

*Research Article*

# **Synchronization between Bidirectional Coupled Nonautonomous Delayed Cohen-Grossberg Neural Networks**

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Based on using suitable Lyapunov function and the properties of  $M$ -matrix, sufficient conditions for complete synchronization of bidirectional coupled nonautonomous Cohen-Grossberg neural networks are obtained. The methods for discussing synchronization avoid complicated error system of Cohen-Grossberg neural networks. Two numerical examples are given to show the effectiveness of the proposed synchronization method.

## **1. Introduction**

It is well known that chaos synchronization has gained considerable attention due to the truth that many benefits of chaos synchronization exist in various engineering fields such as secure communication, image processing, and harmless oscillation generation. Since the pioneering works of Pecora and Carroll [1], chaos synchronization has been widely investigated, and many effective research methods such as feedback control, impulsive control, adaptive control, and important results have been presented in [2–10] and references cited therein. Synchronization of coupled chaotic systems also has received considerable attention [11–13].

Since Cohen and Grossberg proposed the Cohen-Grossberg neural networks (CGNNs) in 1983 [14] and soon the networks have been the subject of active research due to their many applications in various engineering and scientific areas such as pattern recognition, associative memory, and combination, and many dynamic analyses of equilibria or periodic solutions of the networks with delays are investigated [15–20]. Unfortunately, so far, only a few works have been done on synchronization of CGNNs, which remains challenging. In 2010, Li et al. investigated the synchronization of discrete-time CGNNs with delays [6] and Chen discussed

the synchronization of impulsive delayed CGNNs under noise perturbation [7], in 2010, Zhu and Cao discussed adaptive synchronization of delayed CGNNs [21] and in 2012, Gan also discussed adaptive synchronization of delayed CGNNs [22], and in 2011, Yu et al. discuss synchronization of delayed CGNNs via periodically intermittent control [23].

Note that the previous methods in [6, 7, 21–23] have obtained error dynamical system first, and then discuss the error system, but, the error dynamical system of Cohen-Grossberg neural network which comparing the Hopfield neural networks becomes very complex because of existence of the amplification functions of the neural network; hence the results above turn to be quite complex, too. Comparing with foregoing works, the objective of this paper is to study the complete synchronization of nonautonomous CGNNs with mixed time delays by using suitable Lyapunov function and the properties of  $M$ -matrix, and our methods can avoid writing explicit complex error system which leads to complicated conditions for synchronization.

The rest of this paper is organized as follows. The model, some definitions, and assumptions are presented in Section 2. The sufficient conditions for complete synchronization of bidirectional coupled CGNNs are obtained in Section 3. Two examples are given in Section 4 to demonstrate the main results.

## 2. Model Description and Preliminaries

Consider the following bidirectional coupled nonautonomous CGNNs with time delays:

$$\begin{aligned} \dot{x}_i(t) = & -a_i(x_i(t)) \left[ b_i(t, x_i(t)) - \sum_{j=1}^n u_{ij}(t) f_j(x_j(t)) - \sum_{j=1}^n v_{ij}(t) f_j(x_j(t - \tau_{ij}(t))) \right. \\ & \left. - \sum_{j=1}^n w_{ij}(t) \int_0^{+\infty} k_{ij}(s) f_j(x_j(t-s)) ds - I_i(t) \right] \\ & + \sum_{j=1}^n \alpha_{ij}(t) [y_j(t) - x_j(t)], \end{aligned} \quad (2.1)$$

$$\begin{aligned} \dot{y}_i(t) = & -a_i(y_i(t)) \left[ b_i(t, y_i(t)) - \sum_{j=1}^n u_{ij}(t) f_j(y_j(t)) - \sum_{j=1}^n v_{ij}(t) f_j(y_j(t - \tau_{ij}(t))) \right. \\ & \left. - \sum_{j=1}^n w_{ij}(t) \int_0^{+\infty} k_{ij}(s) f_j(y_j(t-s)) ds - I_i(t) \right] \\ & + \sum_{j=1}^n \beta_{ij}(t) [x_j(t) - y_j(t)] \end{aligned} \quad (2.2)$$

for  $1 \leq i \leq n$ ,  $t > 0$ .  $x_i(t)$  and  $y_i(t)$  denote the state variable of the  $i$ th neuron in (2.1) and (2.2),  $f_j(\cdot)$  denote the signal functions of the  $j$ th neuron at time  $t$ ;  $I_i(t)$  denote inputs of the  $i$ th neuron at time  $t$ ;  $a_i(\cdot)$  represent amplification functions;  $b_i(t, \cdot)$  are appropriately behaved functions;  $u_{ij}(t)$ ,  $v_{ij}(t)$ , and  $w_{ij}(t)$  are bounded connection weights of the neural networks, respectively;  $\tau_{ij}(t)$  correspond to the finite speed of the axonal signal transmission, there exist

positive  $\tau_{ij}$  and  $\tau_{ij}^+$  such that  $0 \leq \tau_{ij}(t) \leq \tau_{ij}$  and  $\dot{\tau}_{ij}(t) \leq \tau_{ij}^+ < 1$ ;  $k_{ij}(\cdot)$  correspond to the delay kernel functions, coupled matrix are  $(\alpha_{ij}(t))_{n \times n}$  and  $(\beta_{ij}(t))_{n \times n}$  in which  $\alpha_{ii}(t) \geq 0$  and  $\beta_{ii}(t) \geq 0$  and  $\alpha_{ij}(t), \beta_{ij}(t)$  are bounded on  $[0, +\infty)$ .

Throughout this paper, we assume for system (2.1) and (2.2) that

(H<sub>1</sub>) amplification functions  $a_i(\cdot)$  are continuous and there exist constants  $\underline{a}_i, \bar{a}_i$  such that  $0 < \underline{a}_i \leq a_i(\cdot) \leq \bar{a}_i$  for  $1 \leq i \leq n$ ;

(H<sub>2</sub>) there exist positive continuous and bounded functions  $b_i(t)$  such that

$$\frac{b_i(t, x) - b_i(t, y)}{x - y} \geq b_i(t) > 0 \tag{2.3}$$

for all  $x \neq y, 1 \leq i \leq n$ ;

(H<sub>3</sub>) for activation functions  $f_j(\cdot)$ , there exist positive constants  $L_j$  such that

$$L_j = \sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right| \tag{2.4}$$

for all  $x \neq y, 1 \leq j \leq n$ ;

(H<sub>4</sub>) the kernel functions  $k_{ij}(s)$  are nonnegative continuous function on  $[0, +\infty)$  and satisfy

$$\int_0^{+\infty} s e^{\lambda s} k_{ij}(s) ds < +\infty, \tag{2.5}$$

$$K_{ij}(\lambda) = \int_0^{+\infty} e^{\lambda s} k_{ij}(s) ds$$

are differentiable functions for  $\lambda \in [0, r_{ij}), 0 < r_{ij} < +\infty, K_{ij}(0) = 1$  and  $\lim_{\lambda \rightarrow r_{ij}^-} K_{ij}(\lambda) = +\infty$ .

*Remark 2.1.* A typical example of kernel function is given by  $k_{ij}(s) = (s^r / r!) r_{ij}^{r+1} e^{-r_{ij}s}$  for  $s \in [0, +\infty)$ , where  $r_{ij} \in (0, +\infty), r \in \{0, 1, \dots, n\}$ . These kernel functions are called as the gamma memory filter [24] and satisfy condition (H<sub>4</sub>).

For any bounded function  $s(t)$  on  $[0, +\infty)$ ,  $\underline{s}$  and  $\bar{s}$  denote  $\inf_{t \in [0, +\infty)} \{s(t)\}$  and  $\sup_{t \in [0, +\infty)} \{s(t)\}$ , respectively.

For any  $x(t) = (x_1(t), x_2(t), \dots, x_k(t))^T \in R^k, t > 0$ , define  $\|x(t)\|_1 = \sum_{i=1}^k |x_i(t)|$ , and for any  $\varphi(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_k(s))^T \in R^k, s \in (-\infty, 0]$ , define  $\|\varphi\| = \sup_{s \in (-\infty, 0]} \sum_{i=1}^k |\varphi_i(s)|$ .

Denote

$$C((-\infty, 0], R^k) = \left\{ \varphi : (-\infty, 0] \rightarrow R^k \mid \varphi(s) \text{ is bounded and continuous for } s \in (-\infty, 0] \right\}, \tag{2.6}$$

Then  $C((-\infty, 0], R^k)$  is a Banach space with respect to  $\|\cdot\|$ .

The initial conditions of system (2.1) and (2.2) are given by

$$\begin{aligned}x_i(s) &= \varphi_i(s), & -\infty < s \leq 0, & 1 \leq i \leq n, \\y_i(s) &= \zeta_i(s), & -\infty < s \leq 0, & 1 \leq i \leq n.\end{aligned}\tag{2.7}$$

*Definition 2.2.* System (2.1) and system (2.2) are said to achieve global exponential complete synchronization, if for any solution  $x(t, \varphi_1)$  of system (2.1) and any solution  $y(t, \varphi_2)$  of system (2.2), there exist positive constants  $\lambda$  and  $M$  such that

$$\|y(t, \varphi_2) - x(t, \varphi_1)\|_1 \leq M \|\varphi_2 - \varphi_1\| e^{-\lambda t}, \quad t \geq 0,\tag{2.8}$$

where  $\varphi_2(s) = (\zeta_1(s), \zeta_2(s), \dots, \zeta_n(s))^T$ ,  $\varphi_1(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s))^T \in C((-\infty, 0], \mathbb{R}^n)$ .

*Definition 2.3.* A real matrix  $A = (a_{ij})_{n \times n}$  is said to be a nonsingular  $M$ -matrix if  $a_{ij} \leq 0$  ( $i, j = 1, 2, \dots, n$ ,  $i \neq j$ ), and all successive principle minors of  $A$  are positive.

**Lemma 2.4** (see [25]). *A matrix with nonpositive offdiagonal elements  $A = (a_{ij})_{n \times n}$  is a nonsingular  $M$ -matrix if and only if there exists a vector  $p = (p_i)_{1 \times n} > 0$  such that  $pA > 0$  or  $Ap^T > 0$  holds.*

**Lemma 2.5** (see [26]). *Let  $a < b \leq +\infty$ . Suppose that  $v(t) = (v_1(t), v_2(t), \dots, v_n(t))^T$  satisfies the following differential equality:*

$$\begin{aligned}D^+v(t) &\leq Pv(t) + (Q \otimes \tilde{v}(t))e_n + \int_0^{+\infty} (R \otimes K(s))v(t-s)ds, & t \in [a, b), \\v(a+s) &\in C((-\infty, 0], \mathbb{R}^n), & s \in (-\infty, 0],\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}P &= (p_{ij})_{n \times n}, & p_{ij} \geq 0 \text{ for } i \neq j, \\Q &= (q_{ij})_{n \times n}, & q_{ij} \geq 0, \quad R = (r_{ij})_{n \times n}, & r_{ij} \geq 0, \\ \tilde{v}(t) &= (v_j(t - \tau_{ij}(t)))_{n \times n}, & e_n = (1, 1, \dots, 1)^T \in \mathbb{R}^n, & K(s) = (|k_{ij}(s)|)_{n \times n},\end{aligned}\tag{2.10}$$

and  $\otimes$  means Hadamard product. If initial conditions satisfy

$$v(t) \leq \kappa \xi e^{-\lambda(t-a)}, \quad \kappa \geq 0, \quad t \in (-\infty, a],\tag{2.11}$$

where  $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$  and the positive number  $\lambda$  is determined by the following inequality:

$$[\lambda E + P + Q \otimes \varepsilon(\lambda) + (R \otimes \mathcal{K}(\lambda))] \xi < 0\tag{2.12}$$

in which

$$\varepsilon(\lambda) = \left( e^{\lambda\tau_{ij}} \right)_{n \times n}, \quad \mathcal{K}(\lambda) = (\mathcal{K}_{ij}(\lambda))_{n \times n}, \quad \mathcal{K}_{ij}(\lambda) = \int_0^{+\infty} e^{\lambda s} |k_{ij}(s)| ds, \quad (2.13)$$

and  $E$  is an identity matrix. Then  $v(t) \leq \kappa \xi e^{-\lambda(t-a)}$  for  $t \in [a, b)$ .

### 3. Main Results

**Theorem 3.1.** *Under assumptions (H<sub>1</sub>)–(H<sub>4</sub>), system (2.1) and system (2.2) will achieve global exponential synchronization, if the following conditions hold:*

(H<sub>5</sub>)  $\mathcal{M}_1 = A_1 - C_1$  is a nonsingular  $M$ -matrix, where

$$A_1 = \text{diag} \left( \underline{b}_1 + \frac{\gamma_{11}}{a_1}, \underline{b}_2 + \frac{\gamma_{22}}{a_2}, \dots, \underline{b}_n + \frac{\gamma_{nn}}{a_n} \right), \quad (3.1)$$

$$C_1 = (c_{ij})_{n \times n}, \quad c_{ij} = \left( \bar{u}_{ij} + \bar{v}_{ij} \frac{1}{1 - \bar{\tau}_{ij}^+} + \bar{w}_{ij} \right) L_j + q_{ij},$$

where  $q_{ij} = \bar{\gamma}_{ij}/\underline{a}_i$ ,  $j \neq i$  and  $q_{ii} = 0$ ,  $\gamma_{ij}(t) = \alpha_{ij}(t) + \beta_{ij}(t)$ ,  $i, j = 1, 2, \dots, n$ .

*Proof.* Let  $x(t)$ ,  $y(t)$  be two solutions of system (2.1) with initial value  $\varphi_1 = (\varphi_1, \varphi_2, \dots, \varphi_n)$  and system (2.2) with  $\varphi_2 = (\zeta_1, \zeta_2, \dots, \zeta_n) \in C((-\infty, 0], R^n)$ , respectively.

Denote  $e_i(t) = y_i(t) - x_i(t)$ .

Note that conditions (H<sub>5</sub>),  $\mathcal{M}_1$  is a nonsingular  $M$ -matrix implies that  $\mathcal{M}_1^T$  is a nonsingular  $M$ -matrix. From Lemma 2.4, we know that there exists a vector  $p = (p_1, p_2, \dots, p_n)^T$  such that  $\mathcal{M}_1^T p > 0$ , that is

$$p_i \left( \underline{b}_i + \frac{\gamma_{ii}}{a_i} \right) - \sum_{j=1}^n p_j \left( \bar{u}_{ji} + \frac{\bar{v}_{ji}}{1 - \bar{\tau}_{ji}^+} + \bar{w}_{ji} \right) L_i - \sum_{j=1, j \neq i}^n p_j \frac{\bar{\gamma}_{ji}}{a_j} > 0 \quad (3.2)$$

for  $1 \leq i \leq n$ .

Denote

$$F_i(\theta) = p_i \left( -\frac{\theta}{a_i} + \underline{b}_i + \frac{\gamma_{ii}}{a_i} \right) - \sum_{j=1}^n p_j \left( \bar{u}_{ji} L_i + \bar{v}_{ji} \frac{e^{\theta\tau_{ji}}}{1 - \bar{\tau}_{ji}^+} L_i + \bar{w}_{ji} L_i \int_0^{+\infty} k_{ji}(s) e^{\theta s} ds \right) - \sum_{j=1, j \neq i}^n p_j \frac{\bar{\gamma}_{ji}}{a_j}, \quad (3.3)$$

for  $1 \leq i \leq n$ , which indicates  $F_i(0) > 0$ . Since  $F_i(\theta)$  are continuous and differential on  $[0, r_{ji})$  on  $[0, r_{ji})$  in which  $r_{ji}$  are some positive constants, and  $\lim_{\theta \rightarrow r_{ji}^-} F_i(\theta) = -\infty$  according

to condition (H<sub>4</sub>), furthermore,  $F'_i(\theta) < 0$  for  $\theta \in [0, r_{ji})$ . There exist constants  $\theta_i$  such that  $F_i(\theta_i) = 0$  for  $i = 1, 2, \dots, n$ . So we can choose

$$0 < \lambda \leq \min_{1 \leq i \leq n} \{\theta_i\} \quad (3.4)$$

such that

$$F_i(\lambda) \geq 0. \quad (3.5)$$

Let

$$V_i(t) = e^{\lambda t} \operatorname{sign}(e_i(t)) \int_{x_i(t)}^{y_i(t)} \frac{1}{a_i(s)} ds \quad \text{for } i = 1, 2, \dots, n, \quad (3.6)$$

where  $e_i(t) = y_i(t) - x_i(t)$ .

Calculating the upper right derivative of  $V_i(t)$  along solutions of (2.1) and (2.2), we get

$$\begin{aligned} D^+ V_i(t) &\leq e^{\lambda t} \left\{ \frac{\lambda}{a_i} |e_i(t)| - \operatorname{sign}(e_i(t)) (b_i(t, y_i(t)) - b_i(t, x_i(t))) \right. \\ &\quad + \left| \sum_{j=1}^n u_{ij}(t) (f_j(y_j(t)) - f_j(x_j(t))) \right| \\ &\quad + \left| \sum_{j=1}^n v_{ij}(t) (f_j(y_j(t - \tau_{ij}(t))) - f_j(x_j(t - \tau_{ij}(t)))) \right| \\ &\quad + \left| \sum_{j=1}^n w_{ij}(t) \left( \int_0^{+\infty} k_{ij}(s) f_j(y_j(t-s)) ds - \int_0^{+\infty} k_{ij}(s) f_j(x_j(t-s)) ds \right) \right| \\ &\quad \left. - \frac{\gamma_{ii}}{a_i} |e_i(t)| + \sum_{j=1, j \neq i}^n \frac{\bar{\gamma}_{ij}}{a_i} |e_j(t)| \right\} \\ &\leq e^{\lambda t} \left\{ \left( \frac{\lambda}{a_i} - \underline{b}_i - \frac{\gamma_{ii}}{a_i} \right) |e_i(t)| + \sum_{j=1}^n \bar{u}_{ij} L_j |e_j(t)| + \sum_{j=1}^n \bar{v}_{ij} L_j |e_j(t - \tau_{ij}(t))| \right. \\ &\quad \left. + \sum_{j=1}^n \bar{w}_{ij} L_j \int_0^{+\infty} k_{ij}(s) |e_j(t-s)| ds + \sum_{j=1, j \neq i}^n \frac{\bar{\gamma}_{ij}}{a_i} |e_j(t)| \right\}, \end{aligned} \quad (3.7)$$

where  $\gamma_{ij}(t) = \alpha_{ij}(t) + \beta_{ij}(t)$ .

Now we define a Lyapunov function  $V(t)$  by

$$\begin{aligned}
 V(t) = \sum_{i=1}^n p_i \left\{ V_i(t) + \sum_{j=1}^n \bar{v}_{ij} L_j \frac{e^{\lambda \tau_{ij}}}{1 - \bar{\tau}_{ij}^+} \int_{t-\tau_{ij}(t)}^t |e_j(s)| e^{\lambda s} ds \right. \\
 \left. + \sum_{j=1}^n \bar{w}_{ij} L_j \int_0^{+\infty} k_{ij}(s) \int_{t-s}^t |e_j(\mu)| e^{\lambda(s+\mu)} d\mu ds \right\}.
 \end{aligned}
 \tag{3.8}$$

We can obtain that

$$\begin{aligned}
 D^+ V(t) &\leq e^{\lambda t} \sum_{i=1}^n p_i \left\{ \left( \frac{\lambda}{a_i} - b_i - \frac{\gamma_{ii}}{a_i} \right) |e_i(t)| \right. \\
 &\quad \left. + \sum_{j=1}^n \left( \bar{u}_{ij} + \bar{v}_{ij} \frac{e^{\lambda \tau_{ij}}}{1 - \bar{\tau}_{ij}^+} + \bar{w}_{ij} \int_0^{+\infty} k_{ij}(s) e^{\lambda s} ds \right) \right. \\
 &\quad \left. \times L_j |e_j(t)| + \sum_{j=1, j \neq i}^n \frac{\bar{\gamma}_{ij}}{a_i} |e_j(t)| \right\} \\
 &= -e^{\lambda t} \sum_{i=1}^n F_i(\lambda) |e_i(t)| \leq 0,
 \end{aligned}
 \tag{3.9}$$

which together with (3.6) and (3.8) leads to

$$m_0 e^{\lambda t} \sum_{i=1}^n |e_i(t)| \leq V(t) \leq V(0) \leq M_0 \|\psi_2(l) - \psi_1(l)\|,
 \tag{3.10}$$

where

$$\begin{aligned}
 m_0 &= \min_{1 \leq i \leq n} \left\{ \frac{p_i}{a_i} \right\}, \quad M_0 = \max\{M_1, M_2, M_3\}, \\
 M_1 &= \max_{1 \leq i \leq n} \left( \frac{p_i}{a_i} \right), \\
 M_2 &= \max_{1 \leq i \leq n} \left( \frac{(e^{\lambda \tau} - 1)}{\lambda} \sum_{j=1}^n p_j \bar{v}_{ji} \frac{L_i}{1 - \bar{\tau}_{ji}^+} \right), \\
 M_3 &= \sum_{j=1}^n p_j \max_{1 \leq i \leq n} (\bar{w}_{ji} L_i) \int_0^{+\infty} s e^{\lambda s} \max_{1 \leq i \leq n} k_{ji}(s) ds.
 \end{aligned}
 \tag{3.11}$$

Hence, we obtain that the following inequality holds:

$$\sum_{i=1}^n |e_i(t)| \leq \frac{M_0}{m_0} \|\varphi_2(l) - \varphi_1(l)\| e^{-\lambda t}, \quad t \geq 0, \quad (3.12)$$

that is,

$$\|y(t) - x(t)\|_1 \leq \frac{M_0}{m_0} \|\varphi_2(l) - \varphi_1(l)\| e^{-\lambda t}, \quad t \geq 0. \quad (3.13)$$

This completes the proof.  $\square$

**Theorem 3.2.** *Under assumptions (H<sub>1</sub>)–(H<sub>4</sub>), system (2.2) and system (2.1) will achieve global exponential synchronization, if the following conditions hold:*

(H<sub>6</sub>)  $\mathcal{M}_2 = A_2 - C_2$  is a nonsingular M-matrix, where

$$A_2 = \text{diag} \left( \underline{a}_1 \left( \underline{b}_1 + \frac{\gamma_{11}}{\underline{a}_1} \right), \underline{a}_2 \left( \underline{b}_2 + \frac{\gamma_{22}}{\underline{a}_2} \right), \dots, \underline{a}_n \left( \underline{b}_n + \frac{\gamma_{nn}}{\underline{a}_n} \right) \right), \quad (3.14)$$

$$C_2 = (c_{ij})_{n \times n}, \quad c_{ij} = \bar{a}_j [(\bar{u}_{ij} + \bar{v}_{ij} + \bar{w}_{ij})L_j + q_{ij}],$$

in which  $q_{ij} = \bar{\gamma}_{ij} / \underline{a}_i$ ,  $j \neq i$ ,  $q_{ii} = 0$  and  $\gamma_{ij}(t) = \alpha_{ij}(t) + \beta_{ij}(t)$ ,  $i, j = 1, 2, \dots, n$ .

*Proof.* Let  $x(t), y(t)$  be two solutions of system (2.1) with initial value  $\varphi_1 = (\varphi_1, \varphi_2, \dots, \varphi_n)$  and system (2.2) with  $\varphi_2 = (\zeta_1, \zeta_2, \dots, \zeta_n) \in C((-\infty, 0], R^n)$ , respectively.

Denote  $e_i(t) = y_i(t) - x_i(t)$ .

Let

$$V_i(t) = \text{sign}(e_i(t)) \int_{x_i(t)}^{y_i(t)} \frac{1}{a_i(s)} ds \quad \text{for } i = 1, 2, \dots, n \quad (3.15)$$

and note that

$$\frac{1}{\bar{a}_i} |e_i(t)| \leq V_i(t) \leq \frac{1}{\underline{a}_i} |e_i(t)|. \quad (3.16)$$



Calculating the upper right derivative of  $V_i(t)$  along solutions of (2.1) and (2.2), we can get

$$\begin{aligned}
 D^+V_i(t) &\leq -\underline{b}_i|e_i(t)| + \sum_{j=1}^n \bar{u}_{ij}L_j|e_j(t)| + \sum_{j=1}^n \bar{v}_{ij}L_j|e_j(t - \tau_{ij}(t))| \\
 &\quad + \sum_{j=1}^n \bar{w}_{ij}L_j \int_0^{+\infty} k_{ij}(s)|e_j(t-s)|ds - \frac{\gamma_{ii}(t)}{\underline{a}_i}|e_i(t)| + \sum_{j=1, j \neq i}^n \frac{\bar{\gamma}_{ij}}{\underline{a}_i}|e_j(t)| \\
 &\leq -\underline{b}_i \underline{a}_i V_i(t) + \sum_{j=1}^n \bar{u}_{ij}L_j \bar{a}_j V_j(t) + \sum_{j=1}^n \bar{v}_{ij}L_j \bar{a}_j V_j(t - \tau_{ij}(t)) \\
 &\quad + \sum_{j=1}^n \bar{w}_{ij}L_j \bar{a}_j \int_0^{+\infty} k_{ij}(s)V_j(t-s)ds - \frac{\gamma_{ii} \underline{a}_i}{\underline{a}_i} V_i(t) + \sum_{j=1, j \neq i}^n \frac{\bar{\gamma}_{ij} \bar{a}_j}{\underline{a}_i} V_j(t),
 \end{aligned} \tag{3.17}$$

that is,

$$D^+V(t) \leq AV(t) + (C \otimes \tilde{V}(t))E_n + \int_0^{+\infty} (D \otimes K(s))V(t-s)ds, \tag{3.18}$$

where

$$\begin{aligned}
 V(t) &= (V_1(t), V_2(t), \dots, V_n(t))^T, \quad \tilde{V}(t) = (V_j(t - \tau_{ij}(t)))_{n \times n}, \\
 A &= \text{diag} \left( -\underline{a}_1 \left( \underline{b}_1 + \frac{\gamma_{11}}{\underline{a}_1} \right), -\underline{a}_2 \left( \underline{b}_2 + \frac{\gamma_{22}}{\underline{a}_2} \right), \dots, -\underline{a}_n \left( \underline{b}_n + \frac{\gamma_{nn}}{\underline{a}_n} \right) \right) + H, \\
 H &= (h_{ij})_{n \times n}, \quad h_{ij} = \bar{a}_j \bar{u}_{ij} L_j + \tilde{q}_{ij}, \quad \tilde{q}_{ij} = \frac{\bar{a}_j \bar{\gamma}_{ij}}{\underline{a}_i}, \quad j \neq i, \quad \tilde{q}_{ii} = 0, \quad i, j = 1, 2, \dots, n, \\
 C &= (\tilde{c}_{ij})_{n \times n}, \quad \tilde{c}_{ij} = \bar{a}_j \bar{v}_{ij} L_j, \quad D = (d_{ij})_{n \times n}, \quad d_{ij} = \bar{a}_j \bar{w}_{ij} L_j, \\
 E_n &= (1, 1, \dots, 1)^T \in R^n, \quad K(s) = (k_{ij}(s))_{n \times n}.
 \end{aligned} \tag{3.19}$$

We know from  $(H_6)$  that  $M_2$  is nonsingular  $M$ -matrix, which implies  $-(A + D + C)$  is also a nonsingular  $M$ -matrix. From Lemma 2.4, there exists a vector  $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$  such that  $-(A + D + C)\xi > 0$ , consequently,

$$-\xi_i \underline{a}_i \left( \underline{b}_i + \frac{\gamma_{ii} \underline{a}_i}{\underline{a}_i} \right) + \sum_{j=1}^n \xi_j \bar{a}_j (\bar{u}_{ij} + \bar{v}_{ij} + \bar{w}_{ij}) L_j + \sum_{j=1, j \neq i}^n \xi_j \frac{\bar{a}_j \bar{\gamma}_{ij}}{\underline{a}_i} > 0. \tag{3.20}$$

Consider the function

$$F_i(\theta) = \xi_i \left( \theta - \underline{a}_i \left( \underline{b}_i + \frac{\gamma_i \underline{a}_i}{\underline{a}_i} \right) \right) + \sum_{j=1}^n \xi_j \bar{a}_j \left( \bar{u}_{ij} + \bar{v}_{ij} e^{\theta \tau_{ij}} + \bar{w}_{ij} K_{ij}(\theta) \right) L_j + \sum_{j=1, j \neq i}^n \xi_j \frac{\bar{a}_j \bar{\gamma}_{ij}}{\underline{a}_i}, \quad (3.21)$$

in which  $K_{ij}(\theta) = \int_0^{+\infty} e^{\theta \tau_{ij}} k_{ij}(s) ds$ .

We obtain from condition (H<sub>4</sub>) that  $K_{ij}(0) = 1$ , which, together with (3.20), leads to  $F_i(0) < 0$ , and similar to proof of Theorem 3.1 above, there exist

$$0 < \lambda < \min_{1 \leq i \leq n} \{\theta_i\} \quad (3.22)$$

such that

$$F_i(\lambda) < 0, \quad (3.23)$$

that is,

$$\left[ \lambda E + A + C \otimes \left( e^{\lambda \tau_{ij}} \right)_{n \times n} + D \otimes \mathcal{K}(\lambda) \right] \xi < 0, \quad (3.24)$$

where  $K(\lambda) = (K_{ij}(\lambda))_{n \times n}$  in which  $K_{ij}(\lambda)$  shown in (3.21).

Note that  $V_i(s) \leq 1/\underline{a}_i \|\varphi_2 - \varphi_1\|$  and denote  $\kappa = \|\varphi_2 - \varphi_1\| / \min_{1 \leq i \leq n} \{\underline{a}_i \xi_i\}$ , we have

$$V(t) \leq \kappa \xi e^{-\lambda t}, \quad t \in (-\infty, 0]. \quad (3.25)$$

We have from Lemma 2.5, (3.18), (3.24), and (3.25) that

$$V(t) \leq \kappa \xi e^{-\lambda t}, \quad t \geq 0. \quad (3.26)$$

It follows from (3.16) that

$$|e_i(t)| \leq \bar{a}_i V_i(t) \leq \bar{a}_i \tilde{\kappa} \xi_i \|\varphi_2 - \varphi_1\| e^{-\lambda t} \quad (3.27)$$

in which  $\tilde{\kappa} = 1 / \min_{1 \leq i \leq n} \{\underline{a}_i \xi_i\}$ .

Hence

$$\|y(t) - x(t)\|_1 \leq M \|\varphi_2 - \varphi_1\| e^{-\lambda t}, \quad t \geq 0, \quad (3.28)$$

where  $M = (\sum_{i=1}^n \bar{a}_i \xi_i) / \min_{1 \leq i \leq n} \{\underline{a}_i \xi_i\}$ . This completes the proof.  $\square$

**Corollary 3.3.** Under assumptions  $(H_1)$ – $(H_4)$ , response system (2.2) will be globally exponentially synchronized with master system (2.1) with  $\alpha_{ij} = 0$ ,  $i, j = 1, 2, \dots, n$ , if  $M_1$  in Theorem 3.1 and  $M_2$  in Theorem 3.2 are nonsingular  $M$ -matrices with  $\gamma_{ij}(t) = \beta_{ij}(t)$ .

*Remark 3.4.* If  $a_i(\cdot) = 1$ ,  $b_i(t, x_i(t)) = b_i(t)x_i(t)$  and  $b_i(t, y_i(t)) = b_i(t)y_i(t)$ , system (2.1) and (2.2) reduce to coupled Hopfield neural networks. Both Theorems 3.1 and 3.2 reduce to the simpler cases in which  $\bar{a}_i = \underline{a}_i = 1$ .

*Remark 3.5.* If  $\alpha_{ij}(t) = \beta_{ij}(t) = 0$ ,  $M_1$  in Theorem 3.1 and  $M_2$  in Theorem 3.2 still be nonsingular  $M$ -matrix, both theorems imply the global asymptotical stability of solutions of system (2.1), consequently, system (2.2) and (2.1) achieve complete synchronization for sure. In addition, if system (2.1) and (2.2) reduce to autonomous systems, the results in both theorems above still hold.

## 4. Two Simple Examples

*Example 4.1.* Consider the following coupled CGNNs:

$$\begin{aligned} \dot{x}(t) = & -a(x(t)) \left[ Ax(t) - B \int_0^{+\infty} K(s) \tanh(x(t-s)) ds - I(t) \right] \\ & + K_1(t)(y(t) - x(t)), \end{aligned} \quad (4.1)$$

$$\begin{aligned} \dot{y}(t) = & -a(y(t)) \left[ Ay(t) - B \int_0^{+\infty} K(s) \tanh(y(t-s)) ds - I(t) \right] \\ & + K_2(t)(x(t) - y(t)), \end{aligned} \quad (4.2)$$

where we write system (4.1) and (4.2) in the vector-matrix form and

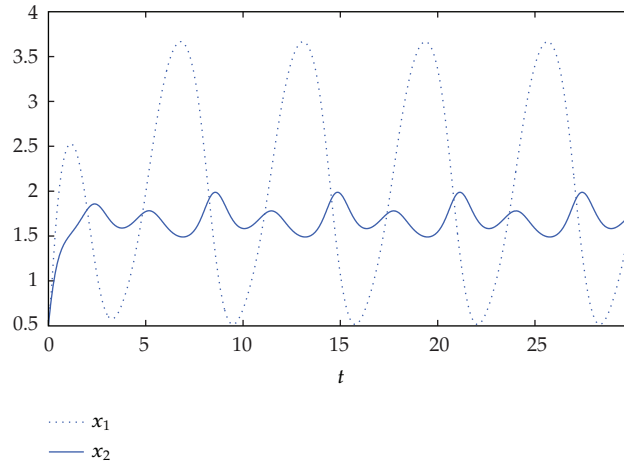
$$\begin{aligned} x(t) &= (x_1(t), x_2(t))^T, \quad y(t) = (y_1(t), y_2(t))^T, \quad I(t) = (2, 2)^T, \\ \tanh(x(t)) &= (\tanh(x_1(t)), \tanh(x_2(t)))^T, \\ \tanh(y(t)) &= (\tanh(y_1(t)), \tanh(y_2(t)))^T, \\ a(x(t)) &= \text{diag}(2 + \sin(x_1(t)), 2 + \sin(x_2(t))), \\ a(y(t)) &= \text{diag}(2 + \sin(y_1(t)), 2 + \sin(y_2(t))), \end{aligned} \quad (4.3)$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \cos(t) \\ 2 & -0 \end{pmatrix}, \quad K(s) = \begin{pmatrix} 0 & e^{-t} \\ e^{-t} & 0 \end{pmatrix},$$

$$K_1(t) = \begin{pmatrix} \alpha_1(t) & 0 \\ 0 & \alpha_2(t) \end{pmatrix},$$

$$K_2(t) = \begin{pmatrix} \beta_1(t) & 0 \\ 0 & \beta_2(t) \end{pmatrix}.$$

Let  $K_1(t) = 0$ . Figure 1 shows the dynamical behaviors of (4.1) with  $K_1(t) = 0$ .



**Figure 1:** Time response of state  $x_1, x_2$  of system (4.1) with initial value  $(0.1, 0.1)$  and  $K_1 = 0$ .

Let  $K_1(t) = \text{diag}(1, 1)$  and  $K_2(t) = \text{diag}(5, 0)$ , it is easy to know  $M_1 = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$  is still a  $M$ -matrix. From Theorem 3.1, we know coupled system (4.1) and system (4.2) achieve complete synchronization, and Figure 2 shows the dynamical behaviors of complete synchronization with initial conditions  $(y_1(s), y_2(s)) = (2, 2)$  and  $(x_1(s), x_2(s)) = (0.1, 0.1)$ .

*Example 4.2.* Consider the following Cohen-Grossberg neural network with discrete delay:

$$\begin{aligned} \dot{x}_1(t) &= -3[x_1(t) - 3.5 \sin(x_2(t - 0.4))], \\ \dot{x}_2(t) &= -2[2x_2(t) - 3 \sin(x_1(t - 0.4))]. \end{aligned} \tag{4.4}$$

System (4.4) is chaotic system. Figure 3 shows the chaotic behaviors of system (4.4) with initial condition  $(0.2, 0.2)$ .

For driving system (4.4), we construct the response system as follows:

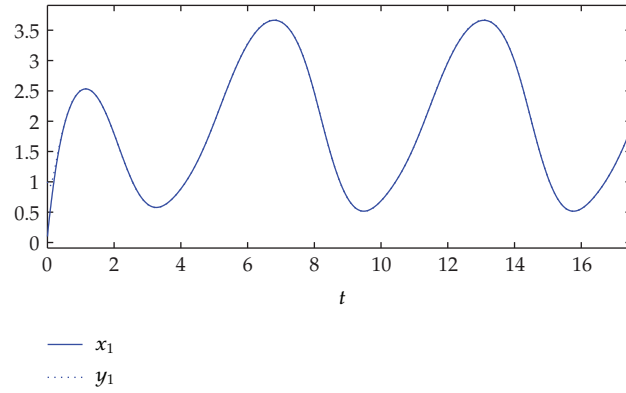
$$\begin{aligned} \dot{y}_1(t) &= -3[y_1(t) - 3.5 \sin(y_2(t - 0.4))] + \alpha_{11}(x_1(t) - y_1(t)), \\ \dot{y}_2(t) &= -2[2y_2(t) - 3 \sin(y_1(t - 0.4))] + \alpha_{22}(x_2(t) - y_2(t)), \end{aligned} \tag{4.5}$$

where  $\alpha_{11} > 0, \alpha_{22} > 0$ .

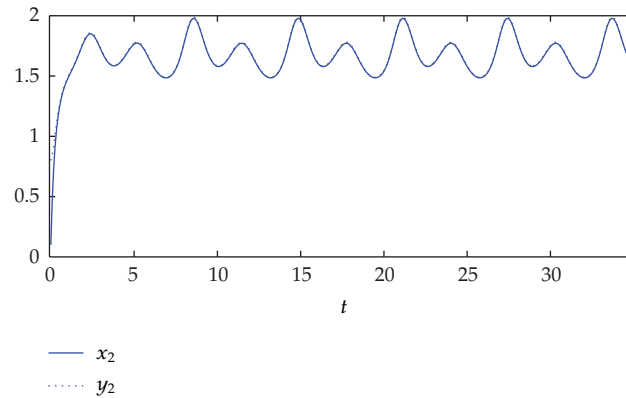
Let  $\alpha_{11} = 9, \alpha_{22} = 0$ . It is easy to know

$$M_1 = \begin{pmatrix} 3 & -3.5 \\ -3 & 4 \end{pmatrix} \tag{4.6}$$

is a  $M$ -matrix. From Corollary 3.3, we know the response system (4.5) achieves complete synchronization with system (4.4), and Figure 4 shows the dynamical behaviors of complete synchronization with initial conditions  $(y_1(s), y_2(s)) = (2, 2)$  and  $(x_1(s), x_2(s)) = (0.2, 0.2)$ .

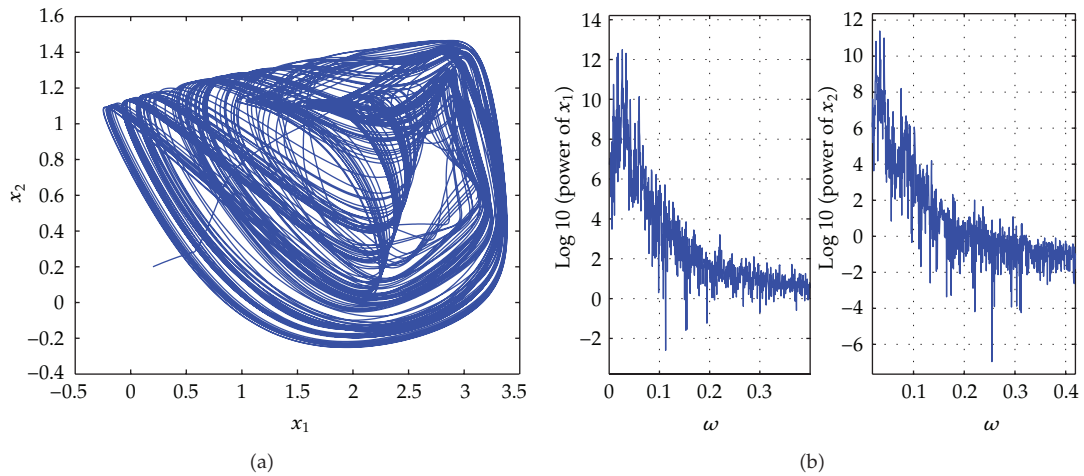


(a)



(b)

**Figure 2:** Complete synchronization of coupled systems (4.1) and (4.2).



**Figure 3:** (a) Phase plot of system (4.4) with initial condition (0.2,0.2). (b) Power spectrums of time series of  $x_1$  and  $x_2$  for system (4.4).

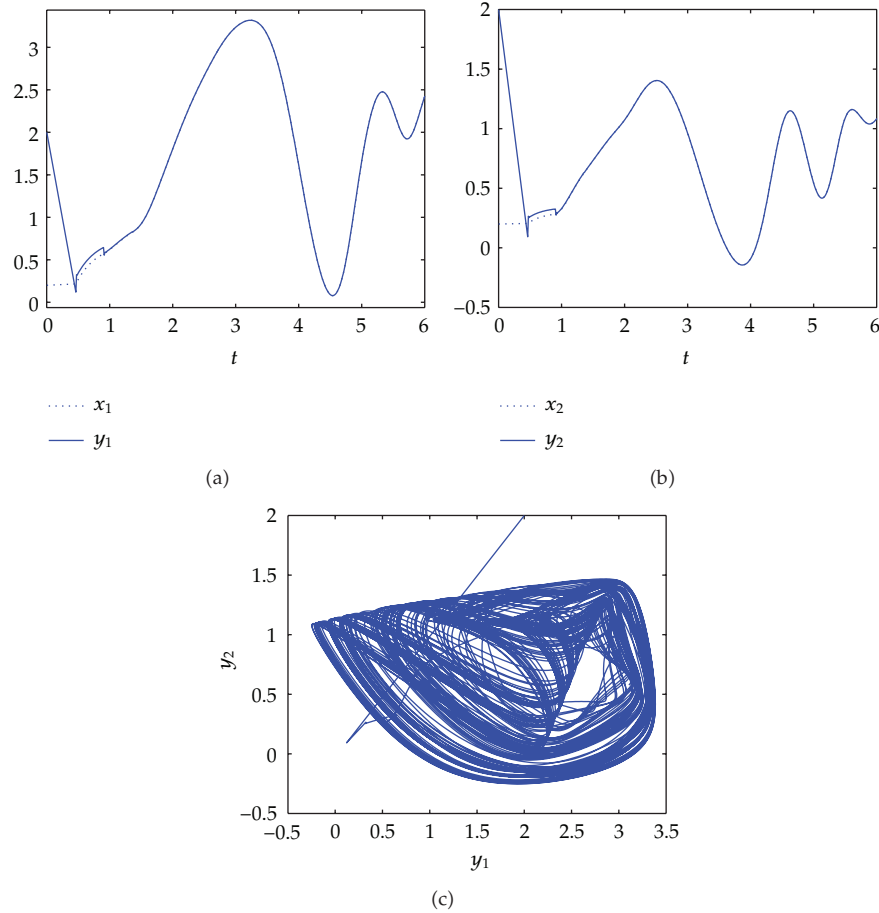


Figure 4: Complete synchronization of the response system (4.5) and the driving system (4.4).

## 5. Conclusions

Based on using suitable Lyapunov function and the properties of  $M$ -matrix, sufficient conditions for complete synchronization of bidirectional coupled CGNNs are directly obtained without writing the explicit error system. Two examples show the effectiveness of the proposed method. Note that  $M_1$  and  $M_2$  must be  $M$ -matrix if we let  $\alpha_{ii}(t) + \beta_{ii}(t)$ , that is,  $\gamma_{ii}(t)$  be big enough to a certain extent such that  $M_1$  and  $M_2$  are strongly diagonally dominant matrix, this shows that our criteria are easy to verify and are useful for synchronization of CGNNs.

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