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Research Article

Viscosity Iterative Schemes for Finding Split Common Solutions of Variational Inequalities and Fixed Point Problems

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We introduce some new iterative schemes based on viscosity approximation method for finding a split common element of the solution set of a pair of simultaneous variational inequalities for inverse strongly monotone mappings in real Hilbert spaces with a family of infinitely nonexpansive mappings. Some strong convergence theorems are also given. Our results generalize and improve some well-known results in the literature and references therein.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let I be the identity mapping on H and C be a nonempty closed convex subset of H. Let $T : C \to H$ be a nonlinear operators. Then the canonical variational inequality problem for the operator T ($(\mathrm{VIP})_T$ or (VIP), for short) is to find $u \in C$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We use the symbol $\mathrm{VI}(C, T)$ to denote the solution set of (VIP), that is

$$\mathrm{VI}(C, T) = \{u \in C : \langle Tu, v - u \rangle \geq 0, \ \forall v \in C\}. \tag{1.2}$$

(VIP) was extensively investigated and generalized to the vector variational inequality problems for single-valued or multivalued maps and contains optimization problems, quasi-variational inequality problems, equilibrium problems, fixed-point problems, complementary problems, bilevel problems, and semi-infinite problems as special cases and applications; see [1–6] and references therein.

Let $S, T : C \to H$ be two nonlinear operators. In [7], some authors have considered the following pair of simultaneous variational inequality problems for operators S and T ($(\mathrm{PSVIP})_{S,T}$, for short):

$$(\mathrm{PSVIP})_{S,T} \text{ Find } u \in C \text{ such that } \langle Su, v-u\rangle \geq 0 \text{ and } \langle Tu, v-u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

An element $u \in C$ is a solution of $(\mathrm{PSVIP})_{S,T}$ if and only if $u \in \mathrm{VI}(C, S) \cap \mathrm{VI}(C, T)$. Clearly, $(\mathrm{PSVIP})_{S,T}$ reduces to (VIP) if $S = T$.

Example 1.1. Let $\mathbb{R}$ with usual inner product and let $a, b \in \mathbb{R}$ with $a < b$. Define two real-valued functions T_1, T_2 by $T_1 x = x^2$, $T_2 x = x^4$, for all $x \in [a, b]$. Then $T_1' x = 2x$, $T_2' x = 4x^3$ and there exists $x_0 \in [a, b]$ such that $T_1 x_0 = \min_{x\in[a,b]} T_1 x$ and $T_2 x_0 = \min_{x\in[a,b]} T_2 x$. If $x_0 \in (a, b)$, then $T_1 x_0 = T_2 x_0 = 0$; if $x_0 = a$, then $T_1' x_0 \geq 0$ and $T_2' x_0 \geq 0$; if $x_0 = b$, then $T_1' x_0 \leq 0$ and $T_2' x_0 \leq 0$. So we have

$$\langle T_1' x_0, x - x_0\rangle = T_1' x_0 (x - x_0) \geq 0, \qquad \langle T_2' x_0, x - x_0\rangle = T_2' x_0 (x - x_0) \geq 0, \quad \forall x \in [a, b] \tag{1.4}$$

or $x_0 \in \mathrm{VI}(C, T_1') \cap \mathrm{VI}(C, T_2')$ which means that x_0 is the solution of $(\mathrm{PSVIP})_{T_1', T_2'}$.

Obviously, the problem $(\mathrm{PSVIP})_{S,T}$ is considered in the same subset of the same space. But many cases, two variational inequality problems often lie in different subset of spaces. So, as a further development of the problem $(\mathrm{PSVIP})_{S,T}$, Censor et al. [8] presented a split variational inequality problem. Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$ and $K \subset H_2$ two closed convex sets. Let $A : H_1 \to H_2$ be a bounded linear operator. $T : C \to H_1$ and $S : K \to H_2$ are two nonlinear operators. The split variational inequality problem for T and S ($(\mathrm{SVIP})_{T,S}$, for short) is defined as follows:

$$\begin{aligned} (\mathrm{PSVIP})_{S,T} \text{ Find } p \in C \text{ such that } \langle Tp, v-p\rangle \geq 0, \quad \forall v \in C, \\ \text{and } u := Ap \in K \text{ solves } \langle Su, w-u\rangle \geq 0, \quad \forall w \in K. \end{aligned} \tag{1.5}$$

It is well known to find a solution of (VIP) or a common element of the solution set of (VIP) and a fixed point of nonlinear operators, which has been studied by many authors (see [9–16]) using all kinds of auxiliary techniques and formulations. In 2005, Iiduka and Takahashi [9] established the following iteration scheme: let $x_1 \in H$ be arbitrary, define

$$x_{n+1} = \alpha_n u + (1-\alpha_n) S_1 P_C (x_n - \lambda_n T x_n), \tag{1.6}$$

where S_1 is a nonexpansive mapping. They proved that the sequence $\{x_n\}$ defined by (1.6) strongly converge to $x^* \in F(S_1) \bigcap \mathrm{VI}(C,T)$, if the coefficient α_n, λ_n satisfy the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \qquad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \tag{1.7}$$

In 2007, Chen et al. [10] studied the following iterative process:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_1 P_C (x_n - \lambda_n T x_n), \tag{1.8}$$

where S_1 is a nonexpansive mapping. If $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, then they proved that $\{x_n\}$ converges strongly to $q \in F(S_1) \bigcap \mathrm{VI}(C,T)$, which solves the variational inequality:

$$\langle fq - q, p - q \rangle \leq 0, \quad \forall p \in F(S_1) \cap \mathrm{VI}(C,T). \tag{1.9}$$

For some split common solution problems, they have been studied by some authors; see [17, 18] and therein references. In this paper, we continue to study the (SVIP) and introduce some new iterative schemes based on viscosity approximation method for finding a common element of the fixed points set of nonexpansive mappings and the split solution set of a pair of variational inequalities for inverse strongly monotone mappings in real Hilbert spaces. Our results are new development of finding a common element of fixed point of nonlinear operators and variational inequality problems.

2. Preliminaries

In this paper, we use symbols $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively. A Banach space $(X, \|\cdot\|)$ is said to satisfy *Opial's condition*, if for each sequence $\{x_n\}$ in X with $x_n \rightharpoonup x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x. \tag{2.1}$$

It is well known that each Hilbert space satisfies Opial's condition; see, for example, [19]. Let $T : X \to X$ be a mapping. In this paper, the set of fixed points of T is denoted by $F(T)$.

A set-valued mapping $T_1 : H \to 2^H$ is said to be *monotone*, if for all $x, y \in H$, $f \in T_1 x$, and $g \in T_1 y$ imply that $\langle f - g, x - y \rangle \geq 0$. A monotone mapping $T_1 : H \to H$ is said to be *maximal*, if the graph $G(T_1)$ of T_1 is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T_1 is maximal, if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(T_1)$ implies that $f \in T_1 x$. Let $T : C \to H$ be a monotone mapping and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$. Define

$$T_1 v = \begin{cases} Tv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{2.2}$$

Then T_1 is maximal monotone and $\mathbf{0} \in T_1 v$ if and only if $v \in \mathrm{VI}(C,T)$, where $\mathbf{0}$ is the zero vector of H; see, for example, [9, 20, 21] for more details.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *projection operator* (or *metric projection*) from H onto C.

Let H_1 and H_2 be two Hilbert spaces. Let $A : H_1 \to H_2$ and $B : H_2 \to H_1$ be two bounded linear operators. B is called the adjoint operator (or adjoint) of A, if for all $z \in H_1$, $w \in H_2$, B satisfies $\langle Az, w\rangle = \langle z, Bw\rangle$. It is known that the adjoint operator of a bounded linear operator on a Hilbert space always exists and is bounded linear and unique. Moreover, it is not hard to show that if B is an adjoint operator of A, then $\|A\| = \|B\|$.

A mapping $T : C \to C$ is said to be

(1) v–expansive if there exists a constant $v > 0$ such that $\|Tx - Ty\| \geq v\|x - y\|$ for all $x, y \in C$. In particular, if $v = 1$, then T is called expansive.

(2) v-strongly monotone if there exists a constant $v > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq v\|x - y\|^2, \quad \forall x, y \in C. \tag{2.3}$$

Clearly, any v-strongly monotone mapping is v-expansive.

(3) u-inverse strongly monotone if there exists a constant $u > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq u\|Tx - Ty\|^2, \quad \forall x, y \in C. \tag{2.4}$$

(4) Relaxed u-cocoercive if there exists a constant $u > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-u)\|Tx - Ty\|^2, \quad \forall x, y \in C. \tag{2.5}$$

(5) Relaxed (u, v)-cocoercive if there exists constants $u, v > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-u)\|Tx - Ty\|^2 + v\|x - y\|^2, \quad \forall x, y \in C. \tag{2.6}$$

Especially, if $u = 0$, then T is v-strongly monotone. So this class of mapping is more general than the class of strongly monotone mapping.

(6) An α-Lipschitz mapping if there exists a constant $\alpha > 0$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$. In particular, if $0 < \alpha < 1$ ($\alpha = 1$, resp.), then T is called a contraction (a nonexpansive mapping, resp.)

Remark ST (see [9]). If T is v-strongly monotone and μ-Lipschitz continuous, that is, $\|Tx - Ty\| \leq \mu\|x - y\|$ for all $x, y \in C$, then T is (v/μ^2)-inverse strongly monotone.

Example 2.1. Let $Tx = -2x$, for all $x \in \mathbb{R}$. Then it is easy to see that for any $x, y \in \mathbb{R}$,

$$\langle Tx - Ty, x - y\rangle = -2|x - y|^2 \geq (-1)|Tx - Ty|^2 + |x - y|^2. \tag{2.7}$$

Hence T is a relaxed $(1, 1)$-cocoercive mapping, but T is not a strongly monotone mapping.

Now, let $\{T_i\}_{i\in\mathbb{N}}$ be a family of infinitely nonexpansive mappings. In [22], a mapping W_n is defined by the following:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I,
\end{aligned} \tag{2.8}$$

where $\{\lambda_i\}_{i\in\mathbb{N}} \subset [0,1]$. Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$.

The following properties for a W-mapping are well known.

Theorem 2.2 (see [22, 23]). *Let C be a nonempty closed convex subset of a Hilbert space E, let $T_1, T_2, \dots$ be a family of infinitely nonexpansive mappings from C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty, and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i \le b < 1$ for any $i \in \mathbb{N}$. Then the following statements hold:*

(1) *W_n is a nonexpansive mapping and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$.*

(2) *For each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

(3) *The mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, $x \in C$, is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2 \dots$.*

Theorem 2.3 (see [23]). *Let C be a nonempty closed convex subset of a Hilbert space H, $T_1, T_2, \dots$ be nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) = \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \le b < 1$ for any $i \in \mathbb{N}$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.9}$$

In particular, if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in C, then $\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0$.

The following results are crucial in this paper.

Lemma 2.4 (see [19]). *For a given $z \in H$, $x \in C$ satisfies the inequality $\langle x - z, y - x \rangle \ge 0$, for all $y \in C$ if and only if $x = P_C(z)$, where P_C is a projection operator from H onto C.*

It is well known that the projection operator P_C is nonexpansive and satisfies

$$\|P_Cx - P_Cy\|^2 \le \langle P_Cx - P_Cy, x - y\rangle, \quad \forall x, y \in H. \tag{2.10}$$

Lemma 2.5 (see [9]). *The element $u \in C$ is a solution of $(VIP)_T$ if and only if $u \in C$ satisfies the relation $u = P_C(u - \rho Tu)$, where P_C is the projection operator, $\rho > 0$ is a constant.*

Lemma 2.6 (see [24]). *Let $\{a_n\}$ be a nonnegative real sequence satisfying the following condition:*

$$a_{n+1} \le (1 - \lambda_n)a_n + \lambda_n b_n, \quad \forall n \ge n_0, \tag{2.11}$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a sequence in $\mathbf{R}$ such that

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \le 0$ *or* $\sum_{n=0}^{\infty} \lambda_n b_n$ *is convergent.*

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 (see [25]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$, then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.8 (see [26]). *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{2.12}$$

Especially, when $E = H$, then $J = I$. So, from Lemma 2.8, one has

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.13}$$

The following result is simple, but it is very useful in this paper.

Lemma 2.9. *Let $\{a_n\}$, $\{b_n\}$ be two nonnegative real sequences. If $\lim_{n\to\infty} a_n = 0$, then $\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} b_n$.*

3. Main Results

In this section, we construct an iteration scheme including a pair of mappings $T : C \to H_1$ and $S : K \to H_2$ which are u-inverse strongly monotone to solve the split variational inequality problem. For the purpose we first give the following Lemmas.

Lemma 3.1 (see page 3 in [9]). *Let $T : C \to H$ be a u-inverse strongly monotone mapping. Then $I - \lambda T$ is nonexpansive for any $\lambda \in [0, 2u]$.*

Example 3.2. Let $Tx = 3x$ for all $x \in \mathbb{R}$ and $u = 1/6$. Since

$$\langle Tx - Ty, x - y\rangle = 3|x - y|^2 \geq u|Tx - Ty|^2, \tag{3.1}$$

T is u-inversely monotone. Let $\lambda \in [0, 1/3] = [0, 2u]$. It is easy to see that

$$|(I - \lambda T)x - (I - \lambda T)y| = (1 - 3\lambda)|x - y| \leq |x - y|. \tag{3.2}$$

So $I - \lambda T$ is nonexpansive for all $\lambda \in [0, 2u]$.

Applying Lemma 3.1, we have the following important result.

Lemma 3.3. *Let $T, S : C \to H$ be two u-inverse strongly monotone mappings and $S_1 : C \to C$ be a nonexpansive mapping. Then for any given sequences $\{r_n\}$ and $\{s_n\}$ in $[0, 2u]$, $P_C(I - s_nT)$, $P_C(I - r_nS)$, $S_1P_C(I - s_nT)$ and $S_1P_C(I - r_nS)$ are all nonexpansive for all $n \in \mathbb{N}$.*

The following conclusion is immediate from Lemma 2.5.

Lemma 3.4. *The element $u \in C$ is a solution of $(SVIP)_{T,S}$ if and only if $u \in C$ satisfies the relation*

$$u = P_C(u - \rho Tu), \qquad Au = P_K(u - \rho SAu), \tag{3.3}$$

where P_C and P_K are the projection operators, $\rho > 0$ is a constant.

Theorem 3.5. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $K \subset H_2$ two nonempty closed convex sets. Let $T : C \to H_1$ and $S : K \to H_2$ be u-inversely monotone. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*. Let $f : C \to C$ be a contraction with contraction constant α. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of infinitely nonexpansive mappings of C into itself and U a nonexpansive mapping of K into itself such that $\Omega = \{p \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap \mathrm{VI}(C, T) : Ap \in F(U) \bigcap \mathrm{VI}(K, S)\} \neq \emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that $0 < \lambda_i \leq \xi < 1$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let W_n be the W-mapping of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily}, \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_K(I - \beta_n S)Ay_n, \\ z_n &= (1 - \theta)x_n + \theta W_n P_C(y_n + rA^*(Ul_n - Ay_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.4}$$

where $r \in (0, 1/\|A\|^2)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ further satisfy the following conditions:

(C_1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C_2) $\{\beta_n\} \subset [a, b]$ and $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = 0$, where $0 < a$, $b < 2u$,

then the following statements hold:

(a) there exists a unique $q \in \Omega$ such that $P_{\Omega} f(q) = q$;

(b) $\{x_n\}$ converges strongly to q.

Proof. Let $p \in \Omega$. By Lemma 3.3, $P_C(I - \beta_n T)$, and $P_K(I - \beta_n S)$ are nonexpansive for all $n \in \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N} \cup \{0\}$, by (3.4) and Lemma 3.4, we obtain the following inequalities:

$$\begin{aligned} \|y_n - p\| &= \|P_C(I - \beta_n T)x_n - P_C(I - \beta_n T)p\| \le \|x_n - p\|, \\ \|l_n - Ap\| &= \|P_K(I - \beta_n S)Ay_n - P_K(I - \beta_n S)Ap\| \le \|Ay_n - Ap\|, \\ \|x_{n+1} - p\| &\le \alpha_n \|f(x_n) - p\| + (1 - \alpha_n)\|z_n - p\| \le \alpha_n \|f(x_n) - p\| + \|z_n - p\|. \end{aligned} \tag{3.5}$$

Let $h_n = P_C(y_n + rA^*(Ul_n - Ay_n))$ for $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} \|W_n h_n - p\|^2 \le \|h_n - p\|^2 &= \|P_C(y_n + rA^*(Ul_n - Ay_n)) - p\|^2 \le \|y_n + rA^*(Ul_n - Ay_n) - p\|^2 \\ &= \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 + 2r\langle y_n - p, A^*(Ul_n - Ay_n)\rangle \\ &\le \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\ &\quad + 2r\langle A(y_n - p) + Ul_n - Ay_n - (Ul_n - Ay_n), Ul_n - Ay_n\rangle \\ &= \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\ &\quad + 2r\left\{\frac{1}{2}\|Ul_n - Ap\|^2 + \frac{1}{2}\|Ul_n - Ay_n\|^2 - \|Ay_n - Ap\|^2 - \|Ul_n - Ay_n\|^2\right\} \\ &\le \|y_n - p\|^2 + r^2\|A^*\|^2\|Ul_n - Ay_n\|^2 - r\|Ul_n - Ay_n\|^2 \\ &= \|y_n - p\|^2 - r\left(1 - r\|A^*\|^2\right)\|Ul_n - Ay_n\|^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \|z_n - p\|^2 &= \|(1 - \theta)(x_n - p) + \theta(W_n h_n - p)\|^2 \\ &\le (1 - \theta)\|x_n - p\|^2 + \theta\|W_n h_n - p\|^2 \\ &\le (1 - \theta)\|x_n - p\|^2 + \theta\|h_n - p\|^2 \\ &\le (1 - \theta)\|x_n - p\|^2 + \theta\|y_n - p\|^2 \\ &\le \|x_n - p\|^2, \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N} \cup \{0\}$. Next, we will show that the conclusion is true by several steps.

Step 1. We show that all $\{x_n\}, \{y_n\}, \{z_n\}, \{Tx_n\}, \{SAy_n\}, \{l_n\}$, and $\{W_n h_n\}$ are bounded.

To prove it, it suffices to show $\{x_n\}$ is bounded. Let $p \in \Omega$. We claim that

$$\|x_n - p\| \leq \mathcal{L} := \max\left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1-\alpha} \right\} \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.8}$$

Indeed, it is obvious that (3.8) is true for $n = 0$. Assume that (3.7) is true for $n = k$, $k \in \mathbb{N}$. Since $\|y_k - p\| \leq \|x_k - p\|$ and $\|z_k - p\| \leq \|x_k - p\|$ by (3.5) and (3.7), it follows from (3.5) that

$$\begin{aligned} \|x_{k+1} - p\| &\leq \alpha_k \|f(x_k) - p\| + (1-\alpha_k)\|z_k - p\| \\ &= \alpha_k \|f(x_k) - f(x_k) + f(x_k) - p\| + (1-\alpha_k)\|z_k - p\| \\ &\leq \alpha_k \alpha \|x_k - p\| + \alpha_k \|f(p) - p\| + (1-\alpha_k)\|x_k - p\| \\ &\leq (1 - \alpha_k(1-\alpha))\|x_k - p\| + \alpha_k \|f(p) - p\| \leq \mathcal{L}, \end{aligned} \tag{3.9}$$

which prove that (3.8) is true for $n = k+1$. By induction, (3.8) holds for all $n \in \mathbb{N} \cup \{0\}$. Hence, by (3.8), we know that $\{x_n\}$ is bounded and so are $\{x_n\}, \{y_n\}, \{z_n\}, \{Tx_n\}, \{SAy_n\}, \{l_n\}, \{h_n\}, \{W_n h_n\}$, $\{l_n\}$, and $\{Ul_n\}$. This also means that there exists a bounded subset $C_1 \subset C$ such that

$$\{x_n\}, \{y_n\}, \{z_n\}, \{Tx_n\}, \{h_n\}, \{W_n h_n\} \subset C_1. \tag{3.10}$$

Step 2. Prove $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

For each $n \in \mathbb{N} \cup \{0\}$, by Lemma 3.1,

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|P_C(I - \beta_{n+1}T)x_{n+1} - P_C(I - \beta_{n+1}T)x_n\| + \|P_C(I - \beta_{n+1}T)x_n - P_C(I - \beta_n T)x_n\| \\ &\leq \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|Tx_n\|. \end{aligned} \tag{3.11}$$

Similarly,

$$\begin{aligned} \|l_{n+1} - l_n\| \leq{}& \|P_K(I - \beta_{n+1}S)Ay_{n+1} - P_K(I - \beta_{n+1}S)Ay_n\| \\ &+ \|P_K(I - \beta_{n+1}S)Ay_n - P_K(I - \beta_n S)Ay_n\| \\ \leq{}& \|Ay_{n+1} - Ay_n\| + |\beta_{n+1} - \beta_n| \|SAy_n\|, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{3.12}$$

Since $h_n = P_C(y_n + rA^*(Ul_n - Ay_n))$, $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\|h_{n+1} - h_n\|^2 &\le \left\| y_{n+1} + rA^*(Ul_{n+1} - Ay_{n+1}) - (y_n + rA^*(Ul_n - Ay_n)) \right\|^2 \\
&\le \left\| y_{n+1} - y_n \right\|^2 + \left\| rA^*(Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n)) \right\|^2 \\
&\quad + 2r\left\langle y_{n+1} - y_n, A^*(Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n)) \right\rangle \\
&\le \left\| y_{n+1} - y_n \right\|^2 + r^2\|A^*\|^2 \left\| Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\|^2 \\
&\quad + 2r\left\langle Ay_{n+1} - Ay_n + Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n), Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\rangle \\
&\quad - 2r\left\langle Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n), Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\rangle \\
&= \left\| y_{n+1} - y_n \right\|^2 + r^2\|A^*\|^2 \left\| Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\|^2 \\
&\quad + 2r\left\{ \frac{1}{2}\|Ul_{n+1} - Ul_n\|^2 + \frac{1}{2}\left\| Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\| - \frac{1}{2}\left\| Ay_{n+1} - Ay_n \right\|^2 \right\} \\
&\quad - 2r\left\| Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\|^2 \\
&= \left\| y_{n+1} - y_n \right\|^2 - r\left(1 - r\|A^*\|^2\right) \left\| Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\|^2 \\
&\quad + r\left\{ \|Ul_{n+1} - Ul_n\|^2 - \left\| Ay_{n+1} - Ay_n \right\|^2 \right\} \\
&= \left\| y_{n+1} - y_n \right\|^2 - r\left(1 - r\|A^*\|^2\right) \left\| Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\|^2 \\
&\quad + r\left|\beta_{n+1} - \beta_n\right| \left(\|Ul_{n+1} - Ul_n\| + \left\| Ay_{n+1} - Ay_n \right\| \right) \left\| SAy_n \right\| \\
&\le \|x_{n+1} - x_n\|^2 + \left|\beta_{n+1} - \beta_n\right| \left(\|Tx_n\| \|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right| \|Tx_n\|^2 \right) \\
&\quad - r\left(1 - r\|A^*\|^2\right) \left\| Ul_{n+1} - Ay_{n+1} - (Ul_n - Ay_n) \right\|^2 \\
&\quad + r\left|\beta_{n+1} - \beta_n\right| \left(\|Ul_{n+1} - Ul_n\| + \left\| Ay_{n+1} - Ay_n \right\| \right) \left\| SAy_n \right\| \\
&\le \|x_{n+1} - x_n\|^2 + \left|\beta_{n+1} - \beta_n\right| M_1,
\end{aligned} \tag{3.13}$$

where M_1 is a constant such that

$$\left(\|Tx_n\| \|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right| \|Tx_n\|^2 \right) + r\left(\|Ul_{n+1} - Ul_n\| + \left\| Ay_{n+1} - Ay_n \right\| \right) \left\| SAy_n \right\| \le M_1 \tag{3.14}$$

for any $n \in \mathbb{N} \cup \{0\}$. Since $\{h_n\} \subset C_1$, for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \|W_{n+1}h_{n+1} - W_n h_n\| &\leq \|W_{n+1}h_{n+1} - Wh_{n+1}\| + \|Wh_{n+1} - Wh_n\| + \|Wh_n - W_n h_n\| \\ &\leq \sup_{x\in C_1}\|W_{n+1}x - Wx\| + \sup_{x\in C_1}\|Wx - W_n x\| + \|h_{n+1} - h_n\|, \end{aligned} \tag{3.15}$$

$$\|W_{n+1}h_{n+1} - W_n h_n\|^2 \leq \|h_{n+1} - h_n\|^2 + \omega_n, \tag{3.16}$$

where

$$\begin{aligned} \omega_n = &\left(\sup_{x\in C_1}\|W_{n+1}x - Wx\| + \sup_{x\in C_1}\|Wx - W_n x\|\right) \\ &\times \left(\sup_{x\in C_1}\|W_{n+1}x - Wx\| + \sup_{x\in C_1}\|Wx - W_n x\| + 2\|h_{n+1} - h_n\|\right). \end{aligned} \tag{3.17}$$

So, we have

$$\begin{aligned} \|z_{n+1} - z_n\|^2 &\leq (1-\theta)\|x_{n+1} - x_n\|^2 + \theta\|W_{n+1}h_{n+1} - W_n h_n\|^2 \\ &\leq (1-\theta)\|x_{n+1} - x_n\|^2 + \theta\|h_{n+1} - h_n\|^2 + \omega_n \\ &\leq (1-\theta)\|x_{n+1} - x_n\|^2 + \theta\|x_{n+1} - x_n\|^2 + \left|\beta_{n+1} - \beta_n\right| M_1 + \omega_n \\ &= \|x_{n+1} - x_n\|^2 + \left|\beta_{n+1} - \beta_n\right| M_1 + \omega_n, \end{aligned} \tag{3.18}$$

for any $n \in \mathbb{N} \cup \{0\}$.

Choose a sequence $\{\overline{y}_n\}$ such that $x_{n+1} = \gamma_n \overline{y}_n + (1-\gamma_n)x_n$, where $\gamma_n = 1-(1-\theta)(1-\alpha_n)$, then we have

$$\overline{y}_n = \frac{\alpha_n f(x_n) + (1-\alpha_n)\theta W_n h_n}{\gamma_n}, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.19}$$

It follows that

$$\begin{aligned} \left\|\overline{y}_{n+1} - \overline{y}_n\right\| \leq &\frac{\alpha_n}{\gamma_n}\|f(x_n)\| + \frac{\alpha_{n+1}}{\gamma_{n+1}}\|f(x_{n+1})\| \\ &+ \frac{(1-\alpha_{n+1})\theta}{\gamma_{n+1}}\|W_{n+1}h_{n+1} - W_n h_n\| + \left|\frac{(1-\alpha_{n+1})\theta}{\gamma_{n+1}} - \frac{(1-\alpha_n)\theta}{\gamma_n}\right|\|W_n h_n\| \\ \leq &(\alpha_n + \alpha_{n+1})M_3 + \frac{(1-\alpha_{n+1})\theta}{\gamma_{n+1}}\|W_{n+1}h_{n+1} - W_n h_n\| + \left|\frac{\alpha_n - \alpha_{n+1}}{\gamma_{n+1}\gamma_n}\right|\theta\|W_n h_n\| \\ \leq &2(\alpha_n + \alpha_{n+1})M_3 + \frac{(1-\alpha_{n+1})\theta}{\gamma_{n+1}}\|W_{n+1}h_{n+1} - W_n h_n\|, \end{aligned} \tag{3.20}$$

where M_3 is a constant such that $\sup_{n\in\mathbb{N}\cup\{0\}}\{\|f(x_n)/\gamma_n\|, \|W_n h_n\|\} \le M_3$. From (3.20), (3.16), and (3.13) we have

$$\begin{aligned}
\|\overline{y}_{n+1}-\overline{y}_n\|^2 &\le 4(\alpha_{n+1}+\alpha_n)^2M_3^2+\frac{(1-\alpha_{n+1})^2\theta^2}{\gamma_{n+1}^2}\|W_{n+1}h_{n+1}-W_nh_n\|^2\\
&\quad+4(\alpha_n+\alpha_{n+1})M_3\frac{(1-\alpha_{n+1})\theta}{\gamma_{n+1}}\|W_{n+1}h_{n+1}-W_nh_n\|\\
&\le 4(\alpha_{n+1}+\alpha_n)^2M_3^2+\frac{(1-\alpha_{n+1})^2\theta^2}{\gamma_{n+1}^2}\|h_{n+1}-h_n\|^2+\frac{(1-\alpha_{n+1})^2\theta^2}{\gamma_{n+1}^2}\omega_n\\
&\quad+4(\alpha_n+\alpha_{n+1})M_3\frac{(1-\alpha_{n+1})\theta}{\gamma_{n+1}}\|W_{n+1}h_{n+1}-W_nh_n\|\\
&\le 4(\alpha_{n+1}+\alpha_n)^2M_3^2+\frac{(1-\alpha_{n+1})^2\theta^2}{\gamma_{n+1}^2}\|x_{n+1}-x_n\|^2+\frac{(1-\alpha_{n+1})^2\theta^2}{\gamma_{n+1}^2}|\beta_{n+1}-\beta_n|M_1\\
&\quad+\frac{(1-\alpha_{n+1})^2\theta^2}{\gamma_{n+1}^2}\omega_n+4(\alpha_n+\alpha_{n+1})M_3\frac{(1-\alpha_{n+1})\theta}{\gamma_{n+1}}\|W_{n+1}h_{n+1}-W_nh_n\|.
\end{aligned} \tag{3.21}$$

Applying the condition (C2), it follows from (3.21) that

$$\limsup_{n\to\infty}\left\{\|\overline{y}_{n+1}-\overline{y}_n\|^2-\|x_{n+1}-x_n\|^2\right\}=0, \tag{3.22}$$

which implies

$$\limsup_{n\to\infty}\{\|\overline{y}_{n+1}-\overline{y}_n\|-\|x_{n+1}-x_n\|\}=0. \tag{3.23}$$

Applying Lemma 2.7, we obtain $\lim_{n\to\infty}\|\overline{y}_n-x_n\| \to 0$ which implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\gamma_n\|\overline{y}_n-x_n\|=0. \tag{3.24}$$

Step 3. Prove $\lim_{n\to\infty}\|Tx_n-Tp\|=\lim_{n\to\infty}\|SAy_n-SAp\|=0$.
For any $n\in\mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|P_C(I-\beta_nT)x_n-P_C(I-\beta_nT)p\|^2\\
&\le \|(I-\beta_nT)x_n-(I-\beta_nT)p\|^2\\
&= \|x_n-p\|^2-2\beta_n\langle x_n-p, Tx_n-Tp\rangle+\beta_n^2\|Tx_n-Tp\|^2\\
&\le \|x_n-p\|^2-\beta_n(2u-\beta_n)\|Tx_n-Tp\|^2.
\end{aligned} \tag{3.25}$$

Similarly,

$$\begin{aligned}\|Ul_n - Ap\|^2 &\leq \|l_n - Ap\|^2 = \|P_K(I - \beta_n S)Ay_n - Ap\|^2 \\ &\leq \|Ay_n - Ap\|^2 - \beta_n(2u - \beta_n)\|SAy_n - SAp\|^2.\end{aligned} \tag{3.26}$$

From (3.5) again, we have

$$\|x_{n+1} - p\|^2 \leq (\alpha_n\|f(x_n) - p\| + \|z_n - p\|)^2 \leq \alpha_n M_4 + \|z_n - p\|^2, \tag{3.27}$$

where M_4 is a constant such that $\sup_{n\in\mathbb{N}\cup\{0\}}\{\alpha_n\|f(x_n) - p\|^2 + 2\|f(x_n) - p\|\|z_n - p\|\} \leq M_4$. It follows that

$$\begin{aligned}0 < \theta\beta_n(2u - \beta_n)\|Tx_n - Tp\|^2 &\leq \theta\|x_n - p\|^2 - \theta\|y_n - p\|^2 \quad (\text{by } (3.24)) \\ &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \quad (\text{by } (3.7)) \\ &\leq \|x_n - p\|^2 + \alpha_n M_4 - \|x_{n+1} - p\|^2 \quad (\text{by } (3.26)) \\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n M_4 \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.28}$$

which yields that $\lim_{n\to\infty}\|Tx_n - Tp\| = 0$ (by the condition $0 < a \leq \beta_n \leq b < 2u$).

For any $n \in \mathbb{N} \cup \{0\}$, by (3.6), (3.7), and (3.27), we have

$$\begin{aligned}\theta r\left(1 - r\|A^*\|^2\right)\|Ul_n - Ay_n\|^2 &\leq \theta\|y_n - p\|^2 - \theta\|W_n h_n - p\|^2 \\ &\leq \theta\|x_n - p\|^2 - \theta\|z_n - p\|^2 \\ &\leq \theta\|x_n - p\|^2 + \alpha_n M_4 - \|x_{n+1} - p\|^2 \\ &= (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n M_4.\end{aligned} \tag{3.29}$$

So,

$$\lim_{n\to\infty}\|Ul_n - Ay_n\| = 0. \tag{3.30}$$

From (3.26) and (3.30) again, we have

$$\begin{aligned}0 < \beta_n(2u - \beta_n)\|SAy_n - SAp\|^2 &\leq \|Ay_n - Ap\|^2 - \|Ul_n - Ap\|^2 \\ &= (\|Ay_n - Ap\| + \|Ul_n - Ap\|)(\|Ay_n - Ap\| - \|Ul_n - Ap\|) \\ &\leq (\|Ay_n - Ap\| + \|Ap\|)\|Ul_n - Ay_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.31}$$

which implies that $\lim_{n\to\infty}\|SAy_n - SAp\| = 0$ (by the condition $0 < a \le \beta_n \le b < 2u$).

On the other hand, since

$$\begin{aligned}
\|l_n - Ap\|^2 &= \|P_K(I-\beta_n S)Ay_n - P_K(I-\beta_n S)Ap\|^2 \\
&\le \langle (I-\beta_n S)Ay_n - (I-\beta_n S)Ap, I_n - Ap\rangle \\
&= \frac{1}{2}\Big\{ \|(I-\beta_n S)Ay_n - (I-\beta_n S)Ap\|^2 \\
&\quad + \|l_n - Ap\|^2 - \|l_n - Ay_n - \beta_n(SAy_n - SAp)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|Ay_n - Ap\|^2 + \|l_n - Ap\|^2 - \|l_n - Ay_n\|^2 \\
&\quad + 2\beta_n \langle l_n - Ay_n, SAy_n - SAp\rangle - \beta_n^2 \|SAy_n - SAp\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|Ay_n - Ap\|^2 + \|l_n - Ap\|^2 - \|l_n - Ay_n\|^2 + 2\beta_n \|l_n - Ay_n\| \|SAy_n - SAp\| \Big\},
\end{aligned} \tag{3.32}$$

we get

$$\|l_n - Ap\|^2 \le \|Ay_n - Ap\|^2 - \|l_n - Ay_n\|^2 + 2\beta_n \|l_n - Ay_n\| \|SAy_n - SAp\|, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.33}$$

By (3.6) and (3.33), we have

$$\begin{aligned}
\|W_n h_n - p\|^2 &\le \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\
&\quad + 2r\left\{ \frac{1}{2}\|Ul_n - Ap\|^2 + \frac{1}{2}\|Ul_n - Ay_n\|^2 - \|Ay_n - Ap\|^2 - \|Ul_n - Ay_n\|^2 \right\} \\
&\le \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\
&\quad + 2r\left\{ \frac{1}{2}\|l_n - Ap\|^2 + \frac{1}{2}\|Ul_n - Ay_n\|^2 - \|Ay_n - Ap\|^2 - \|Ul_n - Ay_n\|^2 \right\} \\
&\le \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\
&\quad + 2r\left\{ -\frac{1}{2}\|l_n - Ay_n\|^2 + \beta_n \|l_n - Ay_n\| \|SAy_n - SAp\| - \frac{1}{2}\|Ul_n - Ay_n\|^2 \right\} \\
&\le \|y_n - p\|^2 + r^2\|A^*\|^2 \|Ul_n - Ay_n\|^2 - r\|Ul_n - Ay_n\|^2 - r\|l_n - Ay_n\|^2 \\
&\quad + 2r\beta_n \|l_n - Ay_n\| \|SAy_n - SAp\| \\
&= \|y_n - p\|^2 + r\left(1 - r\|A^*\|^2\right)\|Ul_n - Ay_n\|^2 - r\|l_n - Ay_n\|^2 \\
&\quad + 2r\beta_n \|l_n - Ay_n\| \|SAy_n - SAp\| \\
&\le \|y_n - p\|^2 - r\|l_n - Ay_n\|^2 + 2r\beta_n \|l_n - Ay_n\| \|SAy_n - SAp\|,
\end{aligned} \tag{3.34}$$

Using (3.7), (3.27), and (3.34), we obtain

$$\begin{aligned}\theta r\|l_n - Ay_n\|^2 &\le \theta\|y_n - p\|^2 - \theta\|W_n h_n - p\|^2 + 2r\theta\beta_n\|l_n - Ay_n\|\|SAy_n - SAp\| \\ &\le \theta\|x_n - p\|^2 - \theta\|z_n - p\|^2 + 2r\theta\beta_n\|l_n - Ay_n\|\|SAy_n - SAp\| \\ &\le \theta\|x_n - p\|^2 + \alpha_n M_4 - \|x_{n+1} - p\|^2 + 2r\theta\beta_n\|l_n - Ay_n\|\|SAy_n - SAp\| \\ &= (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n M_4 + 2r\theta\beta_n\|l_n - Ay_n\|\|SAy_n - SAp\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.35}$$

which implies

$$\lim_{n\to\infty}\|l_n - Ay_n\| = 0. \tag{3.36}$$

According to (3.30) and (3.36), we derive that

$$\lim_{n\to\infty}\|Ul_n - l_n\| = 0. \tag{3.37}$$

Step 4. Prove $\lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|h_n - x_n\| = 0$. Since

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \qquad \lim_{n\to\infty}\|x_{n+1} - z_n\| = \lim_{n\to\infty}\alpha_n\|f(x_n) - z_n\| = 0, \tag{3.38}$$

we have $\lim_{n\to\infty}\|x_n - z_n\| = 0$. For any $n \in \mathbb{N} \cup \{0\}$, since

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(I - \beta_n T)x_n - P_C(I - \beta_n T)p\|^2 \\ &\le \langle (I - \beta_n T)x_n - (I - \beta_n T)p, y_n - p\rangle \\ &= \frac{1}{2}\left\{\|(I - \beta_n T)x_n - (I - \beta_n T)p\|^2 + \|y_n - p\|^2 - \|x_n - y_n - \beta_n(Tx_n - Tp)\|^2\right\} \\ &\le \frac{1}{2}\left\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 + 2\beta_n\langle x_n - y_n, Tx_n - Tp\rangle - \beta_n^2\|Tx_n - Tp\|^2\right\} \\ &\le \frac{1}{2}\left\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 + 2\beta_n\|x_n - y_n\|\|Tx_n - Tp\|\right\},\end{aligned} \tag{3.39}$$

we get

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\beta_n\|x_n - y_n\|\|Tx_n - Tp\|. \tag{3.40}$$

It follows from (3.27), (3.7), and (3.40) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n M_4 + \|z_n - p\|^2 \\ &\le \alpha_n M_4 + (1-\theta)\|x_n - p\|^2 + \theta\|y_n - p\|^2 \\ &\le \alpha_n M_4 + \|x_n - p\|^2 - \theta\|x_n - y_n\|^2 + 2\theta\beta_n\|x_n - y_n\|\|Tx_n - Tp\|,\end{aligned} \tag{3.41}$$

which yields that

$$\begin{aligned}\theta\|x_n - y_n\|^2 &\le \alpha_n M_4 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\beta_n\|x_n - y_n\|\|Tx_n - Tp\| \\ &\le \alpha_n M_4 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + 2\beta_n\|x_n - y_n\|\|Tx_n - Tp\|,\end{aligned} \tag{3.42}$$

By Steps 1–3 and $\lim_{n\to\infty}\alpha_n = 0$, it follows from (3.42) that $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Since $\|h_n - y_n\| \le \|\gamma A^*\|\|l_n - Ay_n\|$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\to\infty}\|h_n - y_n\| = 0$. Using it with $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we get $\lim_{n\to\infty}\|x_n - h_n\| = 0$.

Step 5. Prove $\lim_{n\to\infty}\|W_n x_n - x_n\| = 0$ and $\lim_{n\to\infty}\|Wx_n - x_n\| = 0$.

Indeed, since

$$\|W_n h_n - x_n\| = \frac{1}{\theta}\|z_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.43}$$

we have

$$\|W_n x_n - x_n\| \le \|W_n x_n - W_n h_n\| + \|W_n h_n - x_n\| \le \|x_n - h_n\| + \|W_n h_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.44}$$

By Theorem 2.3, $\lim_{n\to\infty}\|Wx_n - W_n x_n\| = 0$. Since

$$\|Wx_n - x_n\| \le \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\| \quad \forall n, \tag{3.45}$$

we obtain $\lim_{n\to\infty}\|Wx_n - x_n\| = 0$.

Step 6. There exists a unique $q \in \Omega \subset H$ such that $P_\Omega f(q) = q$.

Indeed, for any $x, y \in H$,

$$\|P_\Omega f(x) - P_\Omega f(y)\| \le \|f(x) - f(y)\| \le \alpha\|x - y\|. \tag{3.46}$$

Since $\alpha \in [0,1)$, $P_\Omega f$ is a contraction on H. Applying Banach contraction principle, there exists a unique $q \in H$ such that $q = P_\Omega f(q) \in \Omega$.

Step 7. Prove $\limsup_{n\to\infty}\langle fq - q, x_n - q\rangle \le 0$.

For this purpose, we may choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle fq - q, x_n - q\rangle = \lim_{i\to\infty}\langle fq - q, x_{n_i} - q\rangle. \tag{3.47}$$

Since $\{x_{n_i}\}$ is a bounded sequence, there exists a subsequence of $\{x_{n_i}\}$, which is still denoted by $\{x_{n_i}\}$, such that $x_{n_i} \rightharpoonup x^* \in H$. Therefore, we have

$$\limsup_{i\to\infty}\langle fq-q, x_{n_i}-q\rangle = \langle fq-q, x^*-q\rangle. \tag{3.48}$$

Next we prove $x^* \in \Omega$.

(a) $Wx^* = x^*$. In fact, if $Wx^* \neq x^*$, then we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i}-x^*\| &< \liminf_{i\to\infty}\|x_{n_i}-Wx^*\| \\ &\le \liminf_{i\to\infty}(\|x_{n_i}-Wx_{n_i}\| + \|Wx_{n_i}-Wx^*\|) \\ &\le \liminf_{i\to\infty}(\|x_{n_i}-Wx_{n_i}\| + \|x_{n_i}-x^*\|) \\ &= \liminf_{i\to\infty}\|x_{n_i}-x^*\| \quad (\text{by Step 5 and Lemma 2.9}).\end{aligned} \tag{3.49}$$

This is a contradiction. Hence, $Wx^* = x^*$, which implies that $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ by Theorem 2.2.

(b) Prove $x^* \in \mathrm{VI}(C,T)$. Since $x_{n_i} \rightharpoonup x^*$ and $\lim_{i\to\infty}\|x_{n_i}-y_{n_i}\| = 0$, we have $y_{n_i} \rightharpoonup x^*$. Let

$$T_1x = \begin{cases} Tx + N_Cx, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{3.50}$$

Since T is u-inversely monotone, T is monotone and hence T_1 is a maximal monotone mapping. For any given $(x,z) \in G(T_1)$, since $z - Tx \in N_Cx$ and $y_n \in C$, by the definition of N_C, we have

$$\langle x - y_n, z - Tx\rangle \ge 0 \quad \forall n. \tag{3.51}$$

On the other hand, since $y_n = P_C(I - \beta_n T)x_n$, we have

$$\langle x - y_n, y_n - (x_n - \beta_n T x_n)\rangle \ge 0. \tag{3.52}$$

In particular,

$$\left\langle x - y_n, \frac{y_n - x_n}{\beta_n} + Tx_n\right\rangle \ge 0 \quad \forall n. \tag{3.53}$$

From (3.51) and (3.53) we have

$$\begin{aligned}\langle x - y_{n_i}, z\rangle &\geq \langle x - y_{n_i}, Tx\rangle \\ &\geq \langle x - y_{n_i}, Tx\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\beta_{n_i}} + Tx_{n_i}\right\rangle \\ &= \langle x - y_{n_i}, Tx - Ty_{n_i}\rangle + \langle x - y_{n_i}, Ty_{n_i} - Tx_{n_i}\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\beta_{n_i}}\right\rangle \\ &\geq \langle x - y_{n_i}, Ty_{n_i} - Tx_{n_i}\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\beta_{n_i}}\right\rangle,\end{aligned} \tag{3.54}$$

which implies

$$\langle x - x^*, z\rangle = \lim_{i\to\infty}\langle x - y_{n_i}, z\rangle \geq 0. \tag{3.55}$$

This shows $\mathbf{0} \in T_1 x^*$, that is, $x^* \in \mathrm{VI}(C, T)$.

(c) Prove $Ax^* \in \mathrm{VI}(K, S)$ and $Ax^* \in F(U)$. Let

$$T_2 x = \begin{cases} Sx + N_K x, & x \in K, \\ \emptyset, & x \notin K. \end{cases} \tag{3.56}$$

Then T_2 is a maximal monotone mapping. For any given $(x, z) \in G(T_2)$, since $z - Sx \in N_K x$ and $l_n \in K$, by the definition of N_K we have

$$\langle x - l_n, z - Sx\rangle \geq 0 \quad \forall n. \tag{3.57}$$

On the other hand, since $l_n = P_K(I - \beta_n S)Ay_n$, we have

$$\langle x - l_n, l_n - (Ay_n - \beta_n SAy_n)\rangle \geq 0, \tag{3.58}$$

and hence

$$\left\langle x - l_n, \frac{l_n - Ay_n}{\beta_n} + SAy_n\right\rangle \geq 0 \quad \forall n. \tag{3.59}$$

Since $x_{n_i} \rightharpoonup x^*$ and $\lim_{n\to\infty}\|x_{n_i} - y_{n_i}\| = 0$ and $\lim_{n\to\infty}\|l_{n_i} - Ay_{n_i}\| = 0$, we have $l_{n_i} \rightharpoonup Ax^*$. From (3.57) and (3.59) we have

$$\begin{aligned}\langle x - l_{n_i}, z\rangle &\geq \langle x - l_{n_i}, Sx\rangle \\ &\geq \langle x - l_{n_i}, Sx - Sl_{n_i}\rangle + \langle x - l_{n_i}, Sl_{n_i} - SAy_{n_i}\rangle - \left\langle x - l_{n_i}, \frac{l_{n_i} - Ay_{n_i}}{\beta_{n_i}}\right\rangle \\ &\geq \langle x - l_{n_i}, l_{n_i} - Ay_{n_i}\rangle - \left\langle x - l_{n_i}, \frac{l_{n_i} - Ay_{n_i}}{\beta_{n_i}}\right\rangle.\end{aligned} \tag{3.60}$$

So

$$\langle x - Ax^*, z\rangle = \lim_{i\to\infty}\langle x - l_{n_i}, z\rangle \geq 0. \tag{3.61}$$

This shows $\mathbf{0} \in T_2Ax^*$, that is, $Ax^* \in \mathrm{VI}(K, S)$. In addition, by (3.36), (3.37), and Opial's condition, we can prove easily $Ax^* \in F(U)$.

By (a), (b), and (c), $x^* \in \Omega$ is proved. Hence

$$\limsup_{n\to\infty}\langle fq - q, x_n - q\rangle = \langle fq - q, x^* - q\rangle \leq 0. \tag{3.62}$$

Step 8. Prove $\{x_n\}$ converges strongly to $q = P_\Omega f(q) \in \Omega$.

In fact, by Step 7 we have

$$\limsup_{n\to\infty}\langle fq - q, x_n - q\rangle \leq 0. \tag{3.63}$$

It follows from (3.5) and Lemma 2.8 that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)z_n - q\|^2 \\
&\leq (1-\alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\langle f(x_n) - q, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle f(x_n) - f(q) + f(q) - q, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\alpha\|x_n - q\|\|x_{n+1} - q\| + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n(2-2\alpha))\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + 2\alpha_n\alpha\|x_n - q\|\|x_{n+1} - x_n\| \\
&\quad + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \quad \forall n \in \mathbb{N}\cup\{0\}.
\end{aligned} \tag{3.64}$$

Let $\lambda_n = \alpha_n(2-2\alpha)$, $n \in \mathbb{N}\cup\{0\}$. Then, for any $n \in \mathbb{N}\cup\{0\}$, we obtain

$$\begin{aligned}
\alpha_n^2\|x_n - q\|^2 &= \frac{\lambda_n\alpha_n}{2-2\alpha}\|x_n - q\|^2, \\
2\alpha_n\alpha\|x_n - q\|\|x_{n+1} - x_n\| &= \frac{\lambda_n\alpha}{1-\alpha}\|x_n - q\|\|x_{n+1} - x_n\|, \\
2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle &= \frac{\lambda_n}{1-\alpha}\langle f(q) - q, x_{n+1} - q\rangle.
\end{aligned} \tag{3.65}$$

Set

$$b_n = \frac{\alpha_n}{2-2\alpha}\|x_n - q\|^2 + \frac{\alpha}{1-\alpha}\|x_n - q\|\|x_{n+1} - x_n\| + \frac{1}{1-\alpha}\langle f(q) - q, x_{n+1} - q\rangle. \tag{3.66}$$

The condition (C1) and the boundedness of $\{x_n\}$ ensure $\lim_{n\to\infty}(\alpha_n/(2-2\alpha))\|x_n-q\|^2=0$, Step 2 ensure $\lim_{n\to\infty}(1/(1-\alpha))\alpha\|x_n-q\|\|x_{n+1}-x_n\|=0$, and Step 7 ensure $\lim\sup_{n\to\infty}(1/(1-\alpha))\langle f(q)-q, x_{n+1}-q\rangle\le 0$, so $\lim\sup_{n\to\infty} b_n\le 0$. Applying Lemma 2.6 and the inequality

$$\|x_{n+1}-q\|^2\le(1-\lambda_n)\|x_n-q\|^2+\lambda_n b_n, \tag{3.67}$$

we obtain that $\lim_{n\to\infty}\|x_n-q\|=0$ which means that the sequence $\{x_n\}$ strongly converge to q. This completes the proof of the Theorem 3.5. □

The following convergence theorems can be established by applying Theorem 3.5 with $U=I$.

Corollary 3.6. *Let H_1, H_2 be two real Hilbert spaces and $C\subset H_1$, $K\subset H_2$ two nonempty closed convex sets. Let $T:C\to H_1$ and $S:K\to H_2$ be u-inversely monotone. $A:H_1\to H_2$ is a bounded linear operator with adjoint operator A^*. Let $f:C\to C$ be a contraction with contraction constant α. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of infinitely nonexpansive mappings of C into itself such that $\Omega=\{p\in\bigcap_{i=1}^{\infty}F(T_i)\bigcap \mathrm{VI}(C,T) : Ap\in \mathrm{VI}(K,S)\}\ne\emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that $0<\lambda_i\le\xi<1$, for every $i\in\mathbb{N}$. For each $n\in\mathbb{N}$, let W_n be the W-mapping of C into itself generated by $T_n, T_{n-1},\dots,T_1$ and $\lambda_n,\lambda_{n-1},\dots,\lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered}
x_0=x\in C \quad \text{chosen arbitrarily},\\
y_n=P_C(I-\beta_n T)x_n,\\
l_n=P_K(I-\beta_n S)Ay_n,\\
z_n=(1-\theta)x_n+\theta W_n P_C(y_n+rA^*(l_n-Ay_n)),\\
x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)z_n, \quad \forall n\in\mathbb{N}\cup\{0\},
\end{gathered} \tag{3.68}$$

where $r\in(0,1/\|A\|^2)$ and $\theta\in(0,1)$ are two constants and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in (0,1). If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ further satisfy the following conditions:

(C_1) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(C_2) $\{\beta_n\}\subset[a,b]$ and $\lim_{n\to\infty}|\beta_{n+1}-\beta_n|=0$, where $0<a$, $b<2u$;

then the following statements hold:

(a) there exists a unique $q\in\Omega$ such that $P_\Omega f(q)=q$;

(b) $\{x_n\}$ converges strongly to q.

If $T_i=I$ for all $i\in\mathbb{N}$ in Theorem 3.5, then we have the following result.

Corollary 3.7. *Let H_1, H_2 be two real Hilbert spaces and $C\subset H_1$, $K\subset H_2$ two nonempty closed convex sets. Let $T:C\to H_1$ and $S:K\to H_2$ be u-inversely monotone. $A:H_1\to H_2$ is a bounded linear operator with adjoint operator A^*. Let $f:C\to C$ be a contraction with contraction constant α. Let U be a nonexpansive mapping of K into itself such that $\Omega=\{p\in\mathrm{VI}(C,T) : Ap\in F(U)\bigcap\mathrm{VI}(K,S)\}\ne\emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that*

$0 < \lambda_i \leq \xi < 1$, *for every* $i \in \mathbb{N}$. *For each* $n \in \mathbb{N}$, *let* W_n *be the* W*-mapping of* C *into itself generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily}, \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_K(I - \beta_n S)Ay_n, \\ z_n &= (1-\theta)x_n + \theta P_C(y_n + rA^*(Ul_n - Ay_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.69}$$

where $r \in (0, 1/\|A\|^2)$ *and* $\theta \in (0,1)$ *are two constants and* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are two sequences in (0,1). If* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *further satisfy the following conditions:*

(C_1*)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_2*)* $\{\beta_n\} \subset [a, b]$ *and* $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$, *where* $0 < a$, $b < 2u$;

then the following statements hold:

(a) *there exists a unique* $q \in \Omega$ *such that* $P_\Omega f(q) = q$;

(b) $\{x_n\}$ *converges strongly to* q.

In Theorem 3.5, if $T_i = I$ for all $i \in \mathbb{N}$ and $U = I$, then we obtain Corollary 3.8.

Corollary 3.8. *Let* H_1, H_2 *be two real Hilbert spaces and* $C \subset H_1$, $K \subset H_2$ *two nonempty closed convex sets. Let* $T : C \to H_1$ *and* $S : K \to H_2$ *be* u*-inversely monotone.* $A : H_1 \to H_2$ *is a bounded linear operator with adjoint operator* A^*. *Let* $f : C \to C$ *be a contraction with contraction constant* α. *Let* $\{T_i\}_{i\in\mathbb{N}}$ *be a family of infinitely nonexpansive mappings of* C *into itself and* U *a nonexpansive mapping of* K *into itself such that* $\Omega = \{p \in \mathrm{VI}(C,T) : Ap \in \mathrm{VI}(K,S)\} \neq \emptyset$. *Let* ξ *be a real number and* $\{\lambda_i\}_{i\in\mathbb{N}}$ *be a sequence of real numbers such that* $0 < \lambda_i \leq \xi < 1$, *for every* $i \in \mathbb{N}$. *For each* $n \in \mathbb{N}$, *let* W_n *be the* W*-mapping of* C *into itself generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily}, \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_K(I - \beta_n S)Ay_n, \\ z_n &= (1-\theta)x_n + \theta P_C(y_n + rA^*(l_n - Ay_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.70}$$

where $r \in (0, 1/\|A\|^2)$ *and* $\theta \in (0,1)$ *are two constants and* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are two sequences in* $(0,1)$. *If* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *further satisfy the following conditions:*

(C_1*)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_2*)* $\{\beta_n\} \subset [a, b]$ *and* $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$, *where* $0 < a$, $b < 2u$;

then the following statements hold:

(a) *there exists a unique* $q \in \Omega$ *such that* $P_{\Omega} f(q) = q$;

(b) $\{x_n\}$ *converges strongly to* q.

In Theorem 3.5, if $H_1 = H_2$, $C = K$ and $A = A^* = I$, then we have Corollary 3.9.

Corollary 3.9. *Let* H *be a real Hilbert space and* $C \subset H$ *a nonempty closed convex set. Let* $T : C \to H$ *and* $S : C \to H$ *be* u*-inversely monotone. Let* $f : C \to C$ *be a contraction with contraction constant* α. *Let* $U : C \to C$ *be a nonexpansive mapping and* $\{T_i\}_{i\in\mathbb{N}} : C \to C$ *a family of infinitely nonexpansive mappings such that* $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \bigcap F(U) \bigcap \mathrm{VI}(C,T) \bigcap \mathrm{VI}(K,S) \neq \emptyset$. *Let* ξ *be a real number and* $\{\lambda_i\}_{i\in\mathbb{N}}$ *be a sequence of real numbers such that* $0 < \lambda_i \leq \xi < 1$, *for every* $i \in \mathbb{N}$. *For each* $n \in \mathbb{N}$, *let* W_n *be the* W*-mapping of* C *into itself generated by* $T_n, T_{n-1}, \dots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$
\begin{gathered}
x_0 = x \in C \quad \text{chosen arbitrarily}, \\
y_n = P_C(I - \beta_n T)x_n, \\
l_n = P_C(I - \beta_n S)y_n, \\
z_n = (1-\theta)x_n + \theta W_n(y_n + r(Ul_n - y_n)), \\
x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\},
\end{gathered}
\tag{3.71}
$$

where $r \in (0,1)$ *and* $\theta \in (0,1)$ *are two constants and* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are two sequences in* $(0,1)$. *If* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *further satisfy the following conditions:*

(C_1*)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_2*)* $\{\beta_n\} \subset [a,b]$ *and* $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$, *where* $0 < a$, $b < 2u$;

then the following statements hold:

(a) *there exists a unique* $q \in \Omega$ *such that* $P_{\Omega} f(q) = q$;

(b) $\{x_n\}$ *converges strongly to* q.

In Theorem 3.5, if $H_1 = H_2$, $C = K$ $A = A^* = I$, and $U = I$, then we get the following result.

Corollary 3.10. *Let* H *be a real Hilbert space and* $C \subset H$ *a nonempty closed convex set. Let* $T : C \to H$ *and* $S : C \to H$ *be* u*-inversely monotone. Let* $f : C \to C$ *be a contraction with contraction constant* α. *Let* $\{T_i\}_{i\in\mathbb{N}} : C \to C$ *be a family of infinitely nonexpansive mappings such that* $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \mathrm{VI}(C,T) \bigcap \mathrm{VI}(K,S) \neq \emptyset$. *Let* ξ *be a real number and* $\{\lambda_i\}_{i\in\mathbb{N}}$ *be a sequence of real numbers such that* $0 < \lambda_i \leq \xi < 1$, *for every* $i \in \mathbb{N}$. *For each* $n \in \mathbb{N}$, *let* W_n *be the* W*-mapping of* C *into itself generated by* $T_n, T_{n-1}, \dots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$
\begin{gathered}
x_0 = x \in C \quad \text{chosen arbitrarily}, \\
y_n = P_C(I - \beta_n T)x_n, \\
l_n = P_C(I - \beta_n S)y_n, \\
z_n = (1-\theta)x_n + \theta W_n(y_n + r(l_n - y_n)), \\
x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\},
\end{gathered}
\tag{3.72}
$$

where $r \in (0,1)$ *and* $\theta \in (0,1)$ *are two constants and* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are two sequences in* $(0,1)$. *If* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *further satisfy the following conditions:*

(C_1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_2) $\{\beta_n\} \subset [a,b]$ *and* $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = 0$, *where* $0 < a$, $b < 2u$;

then the following statements hold:

(a) *there exists a unique* $q \in \Omega$ *such that* $P_\Omega f(q) = q$;

(b) $\{x_n\}$ *converges strongly to* q.

Finally, if we let $H_1 = H_2$, $C = K$ $A = A^* = I$, $T_i = I$, for all $i \in \mathbb{N}$ and $U = I$ in Theorem 3.5, then the following result can be established.

Corollary 3.11. *Let* H *be a real Hilbert space and* $C \subset H$ *a nonempty closed convex set. Let* $T : C \to H$ *and* $S : C \to H$ *be* u*-inversely monotone. Let* $f : C \to C$ *be a contraction with contraction constant* α. *Suppose that* $\Omega = \mathrm{VI}(C,T) \bigcap \mathrm{VI}(K,S) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 = x \in C &\quad \text{chosen arbitrarily}, \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_C(I - \beta_n S)y_n, \\ z_n &= (1-\theta)x_n + \theta(y_n + r(l_n - y_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.73}$$

where $r \in (0,1)$ *and* $\theta \in (0,1)$ *are two constants and* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are two sequences in* $(0,1)$. *If* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *further satisfy the following conditions:*

(C_1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_2) $\{\beta_n\} \subset [a,b]$ *and* $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = 0$, *where* $0 < a$, $b < 2u$;

then the following statements hold:

(a) *there exists a unique* $q \in \Omega$ *such that* $P_\Omega f(q) = q$;

(b) $\{x_n\}$ *converges strongly to* q.

Remark 3.12. (a) In [11, 16], the authors gave some algorithms for (u,v)-cocoercive and μ-Lipschitz continuous operator and obtain some strongly convergence theorems; see [11, Theorems 2.1 and 2.2] and [16, Corollary 3.3]. However, the (u,v)-cocoercive and μ-Lipschitz continuous operator considered by [11, 16] is actually a strongly monotone and μ-Lipschitz continuous operator. Then, by Remark ST, such operators studied in [11, 16] are u-inverse strongly monotone. Hence our results obtained in this paper conclude some results in [11, 16] as special cases.

(b) Our results are different from the main results in [9–11, 16] and references therein.

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