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Research Article

A Fundamental Inequality of Algebroidal Function

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By using a new mapping of Ahlfors covering surfaces, a fundamental inequality in the angular domain for the algebroidal function is obtained.

1. Introduction and Main Results

In the field of valued distribution, the fundamental inequality is an important tool. For example, it can be used to investigate the singular direction [1]. Using geometric theory, Tsuji firstly obtained the second fundamental theory in an angular domain and proved the existence of Borel direction [2]. The value distribution theory of meromorphic functions was extended to algebroidal functions last century [3]. In 1983, Lv and Gu proved an inequality of algebroidal function for an angular domain [4]. By the inequality, some results of singular direction are obtained; see [5, 6]. In [7], the authors obtained a more accurate inequality for angular domain. In this paper, we will use a new method to simplify and extent an inequality of Tsuji to algebroidal functions.

First, we recall some definitions from [3].

Suppose that $A_v(z), \ldots, A_0(z)$ are analytic functions with no common zeros in the complex plane. $\Psi(z, W)$ is a bivariate complex function and satisfies

$$\Psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_0(z) = 0. \tag{1.1}$$

For all z in the complex plane, the equation $\Psi(z, W) = 0$ has v complex roots $w_1(z), w_2(z), \ldots, w_v(z)$. Then, (1.1) defines a v-valued algebroidal function $W(z)$; see [3, 8]. If $A_v(z) = 1$, then $W(z)$ is called v-valued integral algebroidal function. If $\Psi(z, W)$ is irreducible, correspondingly $W(z)$ is called v-valued irreducible algebroidal function (note that $W(z)$ is a meromorphic function, if $v = 1$). Now we suppose that $W(z)$ is an irreducible algebriodal function defined by (1.1).

If $A_v(z_0) \neq 0$, and the k-degree equation $\Psi(z_0, W) = 0$ and its partial derivative $(\partial\Psi/\partial W)(z_0, W) = 0$ have no common roots (i.e., z_0 is not a multiple root of $\Psi(z_0, W) = 0$), then z_0 is said to be a regular point. The set of all regular points is called the regular set, denoted by T_W. Its complementary set $S_W := \{z \mid |z| < \infty\} - T_W$ is called the critical set. Obviously, S_w includes all branch points of W(see [3]).

The domain of a v-valued irreducible algebriodal function W is a connected Riemann surface [8], and its single-valued domain is denoted by $\widetilde{R}_z$. A point in $\widetilde{R}_z$ is $\tilde{z}$ and sets lying over $|z| < r$ and $\{\phi_1 < \arg z < \phi_2\}$ $(\phi_1 < \phi_2)$ are $|\tilde{z}| < r$ and $\widetilde{\Omega}(\phi_1, \phi_2)$. Let $n(r, W = a)$ and $n(\Omega(\phi_1, \phi_2), r, W = a)$ be the number of zeros, counted according to their multiplicities, of $W = a$ in $\{|\tilde{z}| < r\}$ and $\{|\tilde{z}| < r\} \bigcap \widetilde{\Omega}(\phi_1, \phi_2)$, respectively. Let $\overline{n}(r, W = a)$ be the number of distinct zeros in $\{|\tilde{z}| < r\}$, and let $n(r, \widetilde{R}_z)$ be the number of branch points in $\{|\tilde{z}| < r\}$. Similarly, we can define $\overline{n}(\Omega(\phi_1, \phi_2), r, W = a)$ and $n(\Omega(\phi_1, \phi_2), r, \widetilde{R}_z)$. Let

$$
\begin{aligned}
S(r, W) &= \frac{1}{\pi} \iint_{|\tilde{z}| \leq r} \left(\frac{|W'(z)|}{1 + |W(z)|^2} \right)^2 dW \quad z = re^{i\theta}, \\
S(\Omega(\phi_1, \phi_2), r, W) &= \frac{1}{\pi} \iint_{\{|\tilde{z}| \leq r\} \bigcap \widetilde{\Omega}(\phi_1, \phi_2)} \left(\frac{|W'(z)|}{1 + |W(z)|^2} \right)^2 dW, \\
T(r, W) &= \frac{1}{v} \int_0^r \frac{S(t, W)}{t} dt, \\
T(\Omega(\phi_1, \phi_2), r, W) &= \frac{1}{v} \int_0^r \frac{S(\Omega(\phi_1, \phi_2), t, W)}{t} dt, \qquad (1.2) \\
N(r, W = a) &= \frac{1}{v} \int_0^r \frac{n(t, W = a) - n(0, W = a)}{t} dt + \frac{n(0, W = a)}{v} \ln r, \\
N\left(r, \widetilde{R}_z\right) &= \frac{1}{v} \int_0^r \frac{n\left(t, \widetilde{R}_z\right) - n\left(0, \widetilde{R}_z\right)}{t} dt + \frac{n\left(0, \widetilde{R}_z\right)}{v} \ln r, \\
m(r, W) &= \frac{1}{2\pi v} \sum_{k=1}^{v} \int_0^{2\pi} \ln^+ \left| w_k\left(re^{i\theta}\right) \right| d\theta.
\end{aligned}
$$

Similarly, we can define $N(\Omega(\phi_1, \phi_2), r, W = a)$, $\overline{N}(\Omega(\phi_1, \phi_2), r, W = a)$ and $N(\Omega(\phi_1, \phi_2), r, \widetilde{R}_z)$. From [3], we know that $T(r, w) = N(r, W = \infty) + m(r, W) + O(1)$ and $N(r, \widetilde{R}_z) \leq 2(v - 1)T(r, W) + O(1)$.

In this paper, we will prove the main theorem.

Theorem 1.1. *Let $W(z)$ be a v-valued algebroidal function in region $\Omega(\phi_1, \phi_2) \triangleq \{|z| \mid \phi_1 < \arg\ z < \phi_2\}$ $(\phi_1 < \phi_2)$. $a_1, a_2, \ldots, a_q$ $(q \geqslant 3)$ are q different complex numbers on the sphere with* *radius* $\delta \in (0,1/2)$. *For* $\phi, \varepsilon^*, \varepsilon$ $(0 < \varepsilon^* < \varepsilon,\ \phi_1 < \phi - \varepsilon < \phi - \varepsilon^* < \phi + \varepsilon^* < \phi + \varepsilon < \phi_2)$, *and* $R > R^* > 2$, *we have*

$$\begin{aligned}(q-2)S(\Omega(\phi-\varepsilon^*,\phi+\varepsilon^*),R^*,W) \\ \leqslant n\Big(\Omega(\phi-\varepsilon,\phi+\varepsilon),R,\widetilde{R}_z\Big) + \sum_{j=1}^{q}\overline{n}(\Omega(\phi-\varepsilon,\phi+\varepsilon),R,W=a_j) \\ + \frac{2^{56}v\pi^{24}\ln R}{\delta^{38}(\varepsilon-\varepsilon^*)(\ln R - \ln R^*)} + (q-2)S\left(\Omega\left(\phi-\varepsilon^*,\phi+\varepsilon^*,\frac{1}{R^*},W\right)\right).\end{aligned} \tag{1.3}$$

By the inequality in Theorem 1.2, we will immediately have the following.

Theorem 1.2. *For a meromorphic function W (a 1-valued algebroidal function with no branch points) defined by* (1.1) *satisfying*

$$\overline{\lim_{R\to\infty}}\frac{T(r,W)}{\ln^2 R} = \infty, \tag{1.4}$$

it has at least one Nevanlinna direction, that is, there exists $\arg z = \phi_0$, *such that* $\sum_{a\in C\bigcup\{\infty\}}\delta(a,\phi_0) \le 2$ *holds for any finitely many deficient value* a, *where*

$$\delta(a,\phi_0) = 1 - \overline{\lim_{\varepsilon\to 0^+}\lim_{R\to\infty}}\frac{\overline{N}(\Omega(\phi_0-\varepsilon,\phi_0+\varepsilon),R,W=a)}{T(\Omega(\phi_0-\varepsilon,\phi_0+\varepsilon),R,W)} > 0. \tag{1.5}$$

2. Some Lemmas

First, it is easy to prove the following.

Lemma 2.1. *Suppose that* $a, b > 0$, *then there exists* $p, q > 0$, *such that*

$$p + iq = \sqrt{\frac{1}{a-bi}}, \tag{2.1}$$

where

$$p = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2(a^2+b^2)}}, \qquad q = \sqrt{\frac{\sqrt{a^2+b^2}-a}{2(a^2+b^2)}}. \tag{2.2}$$

Lemma 2.2. *Suppose that* $A(z) = \int_0^z (dt/\sqrt{t(1-t^2)})$ *and* $B(z) = ((1-z)/(1+z))i$, *then*

(1) *the mapping* $G_1 = A \circ B(z)$ *maps the unit disc* U *to the square* $Q \triangleq \{z = x + iy \mid 0 < x < 2h, 0 < y < 2h\}$, *where* h *is constant;*

(2) G_1 *maps* $\{|z| < r\}$, *where* $0 < r < 1$, *into a symmetrical convex region in* Q*;*

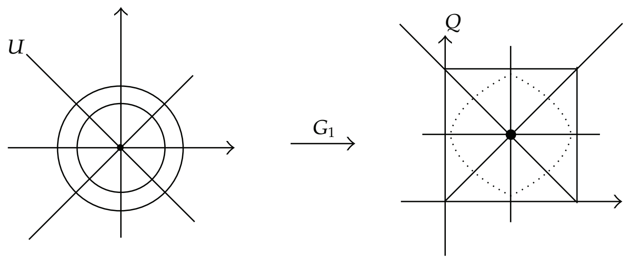

Figure 1

(3) *h in Lemma 2.2 satisfies*

$$h = \int_{\sqrt{2}-1}^{1} \frac{dt}{\sqrt{t(1-t^2)}} = \int_{0}^{\sqrt{2}-1} \frac{dt}{\sqrt{t(1-t^2)}} > 1. \tag{2.3}$$

Proof. Obviously, (1) holds. By the definition of $B(z)$, we have $B(1) = 0$, $B(-1) = \infty$, $B(i) = 1$, $B(-i) = -1$.

In order to compute h, first we prove that G_1 maps $d(0)$, $d(\pi/2)$, $d(\pi/4)$, $d(-\pi/4)$ to four symmetry axes of Q, where $d(\theta) = \{re^{i\theta};\ -1 < r < 1\}$. Let $z = re^{i\theta}$ (θ is fixed), $r \in (-1, 1)$,

$$\begin{aligned} G_1(z) = A \circ B(z) &= \int_0^B \frac{\mathrm{d}t}{\sqrt{t(1-t^2)}} \\ &= -(1+i) \int_1^r \frac{e^{i\theta}\mathrm{d}r}{\sqrt{1-r^4e^{i4\theta}}}. \end{aligned} \tag{2.4}$$

Hence, when $\theta = 0, \pi/4, \pi/2, -\pi/4$, $\arg G_1(re^{i\theta}) = \pi/4, \pi/2, -\pi/4, 0$, respectively, see Figure 1.

Then, we compute h. Since $z = 0$ is the only intersection point of the lines $d(0)$ and $d(\pi/4)$. The center of the square Q, $h + hi$, is that of the curves $G_1(d(0))$ and $G_1(d(\pi/4))$. Then, G_1 conforms 0 onto $h + hi$.

Hence,

$$\begin{aligned} h = G_1\left(e^{i\pi/4}\right) = A\left(\sqrt{2}-1\right) &= \int_0^{\sqrt{2}-1} \frac{dt}{\sqrt{t(1-t^2)}} \\ &= \int_{\sqrt{2}-1}^{1} \frac{dt}{\sqrt{t(1-t^2)}} \\ &> \int_{\sqrt{2}-1}^{1} \frac{dt}{\sqrt{t}} > 1. \end{aligned} \tag{2.5}$$

(2) At last we prove that $G_1(\{|z| < r\})$ is a convex region, see Figure 1.

For a fixed $r \in (0,1)$, by (2.4),

$$G_1\left(re^{i\theta}\right) = \int_1^z \frac{-(1+i)\mathrm{d}z}{\sqrt{1-z^4}} = (1+i)\int_r^1 \frac{dr}{\sqrt{1-r^4}} + \int_0^\theta \frac{(1-i)d\theta}{\sqrt{(r^{-2}-r^2)\cos 2\theta - i(r^{-2}+r^2)\sin 2\theta}}. \tag{2.6}$$

Set $G_1(re^{i\theta}) = x(\theta) + iy(\theta)$. When $\theta \in (0, \pi/4)$, $\cos 2\theta, \sin 2\theta > 0$, by Lemma 2.1,

$$\frac{\partial G_1}{\partial \theta} = \frac{1-i}{\sqrt{(r^{-2}-r^2)\cos 2\theta - i(r^{-2}-r^2)\sin 2\theta}} = (1-i)(p+iq) = (p+q)-(p-q)i \triangleq x' + y'i, \tag{2.7}$$

where $p > q > 0$.

$$\begin{aligned}\frac{\partial^2 G_1}{\partial \theta^2} &= \frac{(1-i)\left[(r^{-2}-r^2)\sin 2\theta + i(r^{-2}+r^2)\cos 2\theta\right]}{\sqrt{\left[(r^{-2}-r^2)\cos 2\theta - i(r^{-2}+r^2)\sin 2\theta\right]^3}} \\ &= \frac{(1-i)(p+qi)\left[(r^{-2}-r^2)\sin 2\theta + i(r^{-2}+r^2)\cos 2\theta\right]}{(r^{-2}-r^2)\cos 2\theta - i(r^{-2}+r^2)\sin 2\theta} \\ &= \frac{(p-q)(r^{-4}-r^4) - 2(p+q)\sin 4\theta}{(r^{-2}-r^2)^2\cos^2 2\theta + (r^{-2}+r^2)^2\sin^2 2\theta} \\ &\quad + i\frac{\left[2(p-q)\sin 4\theta + (p+q)(r^{-4}-r^4)\right)}{(r^{-2}-r^2)^2\cos^2 2\theta + (r^{-2}+r^2)^2\sin^2 2\theta} = x'' + y''i.\end{aligned} \tag{2.8}$$

Hence,

$$\begin{aligned}\frac{dy}{dx} &= \frac{y'}{x'} = -\frac{p-q}{p+q}, \\ \frac{d^2y}{dx^2} &= \frac{y''x' - y'x''}{x'^2} \\ &= \frac{2(r^{-4}-r^4)(p^2+q^2)}{x'^2\left[(r^{-2}-r^2)\cos^2 2\theta + (r^{-2}+r^2)^2\sin 2\theta\right]} > 0.\end{aligned} \tag{2.9}$$

Therefore, the image of $L(r) = \{re^{i\theta};\ 0 < \theta < \pi/4\}$ is a descending convex curve. By the symmetry of square, $G_1(\{|z| = r\})$ is a smooth curve, and $G_1(\{|z| < r < 1\})$ is a convex symmetric figure in the square Q.

Then, we obtain Lemma 2.2. □

Lemma 2.3. *Suppose that mappings*

$$\begin{aligned} C(z) &\triangleq h + i(z-h) \quad (\textit{where } h \textit{ is defined in Lemma 2.2}), \\ D(z,\overline{z}) &\triangleq \frac{x\ln R}{h} + i\frac{y\varepsilon}{h} = \frac{\log R+\varepsilon}{2h}z + \frac{\log R-\varepsilon}{2h}\overline{z}, \\ H(z) &\triangleq e^{z}, \\ G_2(z) &\triangleq C^{-1}\circ D^{-1}\circ H^{-1}(z). \end{aligned} \tag{2.10}$$

Then, the mapping G_2 maps the region $E \triangleq \{1/R < |z| < R\}\bigcap\{|\arg z| \leqslant \varepsilon\}$ into the square Q, and $G_1^{-1}\circ G_2(E^*) \subset \{|z| < r\}$, where $E^* \triangleq \{1/R^* < |z| < R^*\}\bigcap\{|\arg z| \leqslant \varepsilon^*\}$, $R^* \in (2, R)$, $\varepsilon^* \in (0,\varepsilon)$, and

$$0 < 1-r < \min\left\{\frac{(\varepsilon-\varepsilon^*)^2}{2\pi\varepsilon^2}, \frac{(\varepsilon-\varepsilon^*)(\ln R-\ln R^*)}{4\pi\varepsilon\ \ln R}\right\}. \tag{2.11}$$

Proof. The conclusion is equivalent to $G_2(E^*) \subset G_1(\{|z| < r\})$. We prove the lemma by two cases.

Case I. When

$$\frac{\varepsilon^*}{\varepsilon} \geqslant \frac{\ln R^*}{\ln R}, \tag{2.12}$$

then

$$0 \leqslant \arg G_2\left(h - \frac{h\varepsilon^*}{\varepsilon + i(h-h\ \ln R^*/\ln R)}\right) = \arg\left[h\left(\frac{\varepsilon-\varepsilon^*}{\varepsilon} + i\frac{\ln R-\ln R^*}{\ln R}\right)\right] \leqslant \frac{\pi}{4}. \tag{2.13}$$

$G_1(\{|z| < r < 1\})$ is a convex symmetric figure if there exists a $\theta_0 \in (0, \pi/4]$, such that

$$\begin{aligned} \operatorname{Re} G_1\left(re^{i\theta_0}\right) &< h\frac{\varepsilon-\varepsilon^*}{\varepsilon}, \\ \operatorname{Im} G_1\left(re^{i\theta_0}\right) &< h\frac{\ln R-\ln R^*}{\ln R}, \end{aligned} \tag{2.14}$$

then Lemma 2.3 holds, see Figure 2.

In fact, for any $r \in (0,1)$, $\theta \in (0,\pi/4]$,

$$\begin{aligned} G_1\left(re^{i\theta}\right) &= -\int_1^z \frac{(1+i)dz}{\sqrt{1-z^4}} \\ &= \int_1^{e^{i\theta}} \frac{(1+i)dz}{\sqrt{1-z^4}} + \int_{e^{i\theta}}^{re^{i\theta}} \frac{(1+i)dz}{\sqrt{1-z^4}} \triangleq \alpha+\beta, \end{aligned} \tag{2.15}$$

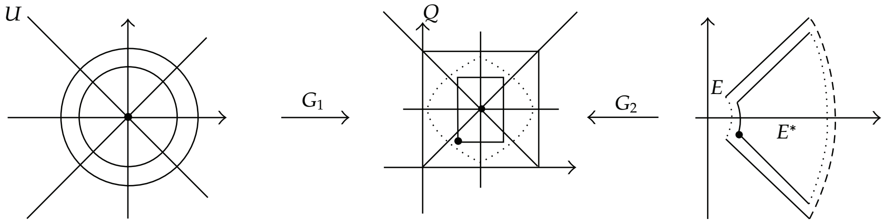

Figure 2

where

$$\begin{aligned}
\alpha &= \int_1^{e^{i\theta}} \frac{(1+i)dz}{\sqrt{1-z^4}} = \int_0^{\theta} \frac{-(1+i)ie^{i\theta}}{\sqrt{1-e^{i4\theta}}} d\theta \\
&= \int_0^{\theta} \frac{d\theta}{\sqrt{\sin 2\theta}} < \frac{\sqrt{\pi}}{2} \int_0^{\theta} \frac{d\theta}{\sqrt{\theta}} = \sqrt{\pi\theta}, \\
\beta &= \int_{e^{i\theta}}^{re^{i\theta}} \frac{(1+i)dz}{\sqrt{1-z^4}} \\
&= (1+i) \int_r^1 \frac{dr}{\sqrt{(1-r^4)\cos 2\theta - i(1+r^4)\sin 2\theta}} \\
&\triangleq (1+i) \int_r^1 \frac{dr}{\sqrt{a-bi}},
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
a &= \left(1-r^4\right)\cos 2\theta < 1, \qquad b = \left(1+r^4\right)\sin 2\theta, \\
a^2+b^2 &= 1+r^8+2r^4\left(\sin^2 2\theta - \cos^2 2\theta\right) \\
&= \left(1-r^4\right) + 4r^4 \sin^2 2\theta \\
&> 4r^4 \sin 2\theta.
\end{aligned} \tag{2.17}$$

By Lemma 2.1, we have

$$\begin{aligned}
\beta &= \int_r^1 (1+i)(p+iq)dr = \int_r^1 [(p-q)+(p+q)i]dr \\
&\triangleq \xi + i\eta,
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
\xi &= \int_r^1 (p-q)\,dr = \int_r^1 \frac{p^2-q^2}{p+q}\,dr \\
&< \int_r^1 \frac{p^2+q^2}{p}\,dr \\
&< \int_r^1 \frac{\sqrt{2}\,dr}{\sqrt{a^2+b^2}} \\
&< \int_r^1 \frac{dr}{r^2\sqrt{2\sin 2\theta}} \\
&< \frac{1-r}{2r\sqrt{\sin 2\theta}} \\
&< \frac{1-r}{2r}\sqrt{\frac{\pi}{\theta}}, \\
\eta &= \int_r^1 (p+q)\,dr < 2\int_r^1 p\,dr \\
&< 2\int_r^1 \frac{dr}{\sqrt[4]{a^2+b^2}} \\
&< \frac{1-r}{r}\sqrt{\frac{\pi}{2\theta}}.
\end{aligned} \tag{2.19}$$

Let

$$\theta_0 = \frac{(\varepsilon-\varepsilon^*)^2}{4\pi\varepsilon^2} < \frac{\pi}{4}. \tag{2.20}$$

Combing (3.1) (*note* that by (3.1), we have $r > \sqrt{2}/2$),

$$\begin{aligned}
\operatorname{Re} G_1\left(re^{i\theta_0}\right) &= \alpha + \xi < \sqrt{\pi\theta_0} + \frac{1-r}{2r}\sqrt{\frac{\pi}{\theta_0}} \\
&< \frac{h(\varepsilon-\varepsilon^*)}{\varepsilon}, \\
\operatorname{Im} G_1\left(re^{i\theta_0}\right) &= \eta < \frac{1-r}{r}\sqrt{\frac{\pi}{2\theta}} \\
&< \frac{1}{2}\frac{\ln R - \ln R^*}{\ln R} \\
&< h\frac{\ln R - \ln R^*}{\ln R}.
\end{aligned} \tag{2.21}$$

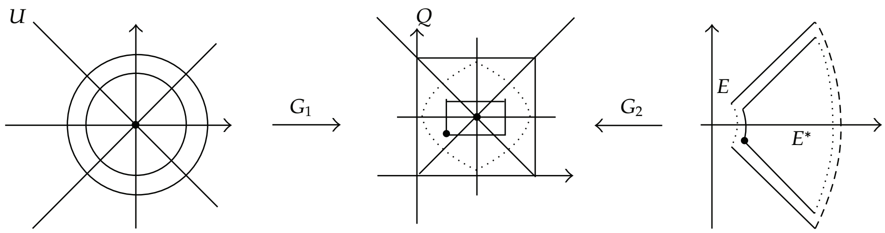

Figure 3

Therefore, a vertex of $G_2(E^*)(h-\varepsilon^* h/\varepsilon, h-h\ln R^*/\ln R) \in G_1(\{|z|<r\})$. By Lemma 2.2, $G_2(E^*) \subset G_1(\{|z|<r\})$.

Case II. When

$$\frac{\varepsilon^*}{\varepsilon} \leq \frac{\ln R^*}{\ln R}, \tag{2.22}$$

since $G_1(\{|z| < r < 1\})$ is a convex symmetric figure, we also have Lemma 2.3, see Figure 3. □

For the convenience of readers, we prove the following lemma again, it can be found in [9].

Lemma 2.4. *(1) Let $G(z) = G_2^{-1} \circ G_1(z)e^{i\phi}$, then*

$$|G_z| + |G_{\overline{z}}| \leqslant \frac{\ln R}{\varepsilon}(|G_z| - |G_{\overline{z}}|). \tag{2.23}$$

(2) Put $s(x,y) = \operatorname{Re} G$, $t(x,y) = \operatorname{Im} G$, then

$$s_x^2 + s_y^2 + t_x^2 + t_y^2 \leqslant \frac{\ln R}{\varepsilon}(s_x t_y - s_y t_x). \tag{2.24}$$

Proof. (1) Since $f \triangleq H^{-1} \circ G_1(z)e^{i\phi}$ and C^{-1} are holomorphic functions, then

$$\left|\overline{f}_z\right| = |f_{\overline{z}}| = \left|\left(C^{-1}\right)_{\overline{D^{-1}}}\right| = 0, \quad \left|\overline{f_{\overline{z}}}\right| = |f_z|. \tag{2.25}$$

For

$$D(z) = \frac{x\ln R}{h} + i\frac{y\varepsilon}{h} \tag{2.26}$$

then

$$D^{-1}\left(f,\overline{f}\right) = \frac{xh}{\ln R} + i\frac{yh}{\varepsilon} = \frac{h\left(f+\overline{f}\right)}{2\ln R} + \frac{h\left(f-\overline{f}\right)}{\varepsilon}. \tag{2.27}$$

Hence,

$$\max\left\{\frac{\left|D_f^{-1}\right| + \left|D_{\overline{f}}^{-1}\right|}{\left|D_f^{-1}\right| - \left|D_{\overline{f}}^{-1}\right|}\right\} = \frac{|\ln R+\varepsilon| + |\ln R-\varepsilon|}{|\ln R+\varepsilon| - |\ln R-\varepsilon|} = \frac{\ln R}{\varepsilon}. \tag{2.28}$$

Thus,

$$\begin{aligned} |G_z| + |G_{\overline{z}}| &= \left|\left(C^{-1}\right)_{D^{-1}} D_f^{-1} f_z\right| + \left|\left(C^{-1}\right)_{D^{-1}} D_{\overline{f}}^{-1} \overline{f}_{\overline{z}}\right| \\ &\leqslant \left|\left(C^{-1}\right)_{D^{-1}} f_z\right| \frac{\ln R}{\varepsilon}\left(\left|D_f^{-1}\right| - \left|D_{\overline{f}}^{-1}\right|\right) \\ &= \frac{\ln R}{\varepsilon}\left(\left|\left(C^{-1}\right)_{D^{-1}} D_f^{-1} f_z\right| - \left|\left(C^{-1}\right)_{D^{-1}} D_{\overline{f}}^{-1} \overline{f}_{\overline{z}}\right|\right). \end{aligned} \tag{2.29}$$

(2) By

$$\begin{aligned} s_x &= s_z + s_{\overline{z}} \\ s_y &= s_z - i s_{\overline{z}}, \end{aligned} \tag{2.30}$$

we have

$$s_z = \frac{s_x - i s_y}{1-i}, \qquad s_{\overline{z}} = \frac{s_y - i s_x}{1-i}. \tag{2.31}$$

Similarly,

$$t_z = \frac{t_x - i t_y}{1-i}, \qquad t_{\overline{z}} = \frac{t_y - i t_x}{1-i}. \tag{2.32}$$

Then,

$$\begin{aligned} |G_z|^2 &= \frac{1}{2}\left[\left(t_y + s_x\right)^2 + \left(t_x - s_y\right)^2\right], \\ |G_{\overline{z}}|^2 &= \frac{1}{2}\left[\left(t_x + s_y\right)^2 + \left(t_y - s_x\right)^2\right]. \end{aligned} \tag{2.33}$$

Therefore,

$$\begin{aligned} s_x^2 + s_y^2 + t_x^2 + t_y^2 &= |G_z|^2 + |G_{\overline{z}}|^2 \\ &\leqslant (|G_z| + |G_{\overline{z}}|)^2 \\ (\text{by } (2.23)) &\leqslant \frac{\ln R}{\varepsilon}\left(|G_z|^2 - |G_{\overline{z}}|^2\right) \\ &\leqslant \frac{\ln R}{\varepsilon}(s_x t_y - s_y t_x). \end{aligned} \tag{2.34}$$

□

Lemma 2.5 (see [10]). *Let F be a connected covering surface on F_0, F_0 is bounded by q different points with radius $\delta \in (0, 1/2)$, then*

$$\max\{0, \rho(F)\} \geqslant (q-2)S - 2^{25}\pi^{11}\delta^{-19}L, \tag{2.35}$$

where L is the length of F and $\rho(F)$ is Euler characteristic of F, $|F|$ is the area of F and

$$S = \frac{|F|}{|F_0|}. \tag{2.36}$$

Lemma 2.6 (see [10]). *Let V be a sphere with radius $1/2$, F_0 be bounded by q different points with radius $\delta \in (0, 1/2)$ and $F_r = W \circ G(F_0)$ then*

$$S = \frac{|F_r|}{|F_0|} = \frac{1}{\pi}\sum_{k=1}^{v}\int_0^r\int_0^{2\pi}\frac{|w_k'|^2(s_x t_y - s_y t_x)r}{\left(1 + |w_k \circ G|^2\right)^2}dr\,d\theta, \tag{2.37}$$

where $s(x,y) = \operatorname{Re} G$, $t(x,y) = \operatorname{Im} G$.

Proof. Suppose that $w_k = u + iv$. Then

$$|F_r| = \sum_{k=1}^{v}\iint_{G(|\tilde{z}|<r)}\frac{1}{\left(1 + |w_k|^2\right)^2}du\,dv, \tag{2.38}$$

where

$$\begin{aligned} du &= (u_s s_x + u_t t_x)dx + (u_s s_y + u_t t_y)dy, \\ dv &= (v_s s_x + v_t t_x)dx + (v_s s_y + v_t t_y)dy. \end{aligned} \tag{2.39}$$

Hence, by the Jacobian determinant, we have

$$\begin{aligned} du\,dv &= \left[\left(u_s^2 + v_s^2\right)(s_x t_y - s_y t_x)\right] dx\,dy, \\ |F_r| &= \sum_{k=1}^{v} \iint_{|\tilde{z}|<r} \frac{(u_s^2 + v_s^2)(s_x t_y - s_y t_x)}{\left(1 + |w_k \circ G|^2\right)^2} dx\,dy \\ &= \sum_{k=1}^{v} \int_0^r \int_0^{2\pi} \frac{|w_k'|^2 (s_x t_y - s_y t_x) r}{\left(1 + |w_k \circ G|^2\right)^2} dr\,d\theta. \end{aligned} \tag{2.40}$$

By $|F_0| = \pi$, we have Lemma 2.6. □

3. Proof of Theorem 1.1

Proof. Set $G(z) = G_2^{-1} \circ G_1(z) e^{i\phi}$. It conforms the unit disc $U = \{|z| < 1\}$ to the sector $E = \{1/R < |W| < R\} \bigcap \{|\arg z - \phi| < \varepsilon\}$ and the interior of $U^* = \{|z| < r\}$ to $E^* = \{1/R^* < |W| < R^*\} \bigcap \{|\arg z - \phi| < \varepsilon^*\}$, where $2 < R^* < R$, $0 < \varepsilon^* < \varepsilon$, and

$$0 < 1 - r < \min\left\{ \frac{(\varepsilon - \varepsilon^*)^2}{2\pi\varepsilon^2}, \frac{(\varepsilon - \varepsilon^*)(\ln R - \ln R^*)}{4\pi\varepsilon \ln R} \right\}. \tag{3.1}$$

Hence $W \circ G$ conforms $\{|z| < r\}$ to the sphere V.

Put $\widetilde{D_r} = \{|\tilde{z}| < r\}$. Then by M. Hurwite Formula, we have

$$\rho\left(\widetilde{D_r}\right) = n\left(r, \widetilde{R}_z\right) - v. \tag{3.2}$$

Put $D_r = \widetilde{D_r} - \{z \mid \prod_{j=1}^{q} (w \circ G(z) - a_j) = 0\}$ and $F_r = W \circ G(F_0)$. Then

$$\begin{aligned} \rho(F_r) = \rho(D_r) &= n\left(r, \widetilde{R}_z\right) - v + \sum_{j=1}^{q} \overline{n}(r, W \circ G = a_j) \\ &\leqslant n\left(1, \widetilde{R}_z\right) - v + \sum_{j=1}^{q} \overline{n}(1, W \circ G = a_j) \triangleq N. \end{aligned} \tag{3.3}$$

By Lemma 2.5, it follows that

$$N \geqslant \rho(F_r) \geqslant (q-2) S(r, W \circ G) - 2^{27}\pi^{11}\delta^{-19} L(r). \tag{3.4}$$

Now we will prove

$$L^2(r) \leqslant 2^4 v \pi r \frac{\ln R}{\varepsilon} \frac{dS(r, W \circ G)}{dr}. \tag{3.5}$$

For any $r \in (0,1)$, $k = 1,2,\ldots,v$ and $\varepsilon > 0$, we have

$$\left|\left|w_k \circ G\left(re^{i\theta}\right)\right| - \left|w_k \circ G^j\right|\right| < \varepsilon, \tag{3.6}$$

where $\theta_j = j\pi/n$ $(j = 1,2,\ldots,2n)$, $\theta \in [\theta_{j-1}, \theta_j]$, $|w_k \circ G^j| = \min\{|w_k \circ G(re^{i\theta})|, \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$.

By (3.6), for any $\theta \in [\theta_{j-1}, \theta_j]$, we have

$$\left|w_k \circ G\left(re^{i\theta}\right)\right|^2 \leqslant \left|w_k \circ G^j\right|^2 + 2\varepsilon\left(\varepsilon + \left|w_k \circ G^j\right|\right). \tag{3.7}$$

Therefore

$$\begin{aligned}\frac{1 + \left|w_k \circ G(re^{i\theta})\right|^2}{1 + \left|w_k \circ G^j\right|^2} &\leqslant 1 + \frac{2\varepsilon^2}{1 + \left|w_k \circ G^j\right|^2} + \frac{2\varepsilon\left|w_k \circ G^j\right|}{1 + \left|w_k \circ G^j\right|^2} \\ &\leqslant \sqrt{2}.\end{aligned} \tag{3.8}$$

Put $w_k = u_k(s,t) + iv_k(s,t)$, $G = s(r,\theta) + it(r,\theta)$. Hence

$$\begin{aligned} L_k(r) &\triangleq \lim_{n\to\infty} \sum_{j=1}^{2n} \frac{\left|w_k \circ G(re^{i\theta_j}) - w_k \circ G(re^{i\theta_{j-1}})\right|}{\sqrt{1 + \left|w_k \circ G(re^{i\theta_j})\right|^2}\sqrt{1 + \left|w_k \circ G(re^{i\theta_{j-1}})\right|^2}} \\ &\leqslant \lim_{n\to\infty} \sum_{j=1}^{2n} \frac{\left\{\left[\int_{\theta_{j-1}}^{\theta_j} ((u_k)_s s_\theta + (u_t)_s t_\theta) d\theta\right]^2 + \left[\int_{\theta_{j-1}}^{\theta_j} ((v_k)_s s_\theta + (v_k)_t t_\theta) d\theta\right]^2\right\}^{1/2}}{1 + \left|w_k \circ G^j\right|^2} \\ &\leqslant \sqrt{2} \lim_{n\to\infty} \sum_{j=1}^{2n} \int_{\theta_{j-1}}^{\theta_j} \frac{\left[((u_k)_s s_\theta + (u_k)_t t_\theta)^2 + ((v_k)_s s_\theta + (v_k)_t t_\theta)^2\right]^{1/2} d\theta}{1 + \left|w_k \circ G_k^j\right|^2} \\ &= 2\int_0^{2\pi} \frac{|w_k'|\left(s_\theta^2 + t_\theta^2\right)^{1/2} d\theta}{1 + |w_k \circ G_k|^2}. \end{aligned} \tag{3.9}$$

By

$$\begin{aligned} s_\theta &= -s_x r \sin\theta + s_y r \cos\theta, \\ t_\theta &= -t_x r \sin\theta + t - t_y r \sin\theta, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} x &= -r\cos\theta, \\ y &= r\sin\theta, \end{aligned} \tag{3.11}$$

we obtain

$$\begin{aligned}
s_\theta^2 + t_\theta^2 &\leqslant 2r^2\left(s_x^2 + s_y^2 + t_x^2 + t_y^2\right), \\
L^2(r) &= \left(\sum_{k=1}^{v} L_k(r)\right)^2 \\
&\leqslant 8\left[\sum_{k=1}^{v}\int_0^{2\pi} \frac{|w_k'|\left(s_x^2 + s_y^2 + t_x^2 + t_y^2\right)^{1/2} r}{1+|w_k\circ G|^2}d\theta\right]^2 \\
(\text{by (2.24)}) &\leqslant 8\frac{\ln R}{\varepsilon}\left[\sum_{k=1}^{v}\int_0^{2\pi} \frac{|w_k'|(s_x t_y - s_y t_x)^{1/2} r}{1+|w_k\circ G|^2}d\theta\right]^2 \\
(\text{Cauchy inequality}) &\leqslant 8v\frac{\ln R}{\varepsilon}\sum_{k=1}^{v}\left[\int_0^{2\pi} \frac{|w_k'|(s_x t_y - s_y t_x)^{1/2} r}{1+|w_k\circ G|^2}d\theta\right]^2 \\
(\text{Schwarz inequality}) &\leqslant 8v\frac{\ln R}{\varepsilon}\sum_{k=1}^{v}\left[\int_0^{2\pi} \frac{|w_k'|^2(s_x t_y - s_y t_x) r}{\left(1+|w_k\circ G|^2\right)^2}d\theta\right]\left[\int_0^{2\pi} r\,d\theta\right] \\
&\leqslant 16v\pi r\frac{\ln R}{\varepsilon}\sum_{k=1}^{v}\int_0^{2\pi} \frac{|w_k'|^2(s_x t_y - s_y t_x) r}{\left(1+|w_k\circ G|^2\right)^2}d\theta \\
(\text{by Lemma 2.6}) &\leqslant 16v\pi r\frac{\ln R}{\varepsilon}\frac{dS(r, W\circ G)}{dr}.
\end{aligned} \tag{3.12}$$

(1) If for all $r' \in (r,1)$

$$S(r, W\circ G) \geqslant \frac{N}{q-2}, \tag{3.13}$$

then by (3.4)

$$\begin{aligned}
\left(S(r', W\circ G) - \frac{N}{q-2}\right)^2 &\leqslant \frac{2^{50}\pi^{22}}{(q-2)^2\delta^{38}}L(r') \\
(\text{by (3.5)}) &\leqslant \frac{2^{54}v\pi^{23}r'\ln R}{(q-2)^2\delta^{38}\varepsilon}\frac{dS(r', W\circ G)}{dr'},
\end{aligned} \tag{3.14}$$

that is,

$$dr' \leqslant \frac{2^{54}v\pi^{23}\ln R}{(q-2)^2\delta^{38}\varepsilon}\frac{dS(r', W\circ G)}{\left(S(r', W\circ G) - N/(q-2)\right)^2}. \tag{3.15}$$

Hence,

$$
\begin{aligned}
1-r &= \int_r^1 dr' \leqslant \frac{2^{54}\upsilon\pi^{23}\ln R}{(q-2)^2\delta^{38}\varepsilon}\int_r^1 \frac{dS(r',W\circ G)}{\left(S(r',W\circ G)-N/(q-2)\right)^2} \\
&\leqslant \frac{2^{54}\upsilon\pi^{23}\ln R}{(q-2)\delta^{38}\varepsilon}\left(\frac{1}{S(r,W\circ G)-N/(q-2)}-\frac{1}{S(R,W\circ G)-N/(q-2)}\right) \\
&< \frac{2^{54}\upsilon\pi^{23}\ln R}{\delta^{38}\varepsilon}\frac{1}{(q-2)S(r,W\circ G)-N}.
\end{aligned}
\tag{3.16}
$$

Therefore

$$
\begin{aligned}
(q-2)S(r,W\circ G) &\leqslant \frac{2^{54}\upsilon\pi^{23}\ln R}{\delta^{38}\varepsilon(1-r)}+N \\
&\leqslant n\left(1,\widetilde{R}_z\right)+\sum_{j=1}^{q}\overline{n}(1,W\circ G=a_j)+\frac{2^{54}\upsilon\pi^{23}\ln R}{\delta^{38}\varepsilon(1-r)}.
\end{aligned}
\tag{3.17}
$$

(2) If there is a $r' \in (r,1)$, such that

$$
(q-2)S(r',W\circ G)-N<0,
\tag{3.18}
$$

then

$$
(q-2)S(r,W\circ G)<(q-2)S(r',W\circ G)<N,
\tag{3.19}
$$

Equation (3.17) holds.

By (3.17) and Lemma 2.3, we have

$$
\begin{aligned}
(q-2)\ S(\Omega(\phi-\varepsilon^*,\phi+\varepsilon^*,R^*,W)) &- (q-2)S\left(\Omega\left(\phi-\varepsilon^*,\phi+\varepsilon^*,\frac{1}{R^*},W\right)\right) \\
&\leqslant (q-2)S(r,W\circ G) \\
(\text{by } (3.1)) &\leqslant n\left(\Omega(\phi-\varepsilon,\phi+\varepsilon),R,\widetilde{R}_z\right)+\sum_{j=1}^{q}\overline{n}(\Omega(\phi-\varepsilon,\phi+\varepsilon),R,W=a_j) \\
&\quad + \frac{2^{56}\upsilon\pi^{24}\ln R}{\delta^{38}(\varepsilon-\varepsilon^*)(\ln R-\ln R^*)}.
\end{aligned}
\tag{3.20}
$$

□

4. Proof of Theorem 1.2

Proof. By the hypothesis of Theorem 1.2, there exists an increasing sequence R_n ($R_n \to \infty$, when $n \to \infty$), such that

$$\lim_{n\to\infty} \frac{T(R_n, W)}{\ln^2 R_n} = +\infty. \tag{4.1}$$

Then, there exist some $\phi_0 \in [0, 2\pi]$, such that for arbitrary $\varepsilon \in (0, \phi_0)$,

$$\overline{\lim_{n\to\infty}} \frac{T(R_n, \phi_0 - \varepsilon, \phi_0 + \varepsilon, W)}{\ln^2 R_n} = +\infty \tag{4.2}$$

holds. We claim that $\arg z = \phi_0$ is the Nevanlinna direction.

Otherwise, for a positive number ε_0, there exist some $a_1, a_2, \ldots, a_q$ $(q \geq 3)$, such that

$$\sum_{j=1}^{q} \delta(a_j, \phi_0) > 2 + 3\varepsilon_0. \tag{4.3}$$

By the definition of $\delta(a_j, \phi_0)$, we have

$$\overline{\lim_{\varepsilon\to 0^+} \lim_{R\to\infty}} \frac{\sum_{j=1}^{q} \overline{N}(\Omega(\phi_0 - \varepsilon, \phi_0 + \varepsilon), R, W = a_j)}{T(\Omega(\phi_0 - \varepsilon, \phi_0 + \varepsilon), R, W)} < q - 2 - 3\varepsilon_0. \tag{4.4}$$

There exists $\varepsilon_1 > 0$, such that for any $\varepsilon \in (0, \varepsilon_1)$,

$$\overline{\lim_{R\to\infty}} \frac{\sum_{j=1}^{q} \overline{N}(\Omega(\phi_0 - \varepsilon, \phi_0 + \varepsilon), R, W = a_j)}{T(\Omega(\phi_0 - \varepsilon, \phi_0 + \varepsilon), R, W)} < q - 2 - 2\varepsilon_0. \tag{4.5}$$

Hence, for $\{R_n\}$ defined earlier, when n is sufficiently large,

$$\begin{aligned} &\sum_{j=1}^{q} \overline{N}(\Omega(\phi_0 - \varepsilon, \phi_0 + \varepsilon), R_n, W = a_j) \\ &\quad < (q - 2 - \varepsilon_0) T(\Omega(\phi_0 - \varepsilon, \phi_0 + \varepsilon), R_n, W). \end{aligned} \tag{4.6}$$

By Theorem 1.1, we have

$$\begin{aligned}
&(q-2)S\left(\Omega\left(\phi-\frac{\varepsilon}{2},\phi+\frac{\varepsilon}{2}\right),R,W\right)\\
&\quad-(q-2)S\left(\Omega\left(\phi-\frac{\varepsilon}{2},\phi+\frac{\varepsilon}{2},1,W\right)\right)\\
&\leqslant\sum_{j=1}^{q}\overline{n}(\Omega(\phi-\varepsilon,\phi+\varepsilon),2R,W=a_j)\\
&\quad+\frac{2^{52}\pi^{24}\ln 2R}{\delta^{38}\varepsilon\ln 2}.
\end{aligned}\tag{4.7}$$

Hence,

$$\begin{aligned}
&(q-2)T\left(\Omega\left(\phi-\frac{\varepsilon}{2},\phi+\frac{\varepsilon}{2}\right),R_n,W\right)\\
&\quad\leqslant\sum_{j=1}^{q}\overline{N}(\Omega(\phi-\varepsilon,\phi+\varepsilon),R_n,W=a_j)\\
&\quad\quad+\frac{2^{54}\pi^{24}\ln^2 2R_n}{\delta^{38}(\varepsilon-\varepsilon^*)\ln 2}+O(1)\\
&\quad<(q-2-\varepsilon_0+O(1))\ln^2 R_n.
\end{aligned}\tag{4.8}$$

Hence,

$$\overline{\lim_{n\to\infty}}\frac{T(\Omega(\phi-\varepsilon/2,\phi+\varepsilon/2),R_n,W)}{\ln^2 R_n}<O(1),\tag{4.9}$$

which contradicts (4.2). Therefore, Theorem 1.2 holds. □

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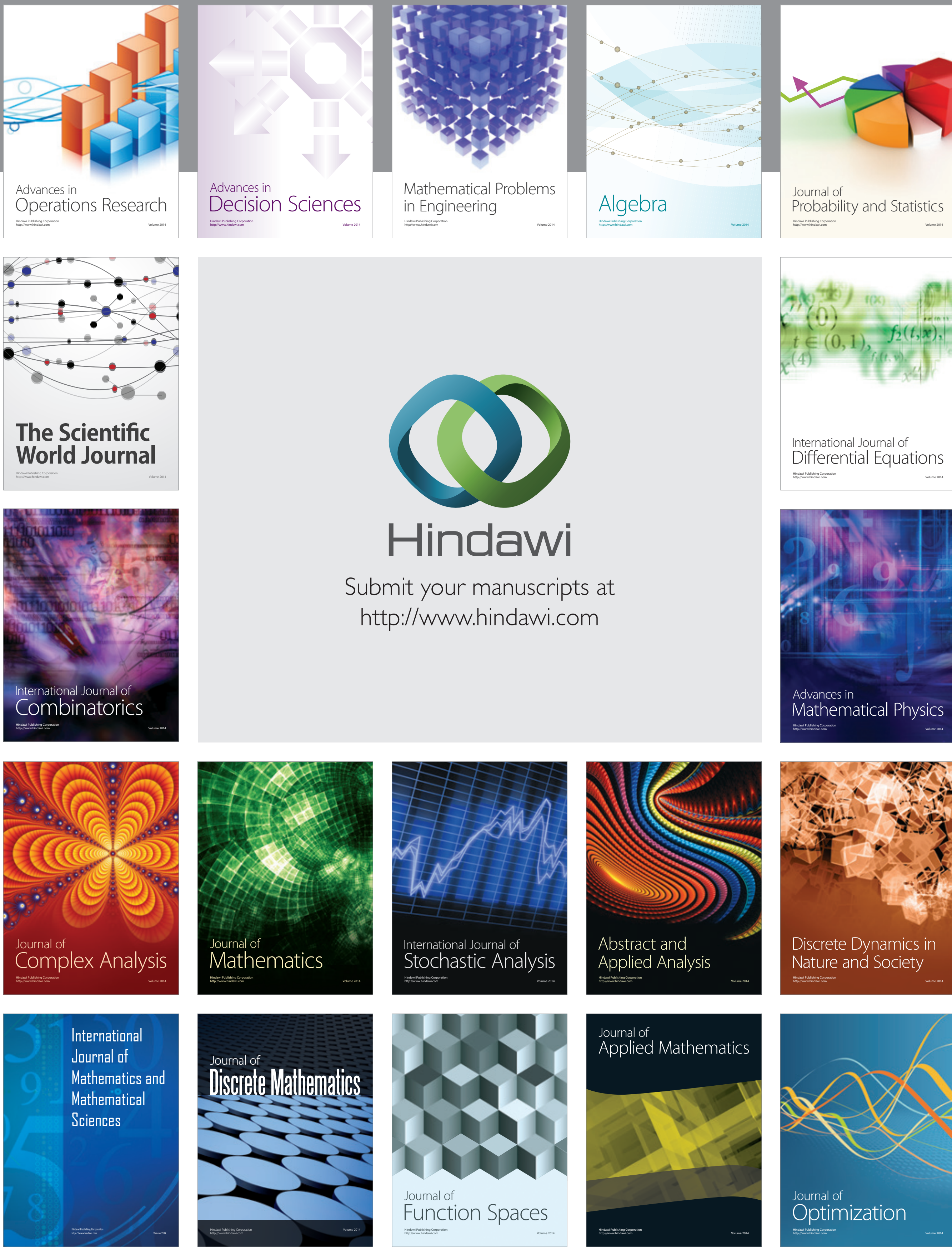
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