 0. \tag{3.19}$$

This implies that every weak cluster point of $\{y_{\tau(n)}\}$ is in the solution set Γ; that is, $\omega_w(y_{\tau(n)}) \subset \Gamma$. So, $\omega_w(x_{\tau(n)}) \subset \Gamma$. On the other hand, we note that

$$\begin{aligned} \left\| y_{\tau(n)} - x_{\tau(n)} \right\| &= \alpha_{\tau(n)} \left\| u - x_{\tau(n)} \right\| \longrightarrow 0, \\ \left\| x_{\tau(n)+1} - y_{\tau(n)} \right\| &\le \frac{\tau_{\tau(n)} f\left(y_{\tau(n)}\right)}{\left\| \nabla f\left(y_{\tau(n)}\right) \right\|} \longrightarrow 0. \end{aligned} \tag{3.20}$$

From which we can deduce that

$$\limsup_{n\to\infty} \langle u - v, x_{\tau(n)} - v \rangle = \max_{\tilde{x} \in \omega_w(x_{\tau(n)})} \langle u - P_\Gamma(u), \tilde{x} - P_\Gamma(u) \rangle \le 0. \tag{3.21}$$

Since $\omega_{\tau(n)} \le \omega_{\tau(n)+1}$, we have from (3.9) that

$$\omega_{\tau(n)} \le \left(1 - 2\alpha_{\tau(n)}\right) \langle u - v, x_{\tau(n)} - v \rangle + 2\alpha_{\tau(n)} \| u - v \|^2. \tag{3.22}$$

Combining (3.21) and (3.22) yields

$$\limsup_{n\to\infty} \omega_{\tau(n)} \le 0, \tag{3.23}$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.24}$$

From (3.15), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)+1} \le \limsup_{n\to\infty} \omega_{\tau(n)}. \tag{3.25}$$

Thus,

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0. \tag{3.26}$$

From Lemma 2.2, we have

$$0 \le \omega_n \le \max\left\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\right\}. \tag{3.27}$$

Therefore, $\omega_n \to 0$. That is, $x_n \to v$. This completes the proof. □

Acknowledgment

The author was supported in part by the Youth Foundation of Taizhou University (2011QN11).

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