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Research Article

On a Nonsmooth Vector Optimization Problem with Generalized Cone Invexity

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By using Clarke's generalized gradients we consider a nonsmooth vector optimization problem with cone constraints and introduce some generalized cone-invex functions called K-α-generalized invex, K-α-nonsmooth invex, and other related functions. Several sufficient optimality conditions and Mond-Weir type weak and converse duality results are obtained for this problem under the assumptions of the generalized cone invexity. The results presented in this paper generalize and extend the previously known results in this area.

1. Introduction

In optimization theory, convexity plays a key role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems; see [1, 2]. Many attempts have been made during the past several decades to relax convexity requirement; see [3–7]. In this endeavor, Hanson [8] introduced invex functions and studied some applications to optimization problem. Subsequently, many authors further weakened invexity hypotheses to establish optimality conditions and duality results for various mathematical programming problems; see, for example, [9–11] and the references cited therein.

Above all, Yen and Sach [12] introduced cone-generalized invex and cone-nonsmooth invex functions. Giorgi and Guerraggio [13] presented the notions of α-K-invex, α-K pseudoinvex, and α-K quasi-invex functions in the differentiable case and derived optimality and duality results for a vector optimization problem over cones. Khurana[14] extended pseudoinvex functions to differentiable cone-pseudoinvex and strongly cone-pseudoinvex functions. Based on this, Suneja et al. [15] defined cone-nonsmooth quasi-invex, cone-nonsmooth pseudoinvex, and other related functions in terms of Clarke's [16] generalized directional derivatives and established optimality and duality results for a nonsmooth vector optimization problem.

On the other hand, Noor [17] proposed several classes of α-invex functions and investigated some properties of the α-preinvex functions and their differentials. Mishra et al. [18] defined strict pseudo-α-invex and quasi-α-invex functions. Mishra et al. [19] further introduced the concepts of nonsmooth pseudo-α-invex functions and established a relationship between vector variational-like inequality and nonsmooth vector optimization problems by using the nonsmooth α-invexity.

In the present paper, by using Clarke's generalized gradients of locally Lipschitz functions we are concerned with a nonsmooth vector optimization problem with cone constraints and introduce several generalized invex functions over cones namely K-α-generalized invex, K-α-nonsmooth invex, and other related functions, which, respectively, extend some corresponding concepts of [12, 13, 15, 17]. Some sufficient optimality conditions for this problem are obtained by using the above defined concepts. Furthermore, a Mond-Weir type dual is formulated and a few weak and converse duality results are established. We generalize and extend some results presented in the literatures on this topic.

2. Preliminaries and Definitions

Throughout this paper, let $\eta : R^n \times R^n \to R^n$ and $\alpha : R^n \times R^n \to R_+ \setminus \{0\}$ be two fixed mappings. $\operatorname{int} K$ and $\overline{K}$ denote the interior and closure of $K \subseteq R^m$, respectively. We always assume that K is a closed convex cone with $\operatorname{int} K \neq \emptyset$.

The positive dual cone K^+ of K is defined as

$$K^+ = \{x^* \in R^m : \langle x^*, x\rangle \geq 0,\ \forall x \in K\}. \tag{2.1}$$

The strict positive dual cone K^{++} of K is given by

$$K^{++} = \{x^* \in R^m : \langle x^*, x\rangle > 0, \forall x \in K \setminus \{0\}\}. \tag{2.2}$$

The following property is from [20], which will be used in the sequel.

Lemma 2.1 (see [20]). *Let $K \subseteq R^m$ be a convex cone with* $\operatorname{int} K \neq \emptyset$. *Then,*

(a) $\forall u^* \in K^+ \setminus \{0\},\ x \in \operatorname{int} K \Rightarrow \langle u^*, x\rangle > 0$;

(b) $\forall u^* \in \operatorname{int} K^+,\ x \in K \setminus \{0\} \Rightarrow \langle u^*, x\rangle > 0$.

A function $\psi : R^n \to R$ is called locally Lipschitz at $u \in R^n$, if there exists $l > 0$ such that

$$\|\psi(x) - \psi(y)\| \leq l\|x - y\|, \tag{2.3}$$

for all x, y in a neighbourhood of u.

A function ψ is called locally Lipschitz on R^n, if it is locally Lipschitz at each point of R^n.

Definition 2.2 (see [16]). Let $\psi : R^n \to R$ be a locally Lipschitz function, then $\psi^\circ(u;v)$ denotes Clarke's generalized directional derivative of ψ at $u \in R^n$ in the direction v and is defined as

$$\psi^\circ(u;v) = \limsup_{y \to u\, t \to 0} \frac{\psi(y+tv) - \psi(y)}{t}. \tag{2.4}$$

Clarke's generalized gradient of ψ at u is denoted by $\partial\psi(u)$ and is defined as

$$\partial\psi(u) = \left\{\xi \in R^n \mid \psi^\circ(u;v) \geq \langle \xi, v\rangle,\ \forall v \in R^n\right\}. \tag{2.5}$$

Let $f : R^n \to R^m$ be a vector-valued function given by $f = (f_1, f_2, \ldots, f_m)$, where $f_i : R^n \to R$, $i = 1, 2, \ldots, m$. Then f is said to be locally Lipschitz on R^n if each f_i is locally Lipschitz on R^n. The generalized directional derivative of a locally Lipschitz function $f : R^n \to R^m$ at $u \in R^n$ in the direction v is given by

$$f^\circ(u;v) = \left\{f_1^\circ(u;v), f_2^\circ(u;v), \ldots, f_m^\circ(u;v)\right\}. \tag{2.6}$$

The generalized gradient of f at u is the set

$$\partial f(u) = \partial f_1(u) \times \partial f_2(u) \times \cdots \times \partial f_m(u), \tag{2.7}$$

where $\partial f_i(u)$ $(i = 1, 2, \ldots, m)$ is the generalized gradient of f_i at u.

Every $A = (a_1, a_2, \ldots, a_m) \in \partial f(u)$ is a continuous linear operator from R^n to R^m and

$$Au = (\langle a_1, u\rangle, \langle a_2, u\rangle, \ldots, \langle a_m, u\rangle) \in R^m, \quad \forall u \in R^n. \tag{2.8}$$

Lemma 2.3 (see [16]). *(a) If $f_i : R^n \to R$ is locally Lipschitz then, for each $u \in R^n$,*

$$f_i^\circ(u;v) = \max\left\{\langle \xi, v\rangle \mid \xi \in \partial f_i(u)\right\}, \quad \forall v \in R^n,\ i = 1, 2, \ldots, m. \tag{2.9}$$

(b) Let f_i $(i = 1, 2, \ldots, m)$ be a finite family of locally Lipschitz functions on R^n, then $\sum_{i=1}^m f_i$ is also locally Lipschitz and

$$\partial\left(\sum_{i=1}^m f_i\right)(u) \subseteq \sum_{i=1}^m \partial f_i(u), \quad \forall u \in R^n. \tag{2.10}$$

Definition 2.4 (see [17]). A function $h : R^n \to R$ is said to be α-invex function at $u \in R^n$ with respect to α and η, if there exist functions α and η such that, for every $x \in R^n$, we have

$$h(x) - h(u) \geq \langle \alpha(x,u)\nabla h(u), \eta(x,u)\rangle. \tag{2.11}$$

In this paper, we consider the following vector optimization problem with cone constraints:

$$\begin{aligned} &K-\min f(x) \\ &\text{s.t.} -g(x)\in Q, \end{aligned} \tag{VP}$$

where $f : R^n \to R^m$, $g : R^n \to R^p$ are locally Lipschitz functions on R^n and K, Q are closed convex cones with nonempty interiors in R^m and R^p, respectively.

Denote $X = \{x \in R^n : -g(x) \in Q\}$ the feasible set of problem (VP).

For each $\lambda \in K^+$ and $\mu \in Q^+$, we suppose that $\lambda f = \lambda \circ f$ and $\mu g = \mu \circ g$ are locally Lipschitz.

Now, we present the concepts of solutions for problem (VP) in the following sense.

Definition 2.5. Let $u \in X$, then

(a) u is said to be a minimum of (VP) if for all $x \in X$,

$$f(u) - f(x) \notin K \setminus \{0\}; \tag{2.12}$$

(b) u is said to be a weak minimum of (VP) if for all $x \in X$,

$$f(u) - f(x) \notin \operatorname{int} K; \tag{2.13}$$

(c) u is said to be a strong minimum of (VP) if for all $x \in X$,

$$f(x) - f(u) \in K. \tag{2.14}$$

Based on the lines of Yen and Sach [12] and Noor [17], we define the notions as follows.

Definition 2.6. Let $f : R^n \to R^m$ be a locally Lipschitz function. f is said to be K-α-generalized invex at $u \in R^n$, if there exist functions α and η such that for every $x \in R^n$ and $A \in \partial f(u)$,

$$f(x) - f(u) - \alpha(x,u)A\eta(x,u) \in K. \tag{2.15}$$

Definition 2.7. Let $f : R^n \to R^m$ be a locally Lipschitz function. f is said to be K-α-nonsmooth invex at $u \in R^n$, if there exist functions α and η such that for every $x \in R^n$,

$$f(x) - f(u) - \alpha(x,u)f^\circ(u;\eta) \in K, \tag{2.16}$$

where

$$f^\circ(u;\eta) = \{f_1^\circ(u;\eta), f_2^\circ(u;\eta), \dots, f_m^\circ(u;\eta)\}. \tag{2.17}$$

Remark 2.8. If $m = 1$, $K = R_+$, and f is differentiable, then K-α-generalized invex and K-α-nonsmooth invex functions become α-invex function [17]; if $\alpha(x,u) \equiv 1$ for all $x, u \in R^n$, then K-α-generalized invex and K-α-nonsmooth invex functions reduce to K-generalized invex and K-nonsmooth invex functions defined by Yen and Sach [12].

Lemma 2.9. *If f is K-α-generalized invex at u with respect to α and η, then f is K-α-nonsmooth invex at u with respect to the same α and η.*

Proof. Since f is K-α-generalized invex at u, then there exist α and η such that for every $x \in R^n$ and $A \in \partial f(u)$

$$f(x) - f(u) - \alpha(x,u)A\eta(x,u) \in K. \tag{2.18}$$

By Lemma 2.3, for each $i \in \{1, 2, \ldots, m\}$, we choose $\tilde{a}_i \in \partial f_i(u)$ such that

$$\langle \tilde{a}_i, \eta \rangle = \max\{\langle a_i, \eta \rangle : a_i \in \partial f_i(u)\} = f_i^\circ(u; \eta). \tag{2.19}$$

Then $\widetilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_m) \in \partial f(u)$ and

$$f(x) - f(u) - \alpha(x,u)\widetilde{A}\eta(x,u) \in K, \tag{2.20}$$

equivalently,

$$f(x) - f(u) - \alpha(x,u)f^\circ(u; \eta) \in K. \tag{2.21}$$

Hence, f is K-α-nonsmooth invex at u with respect to the same α and η. □

The following example shows that converse of the above lemma is not true.

Example 2.10. Let $K = \{(x,y) \mid y \le -x\}$ be a cone in R^2. Assume that $f = (f_1, f_2)$, where

$$f_1(x) = \begin{cases} 2 - x^2, & x \ge 0, \\ 2, & x < 0, \end{cases} \qquad f_2(x) = \begin{cases} -x + 1, & x \ge 0, \\ 1, & x < 0. \end{cases} \tag{2.22}$$

Let $\alpha : R \times R \to R_+ \setminus \{0\}$ and $\eta : R \times R \to R$ be defined as $\alpha(x,u) = 1/2$ and $\eta(x,u) = 2x^2 - u^3$, respectively. Then at $u = 0$,

$$f(x) - f(0) - \frac{1}{2}f^\circ(0; \eta(x,0)) = \begin{cases} (-x^2, -x + x^2), & x \ge 0, \\ (0,0), & x < 0, \end{cases} \in K. \tag{2.23}$$

Hence, f is K-1/2-nonsmooth invex at $u = 0$.

It is easy to verify $\partial f(0) = \{(0, v) \mid -1 \le v \le 0\}$.

Taking $A = (0, -1/4) \in \partial f(0)$ and $x = -1$, we have

$$f(-1) - f(0) - \frac{1}{2} A\eta(-1, 0) = \left(0, \frac{1}{4}\right) \notin K. \tag{2.24}$$

Therefore, f is not K-$1/2$-generalized invex at $u = 0$.

Next, we introduce several related functions of K-α-nonsmooth invex.

Definition 2.11. f is said to be K-α-nonsmooth quasi-invex at $u \in R^n$, if there exist functions α and η such that for every $x \in R^n$,

$$f(x) - f(u) \notin \operatorname{int} K \Longrightarrow -\alpha(x, u) f^\circ(u; \eta(x, u)) \in K. \tag{2.25}$$

Definition 2.12. f is said to be K-α-nonsmooth pseudo-invex at $u \in R^n$, if there exist functions α and η such that for every $x \in R^n$,

$$f(u) - f(x) \in \operatorname{int} K \Longrightarrow -\alpha(x, u) f^\circ(u; \eta(x, u)) \in \operatorname{int} K. \tag{2.26}$$

Definition 2.13. f is said to be strict K-α-nonsmooth pseudo-invex at $u \in R^n$, if there exist functions α and η such that for every $x \in R^n$,

$$f(u) - f(x) \in K \Longrightarrow -\alpha(x, u) f^\circ(u; \eta(x, u)) \in \operatorname{int} K. \tag{2.27}$$

Definition 2.14. f is said to be strong K-α-nonsmooth pseudo-invex at $u \in R^n$, if there exist functions α and η such that for every $x \in R^n$,

$$f(x) - f(u) \notin K \Longrightarrow -\alpha(x, u) f^\circ(u; \eta(x, u)) \in \operatorname{int} K. \tag{2.28}$$

Remark 2.15. If $\alpha(x, u) \equiv 1$ for all $x, u \in R^n$ and f is differentiable, then K-α-nonsmooth pseudo-invex and strong K-α-nonsmooth pseudo-invex functions reduce to K-pseudo-invex and strong K-pseudo-invex functions, defined by Khurana [14].

Remark 2.16. If $m = 1, K = R_+, \alpha(x, u) \equiv 1$ for all $x, u \in R^n$, and f is differentiable, then K-α-nonsmooth quasi-invex functions reduce to quasi-invex functions and K-α-nonsmooth pseudo-invex and strong K-α-nonsmooth pseudo-invex functions reduce to pseudo-invex functions [8].

Remark 2.17. If $\alpha(x, u) \equiv 1$ for all $x, u \in R^n$, then the above definitions reduce to the corresponding definitions [15]. If f is differentiable, then K-α-generalized invex and K-α-nonsmooth pseudo-invex functions reduce to α-K-invex and α-K pseudo-invex functions [13], respectively.

3. Optimality Criteria

In this section, we establish a few sufficient optimality conditions for problem (VP) by using the above defined functions.

Theorem 3.1. *Let f be K-α-generalized invex and g be Q-α-generalized invex at $u \in X$ with respect to the same α and η. We assume that there exist $\lambda \in K^{+}$, $\lambda \neq 0$, $\mu \in Q^{+}$ such that*

$$0 \in \partial(\lambda f)(u) + \partial(\mu g)(u), \tag{3.1}$$

$$\mu g(u) = 0. \tag{3.2}$$

Then u is a weak minimum of (VP).

Proof. By contradiction, we assume that u is not a weak minimum of (VP). Then there exists a feasible solution x of (VP) such that

$$f(u) - f(x) \in \operatorname{int} K. \tag{3.3}$$

From (3.1), it follows that there exist $s \in \partial(\lambda f)(u)$ and $t \in \partial(\mu g)(u)$ such that

$$s + t = 0. \tag{3.4}$$

Since f is K-α-generalized invex and g is Q-α-generalized invex at u, we get

$$f(x) - f(u) - \alpha(x,u)A\eta(x,u) \in K, \quad \forall A \in \partial f(u), \tag{3.5}$$

$$g(x) - g(u) - \alpha(x,u)B\eta(x,u) \in Q, \quad \forall B \in \partial g(u). \tag{3.6}$$

Summing (3.3) and (3.5), we have

$$-\alpha(x,u)A\eta(x,u) \in \operatorname{int} K, \quad \forall A \in \partial f(u). \tag{3.7}$$

As $\lambda \in K^{+}$, $\lambda \neq 0$, from Lemma 2.1, we obtain

$$\alpha(x,u)\lambda A\eta(x,u) < 0, \quad \forall A \in \partial f(u), \tag{3.8}$$

which yields

$$\alpha(x,u)s\eta(x,u) < 0, \quad \text{as } \lambda \neq 0, s \in \partial(\lambda f)(u) = \lambda\partial f(u). \tag{3.9}$$

Considering positivity of $\alpha(x,u)$ and (3.4), one has

$$t\eta(x,u) > 0. \tag{3.10}$$

From $t \in \partial(\mu g)(u) = \mu\partial g(u)$, we deduce

$$t = \mu B^{*}, \text{ for some } B^{*} \in \partial g(u). \tag{3.11}$$

Hence,

$$\mu B^* \eta(x,u) > 0, \quad \text{where } B^* \in \partial g(u). \tag{3.12}$$

By $\mu \in Q^+$, relation (3.6) gives

$$\mu g(x) - \mu g(u) - \mu\alpha(x,u) B\eta(x,u) \geq 0, \quad \forall B \in \partial g(u). \tag{3.13}$$

By virtue of (3.2) and $x \in X$, the above inequality implies

$$-\mu\alpha(x,u) B\eta(x,u) \geq 0, \quad \forall B \in \partial g(u), \tag{3.14}$$

that is,

$$\mu B\eta(x,u) \leq 0, \quad \forall B \in \partial g(u), \tag{3.15}$$

which is a contradiction to (3.12).

Therefore, u is a weak minimum of (VP). □

Theorem 3.2. *Let f be K-α-generalized invex and g be Q-α-generalized invex at $u \in X$ with respect to the same α and η. We assume that there exist $\lambda \in K^{++}$, $\mu \in Q^+$ such that* (3.1) *and* (3.2) *hold. Then u is a minimum of* (*VP*).

Proof. Assume contrary to the result that u is not a minimum of (VP). Then there exists $x \in X$ such that

$$f(u) - f(x) \in K \setminus \{0\}. \tag{3.16}$$

From (3.1), it follows that there exist $s \in \partial(\lambda f)(u)$ and $t \in \partial(\mu g)(u)$ such that

$$s + t = 0. \tag{3.17}$$

Since f is K-α-generalized invex at $u \in X$, we get

$$f(x) - f(u) - \alpha(x,u) A\eta(x,u) \in K, \quad \forall A \in \partial f(u). \tag{3.18}$$

Utilizing (3.16), we deduce

$$-\alpha(x,u) A\eta(x,u) \in K \setminus \{0\}, \quad \forall A \in \partial f(u). \tag{3.19}$$

According to $\lambda \in K^{++}$, we obtain

$$\alpha(x,u)\lambda A\eta(x,u) < 0, \quad \forall A \in \partial f(u). \tag{3.20}$$

Next proceeding on the same lines as in the proof of Theorem 3.1, we obtain a contradiction.
Thus, u is a minimum of (VP). □

Theorem 3.3. *Let f be K-α-nonsmooth pseudo-invex and g be Q-α-nonsmooth quasi-invex at $u \in X$ with respect to the same α and η. We assume that there exist $\lambda \in K^+, \lambda \neq 0, \mu \in Q^+$ such that* (3.1) *and* (3.2) *hold. Then u is a weak minimum of* (*VP*).

Proof. It follows from (3.1) that there exist $s \in \partial(\lambda f)(u)$ and $t \in \partial(\mu g)(u)$ such that

$$s + t = 0. \tag{3.21}$$

Suppose that u is not a weak minimum of (VP). Then there exists $x \in X$ such that

$$f(u) - f(x) \in \operatorname{int} K. \tag{3.22}$$

Since f is K-α-nonsmooth pseudo-invex at $u \in X$, we deduce

$$-\alpha(x,u) f^{\circ}(u; \eta(x,u)) \in \operatorname{int} K. \tag{3.23}$$

By $\lambda \in K^+$, $\lambda \neq 0$ and Lemma 2.1, we obtain

$$\alpha(x,u) \lambda f^{\circ}(u; \eta(x,u)) < 0. \tag{3.24}$$

From $\alpha(x,u) > 0$ and $f_i^{\circ}(u; \eta(x,u)) = \max\{\langle v_i, \eta\rangle : v_i \in \partial f_i(u)\}, i = 1, 2, \dots, m$, we have

$$\lambda A \eta(x,u) < 0, \quad \forall A \in \partial f(u), \tag{3.25}$$

which implies,

$$s\eta(x,u) < 0, \quad \text{where } s \in \partial(\lambda f)(u) = \lambda \partial f(u). \tag{3.26}$$

By $x \in X$ and $\mu \in Q^+$, $-g(x) \in Q$ gives

$$\mu g(x) \leq 0. \tag{3.27}$$

Taking (3.2) into account, one has

$$\mu(g(x) - g(u)) \leq 0. \tag{3.28}$$

Next we prove

$$\mu g^{\circ}(u; \eta(x,u)) \leq 0. \tag{3.29}$$

If $\mu = 0$, inequality (3.29) holds obviously.

If $\mu \neq 0$, from (3.28) and Lemma 2.1, we deduce

$$g(x) - g(u) \notin \operatorname{int} Q. \tag{3.30}$$

Since g is Q-α-nonsmooth quasi-invex at $u \in$ X, we have

$$-\alpha(x,u) g^{\circ}(u; \eta(x,u)) \in Q. \tag{3.31}$$

From $\alpha(x,u) > 0$ and $\mu \in Q^{+}$, it follows that (3.29) also holds.

Similarly, by Lemma 2.3, inequality (3.29) gives

$$\mu B \eta(x,u) \leq 0, \quad \forall B \in \partial g(u), \tag{3.32}$$

which yields,

$$t\eta(x,u) \leq 0, \quad \text{where } t \in \partial(\mu g)(u) = \mu \partial g(u). \tag{3.33}$$

Hence,

$$s\eta(x,u) \geq 0, \tag{3.34}$$

which is in contradiction with (3.26).

Therefore, u is a weak minimum of (VP). □

The following example illustrates the above theorem.

Example 3.4. Consider the vector optimization problem (VP) where $K = \{(x,y) y \geq -x, y \geq x\}$, $Q = \{(x,y) - x \leq y \leq x, x \geq 0\}$, and $f_i, g_i : R \to R$, $i = 1,2$ are defined as

$$f_1(x) = \begin{cases} -x^2 - x, & x > 0, \\ 0, & x \leq 0, \end{cases} \qquad f_2(x) = \begin{cases} -2x + 1, & x > 0, \\ 1, & x \leq 0, \end{cases}$$
$$g_1(x) = \begin{cases} x^3, & x > 0, \\ \dfrac{x}{3}, & x \leq 0, \end{cases} \qquad g_2(x) = \begin{cases} x^2, & x > 0, \\ \dfrac{x}{2}, & x \leq 0. \end{cases} \tag{3.35}$$

Let $\alpha : R \times R \to R_{+} \setminus \{0\}$ and $\eta : R \times R \to R$ be defined as $\alpha(x,u) = 2$ and $\eta(x,u) = (x + 2u)^3$, respectively. It is easily testified that f and g are K-α-nonsmooth pseudo-invex and K-α-nonsmooth quasi-invex at $u = 0$, respectively. The feasible set of (VP) is given by $X = (-\infty, 0]$.

It is also easy to verify $\partial f(0) = [-1,0] \times [-2,0]$, $\partial g(0) = [0,1/3] \times [0,1/2]$.

Taking $\lambda = (1,3) \in K^+$ and $\mu = (3,1) \in Q^+$, we have

$$0 \in \left[-7, \frac{3}{2}\right] = \partial(\lambda f)(0) + \partial(\mu g)(0), \quad \mu g(0) = 0, \tag{3.36}$$

which imply that (3.1) and (3.2) hold.

Therefore, by Theorem 3.3, $u = 0$ is a weak minimum of (VP).

Theorem 3.5. *Let f be strong K-α-nonsmooth pseudo-invex and g be Q-α-nonsmooth quasi-invex at $u \in X$ with respect to the same α and η. We assume that there exist $\lambda \in K^+, \lambda \neq 0, \mu \in Q^+$ such that* (3.1) *and* (3.2) *hold. Then u is a strong minimum of* (*VP*).

Proof. From (3.1), it follows that there exist $s \in \partial(\lambda f)(u)$ and $t \in \partial(\mu g)(u)$ such that

$$s + t = 0. \tag{3.37}$$

Assume that u is not a strong minimum of (VP). Then there exists $x \in X$ such that

$$f(x) - f(u) \notin K. \tag{3.38}$$

Since f is strong K-α-nonsmooth pseudo-invex at u, we deduce

$$-\alpha(x,u) f^{\circ}(u; \eta(x,u)) \in \operatorname{int} K. \tag{3.39}$$

Next proceeding on the same lines as in the proof of Theorem 3.3, we get a contradiction.

Hence u is a strong minimum of (VP). □

Theorem 3.6. *Let f be strict K-α-nonsmooth pseudo-invex and g be Q-α-nonsmooth quasi-invex at $u \in X$ with respect to the same α and η. We assume that there exist $\lambda \in K^+$, $\lambda \neq 0$, $\mu \in Q^+$ such that* (3.1) *and* (3.2) *hold. Then u is a minimum of* (*VP*).

Proof. From (3.1), it follows that there exist $s \in \partial(\lambda f)(u)$ and $t \in \partial(\mu g)(u)$ such that

$$s + t = 0. \tag{3.40}$$

By contradiction, assume that u is not a minimum of (VP). Then there exists $x \in X$ such that

$$f(u) - f(x) \in K \setminus \{0\} \subset K. \tag{3.41}$$

Since f is strict K-α-nonsmooth pseudo-invex at u, we have

$$-\alpha(x,u) f^{\circ}(u; \eta(x,u)) \in \operatorname{int} K. \tag{3.42}$$

Next as in Theorem 3.3 we arrive at a contradiction.

Therefore, u is a minimum of (VP). □

4. Duality

In relation to (VP), we consider the following Mond-Weir type dual problem:

$$\begin{aligned} &K - \max f(y) \\ \text{s.t.}\quad &0 \in \partial(\lambda f)(y) + \partial(\mu g)(y), \\ &\mu g(y) \geq 0, \\ &y \in R^n \quad, \lambda \in K^+, \quad \lambda \neq 0, \quad \mu \in Q^+. \end{aligned} \tag{VD}$$

Denote the feasible set of problem (VD) by G, namely, $G = \{(y, \lambda, \mu) : 0 \in \partial(\lambda f)(y) + \partial(\mu g)(y), \mu g(y) \geq 0, y \in R^n, \lambda \in K^+, \lambda \neq 0, \mu \in Q^+\}$.

Now, we establish weak and converse duality results.

Theorem 4.1 (Weak duality). *Let $x \in X$ and $(y, \lambda, \mu) \in G$. If f is K-α-nonsmooth pseudo-invex and g is Q-α-nonsmooth quasi-invex at y with respect to the same α and η, then*

$$f(y) - f(x) \notin \operatorname{int} K. \tag{4.1}$$

Proof. Since $(y, \lambda, \mu) \in G$, from (VD), it follows that there exist $s \in \partial(\lambda f)(y)$ and $t \in \partial(\mu g)(y)$ such that

$$s + t = 0. \tag{4.2}$$

By contradiction, we assume that $f(y) - f(x) \in \operatorname{int} K$.

Since f is K-α-nonsmooth pseudo-invex at y, we have

$$-\alpha(x, y) f^{\circ}(y; \eta(x, y)) \in \operatorname{int} K. \tag{4.3}$$

By $\lambda \in K^+$, $\lambda \neq 0$ and Lemma 2.1, we get

$$\alpha(x, y) \lambda f^{\circ}(y; \eta(x, y)) < 0. \tag{4.4}$$

From $\alpha(x, y) > 0$ and Lemma 2.3, we deduce

$$\lambda A \eta(x, y) < 0, \quad \forall A \in \partial f(y), \tag{4.5}$$

which yields

$$s\eta(x, y) < 0, \quad \text{where } s \in \partial(\lambda f)(y) = \lambda \partial f(y). \tag{4.6}$$

Using (4.2), we obtain

$$t\eta(x, y) > 0. \tag{4.7}$$

From $t \in \partial(\mu g)(y) = \mu \partial g(y)$, it follows that there exists $B^* \in \partial g(y)$ such that

$$t = \mu B^*. \tag{4.8}$$

Hence,

$$\mu B^* \eta(x, y) > 0, \quad \text{where } B^* \in \partial g(y). \tag{4.9}$$

From $x \in X$ and $(y, \lambda, \mu) \in G$, we find that

$$\mu g(x) \leq 0 \leq \mu g(y). \tag{4.10}$$

Now we claim that

$$\mu g^{\circ}(y; \eta(x, y)) \leq 0. \tag{4.11}$$

If $\mu = 0$, then (4.11) holds trivially.
If $\mu \neq 0$, from (4.10) and Lemma 2.1 we have

$$g(x) - g(y) \notin \operatorname{int} Q. \tag{4.12}$$

As g is Q-α-nonsmooth quasi-invex at y, we obtain

$$-\alpha(x, y) g^{\circ}(y; \eta(x, y)) \in Q, \tag{4.13}$$

which means that (4.11) also holds and is equivalent to

$$\mu B \eta(x, y) \leq 0, \quad \forall B \in \partial g(y), \tag{4.14}$$

which is a contradiction to (4.9). Thus

$$f(y) - f(x) \notin \operatorname{int} K. \tag{4.15}$$

□

Theorem 4.2 (Weak duality). *Let $x \in X$ and $(y, \lambda, \mu) \in G$. If f is K-α-generalized invex and g is Q-α-generalized invex at y with respect to the same α and η, then*

$$f(y) - f(x) \notin \operatorname{int} K. \tag{4.16}$$

Proof. Since $(y, \lambda, \mu) \in G$, from (VD), it follows that there exist $s \in \partial(\lambda f)(y)$ and $t \in \partial(\mu g)(y)$ such that

$$s + t = 0. \tag{4.17}$$

We assume contrary to the result that

$$f(y) - f(x) \in \operatorname{int} K. \tag{4.18}$$

Since f is K-α-generalized invex and g is Q-α-generalized invex at y, we get

$$f(x) - f(y) - \alpha(x,y)A\eta(x,y) \in K, \quad \forall A \in \partial f(y), \tag{4.19}$$

$$g(x) - g(y) - \alpha(x,y)B\eta(x,y) \in Q, \quad \forall B \in \partial g(y). \tag{4.20}$$

Summing (4.18) and (4.19), we have

$$-\alpha(x,y)A\eta(x,y) \in \operatorname{int} K, \quad \forall A \in \partial f(y). \tag{4.21}$$

By $\lambda \in K^{+}$, $\lambda \neq 0$ and Lemma 2.1, we obtain

$$\alpha(x,y)\lambda A\eta(x,y) < 0, \tag{4.22}$$

which yields

$$\alpha(x,y)s\eta(x,y) < 0, \quad \text{where } s \in \partial(\lambda f)(y) = \lambda\partial f(y), \lambda \neq 0. \tag{4.23}$$

Applying positivity of $\alpha(x,y)$ and (4.17), we get

$$t\eta(x,y) > 0. \tag{4.24}$$

By the fact that $t \in \partial(\mu g)(y) = \mu\partial g(y)$, one has

$$t = \mu B^{*}, \quad \text{where } B^{*} \in \partial g(y). \tag{4.25}$$

Hence,

$$\mu B^{*}\eta(x,y) > 0, \quad \text{where } B^{*} \in \partial g(y). \tag{4.26}$$

From $\mu \in Q^{+}$ and (4.20), we obtain

$$\mu g(x) - \mu g(y) - \mu\alpha(x,y)B\eta(x,y) \geq 0, \quad \forall B \in \partial g(y). \tag{4.27}$$

As $x \in X$ and $(y, \lambda, \mu) \in G$, we get

$$\mu g(x) \leq 0 \leq \mu g(y). \tag{4.28}$$

Using the above relation, (4.27) yields

$$-\mu\alpha(x,y)B\eta(x,y) \geq 0, \quad \forall B \in \partial g(y), \tag{4.29}$$

that is,

$$\mu B\eta(x,y) \leq 0, \quad \forall B \in \partial g(y), \tag{4.30}$$

which contradicts (4.26). Therefore,

$$f(y) - f(x) \notin \operatorname{int} K. \tag{4.31}$$

□

Theorem 4.3 (Converse duality). *Let $y \in X$ and $(y,\lambda,\mu) \in G$. Assume that f is K-α-nonsmooth pseud-invex and g is Q-α-nonsmooth quasi-invex at y with respect to the same α and η. Then y is a weak minimum of* (*VP*).

Proof. Since $(y,\lambda,\mu) \in G$, from (VD), it follows that there exist $s \in \partial(\lambda f)(y)$ and $t \in \partial(\mu g)(y)$ such that

$$s + t = 0. \tag{4.32}$$

Assume contrary to the result that y is not a weak minimum of (VP). Then there exists $\overline{x} \in X$ such that

$$f(y) - f(\overline{x}) \in \operatorname{int} K. \tag{4.33}$$

Since f is K-α-nonsmooth pseudo-invex at y, we have

$$-\alpha(\overline{x},y)f^{\circ}(y;\eta(\overline{x},y)) \in \operatorname{int} K. \tag{4.34}$$

By $\lambda \in K^{+}$, $\lambda \neq 0$ and Lemma 2.1, we get

$$\alpha(\overline{x},y)\lambda f^{\circ}(y;\eta(\overline{x},y)) < 0, \tag{4.35}$$

which is equivalent to

$$\lambda A\eta(\overline{x},y) < 0, \quad \forall A \in \partial f(y), \tag{4.36}$$

which yields

$$s\eta(\overline{x},y) < 0, \quad \text{where } s \in \partial(\lambda f)(y) = \lambda\partial f(y),\ \lambda \neq 0. \tag{4.37}$$

Using (4.32), we obtain

$$t\eta(\overline{x}, y) > 0. \tag{4.38}$$

As $t \in \partial(\mu g)(y) = \mu\partial g(y)$, thus $t = \mu B^*$, where $B^* \in \partial g(y)$.

Hence,

$$\mu B^* \eta(\overline{x}, y) > 0, \quad \text{where } B^* \in \partial g(y). \tag{4.39}$$

By $\overline{x} \in X$ and $(y, \lambda, \mu) \in G$, we have

$$\mu g(\overline{x}) \leq 0 \leq \mu g(y). \tag{4.40}$$

By the similar argument to that of Theorem 4.1, we can prove that

$$\mu g^\circ(y; \eta(\overline{x}, y)) \leq 0, \tag{4.41}$$

which is equivalent to

$$\mu B \eta(\overline{x}, y) \leq 0, \quad \forall B \in \partial g(y), \tag{4.42}$$

which is in contradiction with (4.39).

Therefore, y is a weak minimum of (VP). □

Theorem 4.4 (Converse duality). *Let $y \in X$ and $(y, \lambda, \mu) \in G$. Assume that f is K-α-generalized invex and g is Q-α-generalized invex at y with respect to the same α and η. Then y is a weak minimum of (VP).*

Proof. The proof of the above theorem is very similar to the proof of Theorem 3.1, except that for this case we use the feasibility of (y, λ, μ) for (VD) instead of the relations (3.1) and (3.2). □

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References

[1] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*, Wiley-Interscience, Hoboken, NJ, USA, 3rd edition, 2006.
[2] O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, NY, USA, 1969.
[3] C. R. Bector, S. Chandra, and M. K. Bector, "Sufficient optimality conditions and duality for a quasiconvex programming problem," *Journal of Optimization Theory and Applications*, vol. 59, no. 2, pp. 209–221, 1988.

[4] L. B. Dos Santos, R. Osuna-Gómez, M. A. Rojas-Medar, and A. Rufián-Lizana, "Preinvex functions and weak efficient solutions for some vectorial optimization problem in Banach spaces," *Computers & Mathematics with Applications*, vol. 48, no. 5-6, pp. 885–895, 2004.

[5] X. J. Long, "Optimality conditions and duality for nondifferentiable multiobjective fractional programming problems with (C, α, ρ, d)-convexity," *Journal of Optimization Theory and Applications*, vol. 148, no. 1, pp. 197–208, 2011.

[6] H. Slimani and M. S. Radjef, "Nondifferentiable multiobjective programming under generalized d_I-invexity," *European Journal of Operational Research*, vol. 202, no. 1, pp. 32–41, 2010.

[7] G. Yu and S. Liu, "Some vector optimization problems in Banach spaces with generalized convexity," *Computers & Mathematics with Applications*, vol. 54, no. 11-12, pp. 1403–1410, 2007.

[8] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[9] R. N. Kaul and S. Kaur, "Optimality criteria in nonlinear programming involving nonconvex functions," *Journal of Mathematical Analysis and Applications*, vol. 105, no. 1, pp. 104–112, 1985.

[10] C. S. Lalitha, "Generalized nonsmooth invexity in multiobjective programming," *International Journal of Statistics and Management System*, vol. 11, no. 2, pp. 183–198, 1995.

[11] T. W. Reiland, "Generalized invexity for nonsmooth vector-valued mappings," *Numerical Functional Analysis and Optimization*, vol. 10, no. 11-12, pp. 1191–1202, 1989.

[12] N. D. Yen and P. H. Sach, "On locally Lipschitz vector-valued invex functions," *Bulletin of the Australian Mathematical Society*, vol. 47, no. 2, pp. 259–271, 1993.

[13] G. Giorgi and A. Guerraggio, "The notion of invexity in vector optimization: smooth and nonsmooth case," in *Proceedings of the 5th Symposium on Generalized Convexity, Generalized Monotonicity*, Kluwer Academic Publishers, Luminy, France, 1997.

[14] S. Khurana, "Symmetric duality in multiobjective programming involving generalized cone-index functions," *European Journal of Operational Research*, vol. 165, no. 3, pp. 592–597, 2005.

[15] S. K. Suneja, S. Khurana, and Vani, "Generalized nonsmooth invexity over cones in vector optimization," *European Journal of Operational Research*, vol. 186, no. 1, pp. 28–40, 2008.

[16] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1983.

[17] M. A. Noor, "On generalized preinvex functions and monotonicities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 4, article 110, pp. 1–9, 2004.

[18] S. K. Mishra, R. P. Pant, and J. S. Rautela, "Generalized α-invexity and nondifferentiable minimax fractional programming," *Journal of Computational and Applied Mathematics*, vol. 206, no. 1, pp. 122–135, 2007.

[19] S. K. Mishra, S. Y. Wang, and K. K. Lai, "On non-smooth α-invex functions and vector variational-like inequality," *Optimization Letters*, vol. 2, no. 1, pp. 91–98, 2008.

[20] B. D. Craven, *Control and Optimization*, Chapman and Hall Mathematics Series, Chapman and Hall, London, UK, 1995.