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Research Article

Global Analysis for Rough Solutions to the Davey-Stewartson System

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The global well-posedness of rough solutions to the Cauchy problem for the Davey-Stewartson system is obtained. It reads that if the initial data is in H^s with $s > 2/5$, then there exists a global solution in time, and the H^s norm of the solution obeys polynomial-in-time bounds. The new ingredient in this paper is an interaction Morawetz estimate, which generates a new space-time $L^4_{t,x}$ estimate for nonlinear equation with the relatively general defocusing power nonlinearity.

1. Introduction

The Davey-Stewartson system has their origin in fluid mechanics, where it appears as mathematical models for the evolution of weakly nonlinear water waves having one predominant direction of travel, but in which the wave amplitude is modulated slowly in two horizontal directions (see [1]). In dimensionless they read as the following system for the (complex) amplitude $u(t,x,y)$ and the (real) mean velocity potential $v(t,x,y)$

$$\begin{gathered} iu_t + u_{xx} + \mu u_{yy} = a|u|^2 u + buv_x, \\ cv_{xx} + v_{yy} = \left(|u|^2\right)_x, \end{gathered} \tag{1.1}$$

where $i = \sqrt{-1}$, $u = u(t,x,y) : [0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $0 < T \le +\infty$; the parameters μ, a, b, c are real constants. According to signs of the μ and c, these systems may be classified as

$$\text{elliptic-elliptic}: \quad \mu > 0, c > 0, \tag{1.2}$$

$$\text{elliptic-hyperbolic}: \quad \mu > 0, c < 0, \tag{1.3}$$

$$\text{hyperbolic-ellipti}: \quad \mu < 0, c > 0, \tag{1.4}$$

$$\text{hyperbolic-hyperbolic}: \quad \mu < 0, c < 0. \tag{1.5}$$

In the last two decades, the Cauchy problem for the Davey-Stewartson system (1.1) has focused on intense mathematical research. In 1990, Ghidaglia and Saut [2] established the local well-posedness for the Cauchy problem of (1.1) in the cases of (1.2)–(1.4). It reads that for $u_0 \in H^1(\mathbb{R}^2)$, the systems (1.1) have a local solution in time. Hayashi and Hirata [3] studied the initial value problem to the Davey-Stewartson system for the elliptic-hyperbolic case (1.3) in the usual Sobolev space, they proved local existence and uniqueness for the initial data in $H^{5/2}$ $(\mathbb{R}^2)$ whose L^2 norm is sufficiently small. Tsutsumi [4] obtained the L^p-decay estimates of solutions to the systems (1.1) in the elliptic-hyperbolic case (1.3). Hayashi and Saut [5] and Linares and Ponce [6] studied some generalized Davey-Stewartson systems in different spaces, and their main tools are $L^p - L^q$ estimates of solutions to linear Schrödinger equations. These estimates are usually named generalized Strchartz inequality. Guo and Wang [7] studied the Cauchy problem for a generalized Davey-Stewartson system in the elliptic-elliptic case (1.2), and they proved the global well-posedness results for initial data u_0 in $H^s(1 \le s \le 2)$. Recently, Shu and Zhang [8] and Gan and Zhang [9] obtained the sharp conditions of global existence for Davey-Stewartson system in the elliptic-elliptic case (1.2) by constructing a type of cross-constrained variational problem and establishing so-called cross-invariant manifolds generated by the evolution flow. Zhang and Zhu [10] obtained a more precisely sharp criteria of blow-up and global existence. C. Sulem and P. L. Sulem [11] obtained some numerical observations on blow-up solutions. Richards [12] showed the mass concentration phenomenon of blow-up solutions. Li et al. [13] obtained some dynamical properties of blow-up solutions. Recently, Babaoglu and Erbay [14] proposed a generalized Davey-Stewartson system, which was studied in [15–17].

In this paper, we consider the Cauchy problem of (1.1) in the elliptic-elliptic case (without loss of generality, we may take $\lambda = \mu = c = 1$ for simplify),

$$iu_t + u_{xx} + u_{yy} = a|u|^2 u + buv_x, \tag{1.6}$$

$$v_{xx} + v_{yy} = \left(|u|^2\right)_x, \tag{1.7}$$

$$u(0,x) = u_0(x). \tag{1.8}$$

As is well known, the system (1.6)–(1.8) enjoys two useful conservation laws: one is the energy conservation law:

$$H(u,v) = \int_{\mathbb{R}^2} \left(\frac{|u_x|^2 + |u_y|^2 + a|u|^4 + b\left(v_x^2 + v_y^2\right)}{2} \right) dx\, dy, \tag{1.9}$$

and the other is the mass conservation law:

$$M(u) = \int_{\mathbb{R}^2} |u|^2 dx\, dy. \tag{1.10}$$

For details one can see Ghidaglia and Saut [2]. Moreover, one can easily establishes

$$\|u(T)\|_{H^s} \leq C(s, \|u_0\|_{H^s}, T) \tag{1.11}$$

for $s = 1$ (with bounds uniformly in T), and with some additional arguments one can deduce the same claim for $s > 1$. The mass conservation law (1.10) also gives (1.11) for $s = 0$, but unfortunately this does not immediately imply any results for $s > 0$ except in the small mass case.

To make the statement more precise, we denote u, v as the solutions of (1.7)–(1.9). It follows from (1.8) that

$$v_x = \mathcal{F}^{-1}\frac{\xi_1^2}{|\xi|^2}\mathcal{F}|u|^2, \tag{1.12}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inversion. For brevity we denote

$$E(\psi) = \mathcal{F}^{-1}\frac{\xi_1^2}{|\xi|^2}\mathcal{F}\psi. \tag{1.13}$$

Combining (1.8) and (1.9), (1.6)–(1.8) are changed to

$$iu_t + \Delta u = a|u|^2 u + bE\left(|u|^2\right)u, \tag{1.14}$$

$$u(0, x) = u_0(x). \tag{1.15}$$

It is conjectured that the system (1.14)-(1.15) is globally well posed in H^s for all $s \geq 0$ and in particular (1.11) holds for all $s > 0$. This conjecture remains open now. In this paper we aim to prove that the Cauchy problem for (1.1) is globally well posed below the energy norms. That is, we will prove the global well-posedness for initial data $u_0 \in H^s(\mathbb{R}^2)$ with $s < 1$ sufficiently close to one, then we meet the obstacle that there is no conservation law. Indeed, if the initial data is in $H^1(\mathbb{R}^2)$ then it is bounded in $H^1(\mathbb{R}^2)$ for all time and hence the H^s $(s > 1)$ norm is similarly bounded, but if the initial data is only in H^s $(s < 1)$ then the $H^1(\mathbb{R}^2)$ norm may be infinite, and also the conservation of the Hamiltonian appears to be useless. Conservation of the L^2 norm also appears to be unhelpful for this particular problem.

For solutions below the energy threshold the first result was established by Bourgain for nonlinear Schrödinger equation with critical nonlinearity in space dimension two (see [18]). Bourgain came up with the idea of introducing a large frequency parameter N by dividing the solution into the low-frequency portion u_{low} (when $|\xi| \leq N$) and the high frequency portion u_{high} (when $|\xi| \geq N$). The main tool is an extrasmoothing estimate, which shows that if the high frequencies would be merely in $H^s(\mathbb{R}^2)$ for some $s < 1$, then interactions arising from high frequencies were significantly smooth. In fact, they were in the energy class H^1. Moreover, if we denote S_t as the nonlinear flow and $S(t) = e^{it\Delta}u_0$ is the linear group, Bourgain's method shows addition that $(S_t - S(t))u_0 \in H^1(\mathbb{R}^2)$ for all time provided $u_0 \in H^s$, $s > 3/5$. Thus, he showed that the solution is globally well posed with initial data in $H^s(\mathbb{R}^2)$ for any $s > 3/5$. Recently, Kenig, Ponce, Fonseca, Ginibre, Molinet, Pecher [19–23], and Miao and Zhang [24] have developed this methods to study different evolution systems.

Another improvement was given by Colliander et al. in [25, 26], where the authors used the "I-method" that we state below. If $s < 1$, then the energy is infinite and one cannot compare the H^s norm of the solution $u(t)$ with the energy. In order to overcome this difficulty, we introduce the following multiplier I_N:

$$\widehat{I_N f}(\xi) := m(\xi) f(\xi), \tag{1.16}$$

where $m(\xi) := \eta(\xi/N)$, η is a smooth, radial, nonincreasing in $|\xi|$ such that: $\eta(\xi) = 1,\ |\xi| \leq 1$; $\eta(\xi) = (1/|\xi|)^{1-s},\ |\xi| \geq 2$, and $N \gg 1$ is a dyadic number playing the role of a parameter to be chosen. Then we plug this multiplier into the energy which generates to so-called modified energy:

$$H(I_N u) = \int_{\mathbb{R}^2} \left(|I_N u_x|^2 + \left|I_N u_y\right|^2 + \frac{\left(a|I_N u|^4 + bE\left(|I_N u|^2\right) I_N u^2\right)}{2} \right) dx\, dy. \tag{1.17}$$

Note that if $u(t) \in H^s(\mathbb{R}^2)$ then $H(I_N u) < \infty$. Note also that as N goes to infinity, the multiplier I "approaches" the identity operator. Therefore the variant of this smoothed energy is expected to be slow as N goes to infinity. This is the "I-method", originally invented by Colliander et al. [25] to prove global existence for semilinear Schrödinger equations with rough data.

In this paper we design it for the Davey-Stewartson system. The main purpose of this paper is to study that we can lower the value of s to what extent which can also grantees the global existence. In this paper we will prove the following.

Theorem 1.1. *The Cauchy problem* (1.14)-(1.15) *is globally well posed for all* $u_0 \in H^s(\mathbb{R}^2)$, $s > 2/5$ *and* $a + b > 0$. *Moreover, the solution satisfies the following estimate:*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq C(1+T)^{(3s(1-s))/(2(5s-2))+}, \tag{1.18}$$

where the constant C *depends only on the index* s *and* $\|u_0\|_{L^2}$.

Remark 1.2. We view this result as an incremental step towards the conjecture that (1.14)-(1.15) is globally well posed in $H^s(\mathbb{R}^2)$ for all $s \geq 0$.

Remark 1.3. We improve the results obtained by Shen and Guo [27], in which they demonstrated the global existence for $s > 4/7$ for the Cauchy problem (1.14)-(1.15). The technique in their proof mainly depends on the Fourier restriction norm method of Bourgain by showing a generalized estimates of Strichartz type and splitting the data into low- and high-frequency parts. The new ingredient in our proof is a priori interaction Morawetz-type estimate, which generates a new space-time $L^4_{x,t}$ estimates for the "approximate solution" Iu to the nonlinear equation with the relatively general defocusing power nonlinearity, and this technique is motivated by the work given by Colliander et al. in [26].

Remark 1.4. It is worth to remark that Dodson [28–30] proves a frequency-localized interaction Morawetz estimate similar to the estimate made in [31] for considering an L^2-critical initial value problem for cubic nonlinear Schrodinger equation. The major difference between the cubic nonlinear Schrodinger equation and the elliptic-elliptic Davey-Stewartson system (1.14) is the singular integral operator $E(|u|^2)$ in (1.14), which may result in some new difficulties to establish the corresponding frequency localized interaction Morawetz estimate. We hope to solve this problem in a forthcoming paper from the arguments derived by Dodson.

2. Notations and Preliminaries

In this paper, we will often use the notation $A \lesssim B$ whenever there exist some constants K such that $A \leq KB$. Similarly, we will use $A \sim B$ if $A \lesssim B \lesssim A$. We use $A \ll B$ if $A \leq cB$ for some small constant $c > 0$. We use $k\pm$ to denote the real number $k \pm \varepsilon$ for any sufficiently small $\varepsilon > 0$. $\Re z$ and $\Im z$ are the real part and imaginary part of the complex number z, respectively.

We use $\mathcal{S}(\mathbb{R}^4)$ to denote the Schwartz space and $\mathcal{S}'(\mathbb{R}^4)$ to denote its topological dual space. We use $L_x^r(\mathbb{R}^2)$ to denote the usual Lebesgue space of functions $f: \mathbb{R}^2 \to \mathbb{C}$ whose norm

$$\|f\|_{L_x^r} := \left(\int_{\mathbb{R}^2} |f|^r dx\right)^{1/r} \tag{2.1}$$

is finite, with the usual modification in the case $r = \infty$. We also define the space-time space $L_t^q L_x^r$ by

$$\|u\|_{L_{t\in J}^q L_x^r} := \left(\int_J \|u\|_{L_x^r}^q dt\right)^{1/q} \tag{2.2}$$

for any space-time slab $J \times \mathbb{R}^2$, with the usual modification when either q or r are infinity. When $q = r$ we abbreviate $L_t^q L_x^q$ by $L_{t,x}^q$.

Definition 2.1. A pair of exponent (q, r) is called admissible in $\mathbb{R}^2$ if

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \leq q,\ r \leq \infty. \tag{2.3}$$

We recall the known Strichartz estimates [21] (and the reference therein).

Proposition 2.2. *Let (q, r) and $(\tilde{q}, \tilde{r})$ be any two admissible pairs*

$$\begin{gathered} iu_t + \Delta u - f(x,t) = 0, \quad (t,x) \in J \times \mathbb{R}^2 \\ u(x,0) = u_0(x). \end{gathered} \tag{2.4}$$

Then one has the estimate

$$\|u\|_{L_t^q L_x^r(J\times\mathbb{R}^2)} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)} + \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(J\times\mathbb{R}^2)} \tag{2.5}$$

with the prime exponents denoting Hölder dual exponents.

We also define the fractional differential operator $|\nabla|^{\alpha}$ for any real α by

$$\widehat{|\nabla|^{\alpha}u}(\xi) := |\xi|^{\alpha}\widehat{u}(\xi) \tag{2.6}$$

and analogously

$$\widehat{\langle\nabla\rangle^{\alpha}u}(\xi) := \langle\xi\rangle^{\alpha}\widehat{u}(\xi), \tag{2.7}$$

where $\langle a\rangle := \sqrt{1+|a|^2}$. We then define the homogeneous Sobolev space $\dot{H}^s$ and the inhomogeneous Sobolev space H^s by

$$\|u\|_{\dot{H}^s} = \||\nabla|^s u\|_{L_x^2}; \qquad \|u\|_{H^s} = \|\langle\nabla\rangle^s u\|_{L_x^2}. \tag{2.8}$$

We also need some Littlewood-Paley theory. Specifically, let $\varphi(\xi)$ be a smooth bump supported in $|\xi| \le 2$ and equalling one on $|\xi| \le 1$. For each number $N \in 2^{\mathbb{Z}}$ we define the Littlewood-Paley operators:

$$\begin{aligned}
\widehat{P_{\le N}f}(\xi) &:= \varphi\left(\frac{\xi}{N}\right)\widehat{f}(\xi),\\
\widehat{P_{>N}f}(\xi) &:= \left[1-\varphi\left(\frac{\xi}{N}\right)\right]\widehat{f}(\xi),\\
\widehat{P_N f}(\xi) &:= \left[\varphi\left(\frac{\xi}{N}\right)-\varphi\left(\frac{2\xi}{N}\right)\right]\widehat{f}(\xi).
\end{aligned} \tag{2.9}$$

Similarly, we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\le N}$ for $P_{\le N}f$ and similarly for the other operators. We recall the following standard Bernstein and Sobolev type inequalities.

Lemma 2.3. *For any $1 \le 2 \le q \le \infty$ and $s > 0$, one has*

$$\begin{aligned}
\|P_{\ge N}f\|_{L_x^p} &\lesssim N^{-s}\||\nabla|^s P_{\ge N}f\|_{L_x^p},\\
\||\nabla|^s P_{\le N}f\|_{L_x^p} &\lesssim N^{s}\|P_{\le N}f\|_{L_x^p},\\
\||\nabla|^{\pm s} P_N f\|_{L_x^p} &\sim N^{\pm s}\|P_N f\|_{L_x^p},\\
\|P_{\le N}f\|_{L_x^q} &\lesssim N^{1/p-1/q}\|P_{\le N}f\|_{L_x^p},\\
\|P_N f\|_{L_x^q} &\lesssim N^{1/p-1/q}\|P_N f\|_{L_x^p}.
\end{aligned} \tag{2.10}$$

We collect the basic properties of I_N into the following.

Lemma 2.4. *Let $1 < p < \infty$ and $0 \le s < 1$. Then*

$$\|I_N f\|_{L^p} \lesssim \|f\|_{L^p}, \tag{2.11}$$

$$\||\nabla|^s P_{>N} f\|_{L^p} \lesssim N^{s-1}\|\nabla I_N f\|_{L^p}, \tag{2.12}$$

$$\|f\|_{H_x^s} \lesssim \|I_N f\|_{H_x^1} \lesssim N^{1-s}\|f\|_{H_x^s}. \tag{2.13}$$

Proof. For the proof one can see Colliander et al. [25]. □

Now we define the Strichartz norm of functions u

$$\|u\|_{S_T^0} := \sup_{(q,r)\text{ admissible}} \|u\|_{L^q_{t\in[0,T]}L^r_x}. \tag{2.14}$$

Then we introduce the following bilinear smoothing property due to Bourgain [18].

Lemma 2.5. *Let $\psi \in L^2(\mathbb{R}^2)$ such that*

$$\psi_1 = P_{N_j}\psi, \qquad \psi_2 = P_{N_k}\psi. \tag{2.15}$$

Then, for $N_j \le N_k$, the following inequality holds:

$$\left\|e^{it\Delta}\psi_1 \cdot e^{it\Delta}\psi_2\right\|_{L^2_{t,x}(\mathbb{R}^2\times\mathbb{R})} \le C\left(\frac{N_j}{N_k}\right)^{(1/2)-} \|\psi_1\|_2\|\psi_2\|_2. \tag{2.16}$$

That is to say, suppose u solves (1.15)–(1.18) on the time interval $[0,T]$. Let $u_j = P_{N_j}u$, for $j = 1,2$ with $N_1 > N_2$. Then

$$\|u_1u_2\|_{L_T^2L_x^2} \le C\left(\frac{N_2}{N_1}\right)^{(1/2)-} \|u\|^2_{S_T^0}. \tag{2.17}$$

The estimate (2.17) will be also valid if u_j is replaced by $\overline{u_j}$.

We also have the local well posedness result.

Proposition 2.6. *Let us define quantity*

$$\mu([0,T]) := \int_0^T\int_{\mathbb{R}^2} |I_N u|^4 dx\, dt. \tag{2.18}$$

If $\mu([0,T]) < \mu_0$, where μ_0 is some universal constant then for any $s > 0$ the initial value problem (1.15)–(1.18) is locally well-posed and the following estimate is true:

$$Z_I([0,T]) := \sup_{(q,r)\text{ admissible}} \|\langle\nabla\rangle I_N u\|_{L_t^qL_x^r([0,t]\times\mathbb{R}^2)} \lesssim \|\langle\nabla\rangle I_N u_0\|_{L^2}. \tag{2.19}$$

Proof. The proof is standard, see for example [25, 26]. □

Remark 2.7. A modification (2.17) follows using the space-time estimate (2.19): For $N_1 > N_2$, and for solution u of (1.15)–(1.18) satisfies

$$\| \langle \nabla \rangle I u_{N_1} \langle \nabla \rangle I u_{N_2} \|_{L_T^2 L_x^2} \leq C \left(\frac{N_2}{N_1} \right)^{(1/2)-} \| \langle \nabla \rangle I u \|_{S_T^0}^2. \tag{2.20}$$

3. Almost Conservation Laws

In this section we prove the almost conservation of the modified energy $H(I_N u(t))$.

Proposition 3.1. *If the initial data $u_0 \in H^s$ with $s > 1/4$, and $u(t,x)$ solves* (1.14)-(1.15) *for all $t \in [0,T]$ where T is the time that Proposition 2.6 applies, then*

$$\sup_{t \in [0,T]} |H(I_N u(t))| \leq |H(I_N u(0))| + C N^{-(3/2)-} \| I_N \langle \nabla \rangle u(0) \|_{L_x^2}^4 + C N^{-2-} \| I_N \langle \nabla \rangle u(0) \|_{L_x^2}^6. \tag{3.1}$$

In particular when $\| I_N \langle \nabla \rangle u(0) \|_{L_x^2} \lesssim 1$ one has

$$\sup_{t \in [0,T]} |H(I_N u(t))| \lesssim N^{-(3/2)-}. \tag{3.2}$$

Proof. In light of (2.19), it suffices to control the energy increment $|H(I_N u(t)) - H(I_N u(0))|$ for $t \in [0,T]$ in terms of $Z_I([0,T])$. Applying the I_N operator to the system (1.14)-(1.15):

$$\begin{gathered} i I_N u_t + \Delta I_N u = a I_N \left(|u|^2 u \right) + b I_N \left(E \left(|u|^2 \right) u \right), \\ I_N u(0,x) = I_N u_0(x). \end{gathered} \tag{3.3}$$

From now on, we abbreviate I_N as I for simplicity. An elementary calculation shows that $|H(Iu(t)) - H(Iu(0))|$ is controlled by the sum of the space-time integrals:

$$H_1 = \left| \int_0^t \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \widehat{\Delta I \overline{u}}(\xi_1) \widehat{Iu}(\xi_2) \widehat{I\overline{u}}(\xi_3) \widehat{Iu}(\xi_4) \right|, \tag{3.4}$$

$$H_2 = \left| (a+b) \int_0^t \int_{\xi_1'+\xi_4+\xi_5+\xi_6=0} \left(1 - \frac{m(\xi_1')}{m(\xi_4)m(\xi_5)m(\xi_6)} \right) I \left(\widehat{|u|^2 \overline{u}} \right) (\xi_1') \widehat{Iu}(\xi_4) \widehat{I\overline{u}}(\xi_5) \widehat{Iu}(\xi_6) \right|. \tag{3.5}$$

Here we used the properties of operator E for $1 < p < \infty$: (i) $E \in \mathcal{L}(L^p, L^p)$, where $\mathcal{L}(L^p, L^p)$ denotes the space of bounded linear operators from L^p to L^p; (ii) if $u \in H^s$, then $E(u) \in H^s$, $s \in \mathbb{R}$.

We estimate H_1 first. We use u_{N_j} to denote $P_{N_j}u$. When ξ_j is dyadically localized to $\{|\xi| \sim N_j\}$ and we will write $m(\xi_j)$ by m_j. The analysis will not rely upon the complex conjugate structure in the left side of (3.4). Thus, there is symmetry under the interchange of the indices 2–4, and We may assume that $N_2 \geq N_3 \geq N_4$.

Case I. ($N \gg N_2$). Since the convolution hypersurface is $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$, we have $N_1 \ll N$ as well. Hence we get $m(\xi_2 + \xi_3 + \xi_4)/m(\xi_2)m(\xi_3)m(\xi_4) = 1$, and the bound holds trivially.

Case II. $N_2 \gtrsim N \gg N_3 > N_4$, for $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$, we have $N_1 \sim N_2$. Applying the mean value theorem, we deduce that

$$1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} = \frac{m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)} \leq \frac{\nabla m(\xi_2)(\xi_3 + \xi_4)}{m(\xi_2)} \leq \frac{N_3}{N_2}. \tag{3.6}$$

Moreover, H_1 is controlled by

$$\frac{N_3}{N_2}\left(\frac{N_3}{N_1}\right)^{1/2}\left(\frac{N_4}{N_2}\right)^{1/2}\frac{\langle N_1\rangle}{\langle N_2\rangle\langle N_3\rangle\langle N_4\rangle}\|Iu\|^4_{Z_I([0,T])} \leq N^{-(3/2)-}(N_1N_2\langle N_3\rangle\langle N_4\rangle)^{0-}Z_I([0,T])^4. \tag{3.7}$$

Case III: ($N_2 \geq N_3 \geq N$). We have the bound on the symbol:

$$\left|1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right| \leq \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}. \tag{3.8}$$

If $N_1 \sim N_2 \geq N_3 \gtrsim N$, then we bounded H_1 by renormalizing the derivatives and multiplier, paring $u_{N_1}u_{N_3}$ and $u_{N_2}u_{N_4}$ and using Lemma 2.5:

$$\frac{m_1}{m_2m_3m_4}\left(\frac{N_3}{N_1}\right)^{1/2}\left(\frac{N_4}{N_2}\right)^{1/2}\frac{\langle N_1\rangle}{\langle N_2\rangle\langle N_3\rangle\langle N_4\rangle}Z_I([0,T])^4. \tag{3.9}$$

We write this bound as

$$\frac{m_1}{m_2N_2^{1/2}m_3N_3^{1/2}m_4N_4^{1/2}N_1^{1/2}}Z_I([0,T])^4. \tag{3.10}$$

Since $m(x)$ is bounded from above by 1 and $m(x)\langle x\rangle^p$ for $p > 1 - s$ is nondecreasing and bounded from above by 1, for $s \geq 1/2$, we bound

$$\frac{m_1}{m_2N_2^{1/2}m_3N_3^{1/2}m_4N_4^{1/2}N_1^{1/2}} \leq \frac{1}{N^{(3/2)+}}, \tag{3.11}$$

here we used the fact that $m_i N_i^{1/2} \geq m(N)N^{1/2} = N^{1/2}$ for $i = 2, 3$. For $s < 1/2$, by using the definition of $m(\xi)$:

$$\frac{m_1}{m_2 N_2^{1/2} m_3 N_3^{1/2} m_4 N_4^{1/2} N_1^{1/2}} \leq \frac{N_2^{1/2-s} N_3^{1/2-s} N_4^{1/2-s}}{N^{2-2s} N_1^{1/2} N_1^{1-s}} \leq \frac{N_1^{1/2-2s}}{N^{2-2s} N_1^{1/2}}. \tag{3.12}$$

Using the facts $1/2 > s > 1/4$ and $N_2 \geq N_3 \geq N$,

$$\frac{N_1^{1/2-2s}}{N^{2-2s} N_1^{1/2}} \leq N^{-(3/2)-}. \tag{3.13}$$

Hence

$$H_1 \leq N^{-(3/2)-} (N_1 N_2 N_3 \langle N_4 \rangle)^{0-} Z_I([0,T])^4. \tag{3.14}$$

If $N_2 \sim N_3 \geq N$, then paring $u_{N_1} u_{N_2}$ and $u_{N_3} u_{N_4}$ and using Lemma 2.5 again, a similar analysis leads to the bound:

$$\begin{aligned} &\frac{m_1}{m_2 m_3 m_4} \left(\frac{N_1}{N_2}\right)^{1/2} \left(\frac{N_4}{N_3}\right)^{1/2} \frac{\langle N_1 \rangle}{\langle N_2 \rangle \langle N_3 \rangle \langle N_4 \rangle} Z_I([0,T])^4 \\ &\quad \leq N^{-3/2-} (N_1 N_2 N_3 \langle N_4 \rangle)^{0-} Z_I([0,T])^4. \end{aligned} \tag{3.15}$$

Now we turn to give the bound for the term H_2. it required 6-linear estimate for (3.5). We write m_{123} to denote $m(\xi_1 + \xi_2 + \xi_3)$ and use N_{123} to denote the size of $\xi_1 + \xi_2 + \xi_3$. By symmetry, we may assume $N_4 \geq N_5 \geq N_6$. We carry out a case by case analysis.

Case I. ($N \gg N_4$). On $\xi_1' + \xi_4 + \xi_5 + \xi_6 = 0$, this forces $N_{123} \sim N_4$ as well. In this case, $m(\xi_2 + \xi_3 + \xi_4)/m(\xi_2)m(\xi_3)m(\xi_4) = 1$, and the bound holds trivially.

Case II. ($N_4 \gtrsim N \geq N_5$). On $\xi_1' + \xi_4 + \xi_5 + \xi_6 = 0$, $N_{123} \sim N_4$ in this case. By the mean value theorem,

$$\left| \left[1 - \frac{m_{123}}{m_4 m_5 m_6} \right] \right| = \left| \frac{m_4 - m_{456}}{m_4} \right| \leq \frac{N_5}{N_4}. \tag{3.16}$$

Applying the Cauchy-Schwartz inequality and the above multiplier bound to (3.5), we deduce that

$$(N_4 \langle N_5 \rangle)^{-1} \frac{N_5}{N_4} \| P_{123} I(u_{N_1} u_{N_2} u_{N_3}) I u_{N_6} \|_{L^2_{t,x}} \| I \langle \nabla \rangle u_{N_4} I \langle \nabla \rangle u_{N_5} \|_{L^2_{tx}}. \tag{3.17}$$

By Hölder inequality and Lemma 2.5, we control the above expression by

$$(N_4 \langle N_5 \rangle)^{-1} \left(\frac{N_5}{N_4}\right)^{3/2} \| P_{123} I(u_{N_1} u_{N_2} u_{N_3}) \|_{L^2_{t,x}} \| I u_{N_6} \|_{L^\infty_{t,x}} Z_I([0,t])^2. \tag{3.18}$$

By the Sobolev's inequality, we have

$$\|Iu_{N_6}\|_{L^\infty_{t,x}} \leq Z_I([0,t]). \tag{3.19}$$

It follows from Colliander et al. in [26] that

$$\|P_{123}I(u_{N_1}u_{N_2}u_{N_3})\|_{L^2_{t,x}} \leq \langle N_1\rangle^{-1/2}Z_I([0,t])^3. \tag{3.20}$$

We use (3.7)–(3.15) to complete the Case II analysis. H_2 is bounded by

$$\begin{aligned}(N_4\langle N_5\rangle)^{-1}&\left(\frac{N_5}{N_4}\right)^{3/2}\langle N_1\rangle^{-1/2}Z_I([0,t])^6\\ &\leq N^{-2-}\left(\prod_{i=1}^{6}\langle N_i\rangle\right)^{0-}Z_I([0,t])^6.\end{aligned} \tag{3.21}$$

Case III ($N_4 \geq N_5 \geq N$). We have the bound on the symbol

$$\left|1-\frac{m_{123}}{m_4m_5m_6}\right| \leq \frac{m_{123}}{m_4m_5m_6}. \tag{3.22}$$

Similar steps leads to the bound

$$H_2 \leq N^{-2-}\left(\prod_{i=1}^{6}\langle N_i\rangle\right)^{0-}Z_I([0,t])^6. \tag{3.23}$$

Combine the estimates for H_1 and H_2, we can complete the proof of Proposition 3.1. □

Remark 3.2. One can see that the proof of Proposition 3.1 closely follows the proof from Colliander [32]. However, the proof in this paper provides some clarity to the final stages of the proof in [32] and the necessary restrictions on s.

4. The Interaction Morawetz Inequality

In this section we develop a prior two-particle interaction Morawetz inequality of solutions to the Cauchy problem (1.14)-(1.15). This prior control will be fundamental to our analysis.

We first recall the generalized viriel identity [33].

Proposition 4.1. *If β is convex and real valued, and u is a smooth solution to* (1.14)-(1.15) *on* $[0,T]\times\mathbb{R}^4$*, then the following inequality holds:*

$$\int_0^t\int_{\mathbb{R}^4}(-\Delta\Delta\beta)|u(x,t)|^2dx\,dt + \int_0^t\int_{\mathbb{R}^4}(\Delta\beta)|u(x,t)|^4dx\,dt \lesssim \sup_{t\in[0,T]}\left|M_\beta(t)\right|, \tag{4.1}$$

where $M_a(t)$ is the Morawetz action given by

$$M_\beta(t) = 2\int_{\mathbb{R}^2} \nabla\beta(x) \cdot \mathcal{J}(\overline{u}\nabla u)dx. \tag{4.2}$$

Proof. Since β is convex and real valued and $a+b > 0$, by the fundamental theorem of calculus we can easily deduce the result. In the case of a solution to an equation with a nonlinearity which is not associated to a defocusing potential, the following corollary holds. □

Corollary 4.2. *Let $\beta : \mathbb{R}^4 \to \mathbb{R}$ be convex and u be a smooth solution to the equation:*

$$iu_t + \Delta u = \mathcal{N}. \tag{4.3}$$

Then, the following inequality holds:

$$\int_0^t \int_{\mathbb{R}^4} (-\Delta\Delta\beta)|u(x,t)|^2 dx\,dt + 2\int_0^t \int_{\mathbb{R}^4} \nabla\beta \cdot \{\mathcal{N}, u\}_p dx\,dt \lesssim \sup_{t\in[0,T]} \left|M_\beta(T) - M_\beta(t)\right|, \tag{4.4}$$

where $M_\beta(t)$ is the Morawetz action corresponding to u and $\{\cdot\}_p$ is the momentum bracket defined by

$$\{f,g\}_p = \mathcal{R}\left(f\overline{\nabla g} - g\overline{\nabla f}\right). \tag{4.5}$$

Now we give the interaction Morawetz inequality, although the results presented here are well known to experts, it seems to us that simple, self-contained proofs are often difficult to locate, so we present them for the convenience of the reader.

Proposition 4.3. *Let u be solution to the Cauchy problem of* (1.14)-(1.15)*, then the following $L^4_{x,t}$ space-time estimate holds*

$$\|u\|^4_{L^4_T L^4_x} \lesssim T^{1/3} \sup_{t\in[0,T]} \|u\|^3_{L^\infty_t L^2_x} \|u\|_{L^\infty_t \dot{H}^1_x} + T^{1/3}\|u\|^4_{L^\infty_t L^2_x}. \tag{4.6}$$

Proof. The proof of the Proposition 4.3 is similar to that in Colliander et al. [26]. Now we choose $\beta(x_1,x_2) = (1/2M)x^2(1-\log(x/M))$ if $|x| < M/\sqrt{e}$; $\beta(x_1,x_2) = 50x$ if $|x| > M$; then, $\beta(x_1,x_2)$ is smooth and convex for all $x \in \mathbb{R}^2$. We apply the generalized viriel identity with the weight $\beta(x_1,x_2)$ and the tensor product $u(x_1,x_2,t) = u_1(x_1,t) \otimes u_2(x_2,t) = u_1(x_1,t)u_2(x_2,t)$, where $u_1(x_1,t), u_2(x_2,t)$ are solutions with $(x_1,x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ to (1.14)-(1.15). It is not hard to see that the tensor product satisfies the equation:

$$iu_t + \Delta_4 u = f(u), \qquad (t,x) \in [0,T] \times \mathbb{R}^4, \tag{4.7}$$

here

$$f(u) = a|u_1|^2 u + bE\left(|u_1|^2\right)u + a|u_2|^2 u + bE\left(|u_2|^2\right)u, \tag{4.8}$$

and Δ_4 is the Laplace in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$.

Then we conclude that

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} (-\Delta\Delta\beta(x_1,x_2))|u_1(x_1,t)|^2|u_2(x_2,t)|^2 dx_1\,dx_2\,dt \lesssim 2\sup_{[0,T]}\left|M_\beta^{\otimes 2}(t)\right|, \tag{4.9}$$

where

$$M_\beta^{\otimes 2}(t) = 2\int_{\mathbb{R}^2\otimes\mathbb{R}^2} \nabla\beta(x)\cdot \mathcal{J}\left(\overline{u_1\bigotimes u_2}(x)\nabla u_1\bigotimes u_2(x)\right)dx. \tag{4.10}$$

Note that the definition of $\beta(x_1,x_2)$ implies

$$-\Delta\Delta\beta(x_1,x_2) = \frac{2}{M}\delta_{x_1=x_2} + O\left(\frac{1}{M^3}\right). \tag{4.11}$$

Thus

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} (-\Delta\Delta\beta(x_1,x_2))|u(x_1,t)|^2|u(x_2,t)|^2 dx_1\,dx_2\,dt\\ &\quad= \frac{2}{M}\int_0^T\int_{\mathbb{R}^4}|u(x,t)|^4 dx\,dt + O\left(\frac{1}{M^3}\right)\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^2}|u(x_1,t)|^2|u(x_2,t)|^2dx_1\,dx_2\,dt.\end{aligned} \tag{4.12}$$

It follows from the Fubuni's theorem that

$$\frac{C}{M^3}\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^2}|u(x_1,t)|^2|u(x_2,t)|^2dx_1\,dx_2\,dt \lesssim \frac{CT}{M^3}\|u\|^4_{L_t^\infty L_x^2}. \tag{4.13}$$

On the other hand,

$$\sup_{[0,T]}\left|M_\beta^{\bigotimes 2}(t)\right| \lesssim \sup_{t\in[0,T]}\|u\|^3_{L_t^\infty L_x^2}\|u\|_{L_t^\infty \dot{H}_x^1}. \tag{4.14}$$

Picking $M \sim T^{1/3}$, we get

$$\|u\|^4_{L_T^4 L_x^4} \lesssim T^{1/3}\sup_{t\in[0,T]}\|u\|^3_{L_t^\infty L_x^2}\|u\|_{L_t^\infty \dot{H}_x^1} + T^{1/3}\|u\|^4_{L_t^\infty L_x^2}. \tag{4.15}$$

□

Remark 4.4. For the common Morawetz inequality, the nonlinear term (the second term in (4.1)) has played the central role in the scattering theory for the nonlinear Schrödinger equation, and the first term in (4.1) did not play a big role in these works. But now by taking advantage of the first term, we can obtain a global prior estimate for defocusing nonlinearity, and we mention that the heart of the matter is that

$$-\Delta\Delta\beta(x_1,x_2) = \frac{2}{M}\delta_{x_1=x_2} + O\left(\frac{1}{M^3}\right). \tag{4.16}$$

This idea first appeared in [31].

Now, we consider the solution Iu of

$$iIu_t + \Delta Iu = aI\left(|u|^2 u\right) + bI\left(E\left(|u|^2\right)u\right). \tag{4.17}$$

If Iu does not solve (4.7) but the following equation:

$$iIu_t + \Delta Iu = a|Iu|^2 Iu + bE\left(|Iu|^2\right)Iu, \tag{4.18}$$

then the calculations that we have done above would reveal that

$$\|Iu\|^4_{L^4_T L^4_x} \lesssim T^{1/3} \sup_{[0,T]} \|Iu\|^3_{L^\infty_t L^2_x} \|Iu\|_{L^\infty_t \dot{H}^1_x} + T^{1/3} \|Iu\|^4_{L^\infty_t L^2_x}, \tag{4.19}$$

of course this is not the case. We may rewrite (4.17) as

$$\begin{aligned} iIu_t + \Delta Iu = {} & a\left(|Iu|^2 Iu\right) + b\left(E\left(|Iu|^2\right)Iu\right) + I\left(a|u|^2 u + b\left(E\left(|u|^2\right)u\right)\right) \\ & - a\left(|Iu|^2 Iu\right) - b\left(E\left(|Iu|^2\right)Iu\right) = G(Iu) + (IG(u) - G(Iu)). \end{aligned} \tag{4.20}$$

For what follows we abbreviate $u_i = u(x_i)$ where u_i is the solution of

$$iu_t + \Delta u = a|u|^2 u + bE\left(|u|^2\right)u, \quad (t, x_i) \in [0,T] \times \mathbb{R}^2. \tag{4.21}$$

We aim to prove the following theorem.

Theorem 4.5. *Let Iu be a solution to* (4.7), *then*

$$\begin{aligned} \|Iu\|^4_{L^4_T L^4_x} \lesssim {} & T^{1/3} \sup_{[0,T]} \|Iu\|^3_{L^2_x} \|Iu\|_{\dot{H}^1_x} + T^{1/3} \|u_0\|^4_{L^2_x} \\ & + T^{1/3} \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla\beta \cdot \{\mathcal{N}_{\text{bad}}, Iu(x_1,t) Iu(x_2,t)\}_p dx_1\, dx_2\, dt, \end{aligned} \tag{4.22}$$

where

$$\mathcal{N}_{\text{bad}} = \sum_{i-1}^{2} (IG(u_i) - G(Iu_i)) \prod_{j=1, i \neq j}^{2} Iu_j. \tag{4.23}$$

In particular, on a time interval T_k where the local well-posedness Proposition 2.6 holds one has that

$$\int_{T_k} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla\beta \cdot \{\mathcal{N}_{\text{bad}}, Iu(x_1,t) Iu(x_2,t)\}_p dx_1\, dx_2\, dt \lesssim \frac{1}{N^{1-}} Z_I^6(J_k). \tag{4.24}$$

Proof. According to Corollary 4.2,

$$\int_0^T \int_{\mathbb{R}^2} (-\Delta\Delta\beta)|Iu(x,t)|^2 dx\, dt \lesssim 2\sup_{[0,T]}\left|\int_{\mathbb{R}^2} \nabla\beta \cdot \mathcal{J}\left(\overline{Iu}(x)\nabla Iu\right)dx\right| + \left|\int_0^T \int_{\mathbb{R}^2} \nabla\beta \cdot \{IG(u_i) - G(Iu_i)\}_p dx\, dt\right|. \tag{4.25}$$

Set

$$IU = I \bigotimes I\left(u(x_1,t) \bigotimes u(x_2,t)\right) = \prod_{j=1}^{2} Iu(x_j,t). \tag{4.26}$$

If u solves (4.3) for $n = 2$, then IU solves (4.3) for $n = 4$, with right-hand side $\mathcal{N}_I$ given by

$$\mathcal{N}_I = \sum_{i=1}^{2}\left(I(\mathcal{N}_i) \prod_{j=1, i\neq j}^{2} Iu_j \right). \tag{4.27}$$

Now we decompose $\mathcal{N}_I$ as good part and bad part. The good part creates a positive term that we ignore. The bad term produces the error term. Now we have the bound:

$$\|Iu\|^4_{L^4_T L^4_x} \lesssim T^{1/3} \sup_{[0,T]} \|Iu\|^3_{L^2_x} \|Iu\|_{H^1_x} + T^{1/3}\|u_0\|^4_{L^2_x} + T^{1/3} \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} \nabla\beta \cdot \{\mathcal{N}_{\text{bad}}, Iu(x_1,t) Iu(x_2,t)\}_p dx_1\, dx_2\, dt, \tag{4.28}$$

where we have used the fact that $\|Iu\|_{L^2} \lesssim \|u\|_{L^2} = \|u_0\|_{L^2}$. Remark that $\nabla\beta$ is a real valued, thus

$$\nabla\beta \cdot \mathcal{R}\left(f\nabla\overline{g} - g\nabla\overline{f}\right) = \mathcal{R}\left(\nabla\beta \cdot \left(f\nabla\overline{g} - g\nabla\overline{f}\right)\right) \tag{4.29}$$

and that $\nabla = (\nabla_{x_1}, \nabla_{x_2})$. We now compute the dot product under the integral in (4.20), that is,

$$\mathcal{R}\left\{ \sum_{i=1}^{2} \nabla_{x_i}\beta\left(\mathcal{N}_{\text{bad}}\left(\nabla_{x_i}\left(\overline{Iu_1 Iu_2}\right)\right) - Iu_1 Iu_2 \nabla_{x_i}\overline{\mathcal{N}_{\text{bad}}}\right)\right\}. \tag{4.30}$$

Recall that

$$\mathcal{N}_{\text{bad}} = \sum_{i-1}^{2} (IG(u_i) - G(Iu_i)) \prod_{j=1, i\neq j}^{2} Iu_j. \tag{4.31}$$

Using the definition of $\mathcal{N}_{\text{bad}}$ and the fact that ∇_{x_1} acts only on Iu_1, we have

$$\begin{aligned}&\mathcal{N}_{\text{bad}}\left(\nabla_{x_i}\left(\overline{Iu_1Iu_2}\right)\right)-Iu_1Iu_2\left(\nabla_{x_i}\overline{\mathcal{N}}_{\text{bad}}\right)\\&\quad=\left[(I(\mathcal{N}_1)-\mathcal{N}(Iu_1))\nabla_{x_1}\overline{Iu_1}-\nabla_{x_1}\left(I\mathcal{N}_1-\overline{\mathcal{N}}(Iu_1)\right)Iu_1\right]|Iu_2|^2.\end{aligned}\tag{4.32}$$

Analogously, we can see that the second part is given by

$$\mathcal{R}\left\{\nabla_{x_2}\beta\left[(I(\mathcal{N}_2)-\mathcal{N}(Iu_2))\nabla_{x_2}\overline{Iu_2}-\nabla_{x_2}\left(I\overline{\mathcal{N}}_2-\overline{\mathcal{N}}(Iu_2)\right)Iu_2\right]|Iu_1|^2\right\}.\tag{4.33}$$

We have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^4}\left|\nabla_{x_1}\beta\right||I(\mathcal{N}_1)-\mathcal{N}(Iu_1)||\nabla_{x_1}Iu_1||Iu_2|^2dx_1\,dx_2\,dt\\&\quad\le\int_0^T\int_{\mathbb{R}^2}|I(\mathcal{N}_1)-\mathcal{N}(Iu_1)||\nabla_{x_1}Iu_1|dx_1\,dt\|Iu\|^2_{L_t^\infty L_x^2}\\&\quad\le\left(\int_0^T\int_{\mathbb{R}^2}|I(\mathcal{N}_1)-\mathcal{N}(Iu_1)||\nabla_{x_1}Iu_1|dx\,dt\right)Z_I^2\\&\quad\le\|I(\mathcal{N}_1)-\mathcal{N}(Iu_1)\|_{L_t^1L_x^2}\|\nabla Iu\|_{L_t^\infty L_x^2}Z_I^2\\&\quad\le\|I(\mathcal{N}_1)-\mathcal{N}(Iu_1)\|_{L_t^1L_x^2}Z_I^3.\end{aligned}\tag{4.34}$$

Here we used the fact that the pair $(\infty,2)$ is admissible and $|\nabla_{x_1}\beta|\lesssim 1$. By a similar way we can deduce that

$$\int_0^T\int_{\mathbb{R}^4}\left|\nabla_{x_1}\beta\right||\nabla_{x_1}I(\mathcal{N}_1)-\mathcal{N}(Iu_1)||Iu_1||Iu_2|^2dx_1\,dx_2\,dt\le\|\nabla_x(I(\mathcal{N})-\mathcal{N}(Iu))\|_{L_t^1L_x^2}Z_I^3.\tag{4.35}$$

Hence, we only need to estimate $\|\nabla_x(I(\mathcal{N})-\mathcal{N}(Iu))\|_{L_t^1L_x^2}$. Observe that $\mathcal{N}=a(|u|^2u)+b(E(|u|^2)u)$, and

$$\left|\nabla_x\left(I(\widehat{\mathcal{N})-\mathcal{N}}(Iu)(\xi)\right)\right|\le C\int_{\xi=\xi_1+\xi_2+\xi_3}|\xi||m(\xi)-m(\xi_1)m(\xi_2)m(\xi_3)|\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)d\xi_1\,d\xi_2\,d\xi_3.\tag{4.36}$$

Using the fact that $a+b>0$ and the properties of operator E, we have

$$\begin{aligned}
\|\nabla_x(I(\mathcal{N})-\mathcal{N}(Iu))\|_{L_t^1L_x^2} &= \|\nabla_x(I(\mathcal{N})-\mathcal{N}(Iu))\|_{L_t^1L_\xi^2} \\
&\leq C\sum_{N_1,N_2,N_3}\left\|\int_{\xi=\xi_1+\xi_2+\xi_3}|\xi||m(\xi)-m(\xi_1)m(\xi_2)m(\xi_3)|\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_1\,d\xi_2\,d\xi_3\right\|_{L_t^1L_\xi^2} \\
&= C\sum_{N_1,N_2,N_3}\left\|\int_{\xi=\xi_1+\xi_2+\xi_3}|\xi|\frac{|m(\xi)-m(\xi_1)m(\xi_2)m(\xi_3)|}{m(\xi_1)m(\xi_2)m(\xi_3)}\widehat{I}u(\xi_1)\widehat{I}u(\xi_2)\widehat{I}u(\xi_3)d\xi_1\,d\xi_2\,d\xi_3\right\|_{L_t^1L_\xi^2}.
\end{aligned} \tag{4.37}$$

There is symmetry under interchange of the indices 1, 2, 3. We may assume that

$$N_1\geq N_2\geq N_3. \tag{4.38}$$

Let

$$\sigma(\xi_1,\xi_2,\xi_3)=|\xi|\frac{|m(\xi)-m(\xi_1)m(\xi_2)m(\xi_3)|}{m(\xi_1)m(\xi_2)m(\xi_3)}. \tag{4.39}$$

We carry out a case by case analysis for (4.28).

Case I. $N_1\ll N$, this force $\sigma(\xi_1,\xi_2,\xi_3)=0$, then there no contribution to (4.25).

Case II. $N_1\gtrsim N\gg N_2$, we have

$$\begin{aligned}
&\left\|\int_{\xi=\xi_1+\xi_2+\xi_3}\sigma(\xi_1,\xi_2,\xi_3)\widehat{Iu}(\xi_1)\widehat{Iu}(\xi_2)\widehat{Iu}(\xi_3)d\xi_1\,d\xi_2\,d\xi_3\right\|_{L_t^1L_\xi^2} \\
&\quad=\frac{1}{N}\left\|\int_{\xi=\xi_1+\xi_2+\xi_3}\frac{N}{\xi_1\xi_2}\sigma(\xi_1,\xi_2,\xi_3)\widehat{\nabla Iu}(\xi_1)\widehat{\nabla Iu}(\xi_2)\widehat{Iu}(\xi_3)d\xi_1\,d\xi_2\,d\xi_3\right\|_{L_t^1L_\xi^2} \\
&\quad\lesssim\frac{1}{N}\|\nabla Iu_1\|_{L_t^3L_x^6}\|\nabla Iu_2\|_{L_t^3L_x^6}\|Iu_1\|_{L_t^3L_x^6},
\end{aligned} \tag{4.40}$$

where we have used the fact that $|\xi|\sim N_1$ and by mean value theorem that

$$\frac{|m(\xi)-m(\xi_1)m(\xi_2)m(\xi_3)|}{m(\xi_1)m(\xi_2)m(\xi_3)}\lesssim\frac{N_3}{N_2}. \tag{4.41}$$

Therefore,

$$\frac{N}{\xi_1\xi_2}\sigma(\xi_1,\xi_2,\xi_3)\lesssim\frac{N}{\xi_1\xi_2}\frac{N_2}{N_1}\xi\leq 1. \tag{4.42}$$

Case III ($N_1 \geq N_2 \gtrsim N \gg N_3$). We also have

$$\begin{aligned}
&\left\| \int_{\xi=\xi_1+\xi_2+\xi_3} \sigma(\xi_1,\xi_2,\xi_3)\widehat{Iu}(\xi_1)\widehat{Iu}(\xi_2)\widehat{Iu}(\xi_3)d\xi_1\, d\xi_2\, d\xi_3 \right\|_{L_t^1 L_\xi^2} \\
&\quad = \frac{1}{N}\left\| \int_{\xi=\xi_1+\xi_2+\xi_3} \frac{N}{\xi_1\xi_2}\sigma(\xi_1,\xi_2,\xi_3)\widehat{\nabla Iu}(\xi_1)\widehat{\nabla Iu}(\xi_2)\widehat{Iu}(\xi_3)d\xi_1\, d\xi_2\, d\xi_3 \right\|_{L_t^1 L_\xi^2} \\
&\quad \lesssim \frac{1}{N}\|\nabla Iu_1\|_{L_t^3 L_x^6}\|\nabla Iu_2\|_{L_t^3 L_x^6}\|Iu_1\|_{L_t^3 L_x^6}.
\end{aligned} \tag{4.43}$$

Case IV ($N_1 \geq N_2 \geq N_3 \gtrsim N$). We estimate as follows:

$$\left\| \int_{\xi=\xi_1+\xi_2+\xi_3} \sigma(\xi_1,\xi_2,\xi_3)\widehat{Iu}(\xi_1)\widehat{Iu}(\xi_2)\widehat{Iu}(\xi_3)d\xi_1\, d\xi_2\, d\xi_3 \right\|_{L_t^1 L_\xi^2} \lesssim \frac{1}{N^2}\prod_{j=1}^{3}\|\nabla Iu_j\|_{L_t^3 L_x^6}, \tag{4.44}$$

where we have used the estimate

$$\sigma(\xi_1,\xi_2,\xi_3)\frac{N^2}{\xi_1\xi_2\xi_3} \leq N^{-1+3s}(N_1N_2N_3)^{-s} \leq 1. \tag{4.45}$$

Finally, since the pair (3, 6) is admissible, we can get

$$\|\nabla_x(I(\mathcal{N}) - \mathcal{N}(Iu))\|_{L_t^1 L_x^2} \lesssim \frac{1}{N}Z_I^3. \tag{4.46}$$

Combining (4.22), (4.24), and (4.40), we complete the proof Theorem 4.5. □

5. The Proof of Theorem 1.1

The idea is followed from [26, 34]. The first observation is the fact that if $u(t,x)$ is a solution of the Cauchy problem (1.14)-(1.15), we can scale it and obtain a new solution, namely, the scale function

$$u^\lambda(x) = \frac{1}{\lambda}u\left(\frac{x}{\lambda},\frac{t}{\lambda^2}\right) \tag{5.1}$$

satisfies the same equation with initial data $u_0^\lambda = (1/\lambda)u_0(x/\lambda)$. This scaling preserves the L^2 norms of $u(t)$ as well as the $L_{t,x}^4$ space-time norm. From estimate (2.13), we have

$$\left\|Iu_0^\lambda\right\|_{H^1} \lesssim N^{1-s}\left\|u_0^\lambda\right\|_{H^s} \lesssim N^{1-s}\lambda^{-s}. \tag{5.2}$$

We can choose the parameter λ in the manner

$$\lambda \sim N^{(1-s)/s}, \tag{5.3}$$

so that $\|Iu_0^\lambda\|_{H^1} = O(1)$.

Then, for any T_0 (arbitrarily large), define

$$\Gamma = \left\{ 0 < t < \lambda^2 T_0 : \left\| I u^{\lambda} \right\|_{L_t^4 L_x^4([0,t]\times\mathbb{R}^2)} \le \delta N^{1/8} t^{1/12} \right\}, \tag{5.4}$$

where δ is a constant to be chosen later. We claim that Γ is the whole interval $[0, \lambda^2 T_0]$. Indeed, if Γ is not the whole interval $[0, \lambda^2 T_0]$, then using the fact that

$$\left\| I u^{\lambda} \right\|_{L_t^4 L_x^4([0,t]\times\mathbb{R}^2)} \tag{5.5}$$

is a continuous function of time, there exist some $T \in [0, \lambda^2 T_0]$ with the properties,

$$\left\| I u^{\lambda} \right\|_{L_t^4 L_x^4([0,T]\times\mathbb{R}^2)} \ge \delta N^{1/8} t^{1/12}, \tag{5.6}$$

$$\left\| I u^{\lambda} \right\|_{L_t^4 L_x^4([0,T]\times\mathbb{R}^2)} \le 2\delta N^{1/8} t^{1/12}. \tag{5.7}$$

Now we divide the time interval $[0, T_0]$ into subintervals J_k, $k = 1, \ldots, L$ in such a way that

$$\left\| I u^{\lambda} \right\|^4_{L_t^4 L_x^4([0,J_k]\times\mathbb{R}^2)} \le \mu_0, \tag{5.8}$$

where μ_0 is as the same as in Proposition 2.6. This is possible because of (5.7). Then, the number of the slices, which we will call L, is most like

$$L \sim \frac{\left(2\delta N^{1/8} t^{1/12}\right)^4}{\mu} \sim \frac{(2\delta)^4 N^{1/2} T^{1/3}}{\mu}. \tag{5.9}$$

According to Propositions 2.6 and 3.1, we have that, for any $1/3 < s < 1/2$

$$\sup_{t\in[0,T]} H\left(I u^{\lambda}\right) \lesssim H\left(I u_0^{\lambda}\right) + \frac{L}{N^{3/2}}, \tag{5.10}$$

for our choice of λ, $H(I u_0^{\lambda}) \lesssim 1$. Noting that $2/5 > 1/4$, we can apply the Proposition 3.1. In order to guarantee that

$$H\left(I u^{\lambda}\right) \lesssim 1 \tag{5.11}$$

for all $t \in [0, T]$, we require

$$L \lesssim N^{3/2}. \tag{5.12}$$

Since $T \le \lambda^2 T_0$, this is fulfilled as long as

$$\frac{(2\delta)^4 N^{1/2} \lambda^{2/3} T_0^{1/3}}{\mu_0} \sim N^{3/2}, \tag{5.13}$$

note (5.3), that is to say we have

$$T_0^{1/3}\frac{(2\delta)^4}{\mu_0}\sim N^{(5s-2)/3s}. \tag{5.14}$$

If $s>2/5$, we have that T_0 is arbitrarily large if we send N to infinity.

We use Theorem 4.5 to show that (5.6) is not true. Recall the estimates (4.22)–(4.24), we have

$$\begin{aligned}\left\|Iu^{\lambda}\right\|_{L_t^4L_x^4}^4 &\lesssim T^{1/3}\sup_{t\in[0,T]}\left\|Iu^{\lambda}\right\|_{\dot H^1}^2\left\|Iu^{\lambda}\right\|_{L^2}^2+T^{1/3}\left\|u^{\lambda}\right\|_{L^2}^4+T^{1/3}L\frac{1}{N}\\ &\lesssim N^{1/2}T^{1/3},\end{aligned} \tag{5.15}$$

which implies that

$$\left\|Iu^{\lambda}\right\|_{L_t^4L_x^4([0,T]\times\mathbb{R}^2)}\lesssim N^{1/8}t^{1/12}. \tag{5.16}$$

This estimate contradicts to (5.6) for suitable choice of δ (namely, we choose $\delta\gg 1$). Therefore $\Gamma=[0,\lambda^2T_0]$, and T_0 can be chosen arbitrarily large. In addition,

$$\begin{aligned}\|u(T_0)\|_{H^s} &\lesssim \|u_0\|_{L^2}+\|u(T_0)\|_{\dot H^s}=\|u_0\|_{L^2}+\lambda^s\left\|u^{\lambda}\left(\lambda^2T_0\right)\right\|_{\dot H^s}\\ &\lesssim \lambda^s\left\|Iu^{\lambda}\left(\lambda^2T_0\right)\right\|_{H^1}\lesssim\lambda^s\lesssim N^{1-s}\lesssim T_0^{3s(1-s)/2(5s-2)}.\end{aligned} \tag{5.17}$$

Since T_0 is arbitrarily large, the priori bound on the H^s norm concludes the global well-posedness of the Cauchy Problem (1.14)-(1.15).

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References

[1] A. Davey and K. Stewartson, "On three-dimensional packets of surface waves," *Proceedings of the Royal Society A*, vol. 338, no. 1613, pp. 101–110, 1974.

[2] J. M. Ghidaglia and J. C. Saut, "On the initial value problem for the Davey-Stewartson systems," *Nonlinearity*, vol. 3, no. 2, pp. 475–506, 1990.

[3] N. Hayashi and H. Hirata, "Local existence in time of small solutions to the elliptic-hyperbolic Davey-Stewartson system in the usual Sobolev space," *Proceedings of the Edinburgh Mathematical Society II*, vol. 40, no. 3, pp. 563–581, 1997.

[4] M. Tsutsumi, "Decay of weak solutions to the Davey-Stewartson systems," *Journal of Mathematical Analysis and Applications*, vol. 182, no. 3, pp. 680–704, 1994.

[5] N. Hayashi and J. C. Saut, "Global existence of small solutions to the Davey-Stewartson and the Ishimori systems," *Differential and Integral Equations*, vol. 8, no. 7, pp. 1657–1675, 1995.

[6] F. Linares and G. Ponce, "On the Davey-Stewartson systems," *Annales de l'Institut Henri Poincaré C*, vol. 10, no. 5, pp. 523–548, 1993.

[7] B. L. Guo and B. X. Wang, "The Cauchy problem for Davey-Stewartson systems," *Communications on Pure and Applied Mathematics*, vol. 52, no. 12, pp. 1477–1490, 1999.

[8] J. Shu and J. Zhang, "Sharp conditions of global existence for the generalized Davey-Stewartson system," *IMA Journal of Applied Mathematics*, vol. 72, no. 1, pp. 36–42, 2007.

[9] Z. Gan and J. Zhang, "Sharp threshold of global existence and instability of standing wave for a Davey-Stewartson system," *Communications in Mathematical Physics*, vol. 283, no. 1, pp. 93–125, 2008.

[10] J. Zhang and S. H. Zhu, "Sharp blow-up criteria for the Davey-Stewartson system in $\mathbb{R}^3$," *Dynamics of Partial Differential Equations*, vol. 8, no. 3, pp. 239–260, 2011.

[11] C. Sulem and P. L. Sulem, *The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse*, vol. 139 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1999.

[12] G. Richards, "Mass concentration for the Davey-Stewartson system," *Differential and Integral Equations*, vol. 24, no. 3-4, pp. 261–280, 2011.

[13] X. G. Li, J. Zhang, S. Y. Lai, and Y. H. Wu, "The sharp threshold and limiting profile of blow-up solutions for a Davey-Stewartson system," *Journal of Differential Equations*, vol. 250, no. 4, pp. 2197–2226, 2011.

[14] C. Babaoglu and S. Erbay, "Two-dimensional wave packets in an elastic solid with couple stresses," *International Journal of Non-Linear Mechanics*, vol. 39, no. 6, pp. 941–949, 2004.

[15] C. Babaoglu, A. Eden, and S. Erbay, "Global existence and nonexistence results for a generalized Davey-Stewartson system," *Journal of Physics A*, vol. 37, no. 48, pp. 11531–11546, 2004.

[16] A. Eden and S. Erbay, "Standing waves for a generalized Davey-Stewartson system," *Journal of Physics A*, vol. 39, no. 43, pp. 13435–13444, 2006.

[17] A. Eden, T. B. Gürel, and E. Kuz, "Focusing and defocusing cases of the purely elliptic generalized Davey-Stewartson system," *IMA Journal of Applied Mathematics*, vol. 74, no. 5, pp. 710–725, 2009.

[18] J. Bourgain, "Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity," *International Mathematics Research Notices*, vol. 1998, no. 5, pp. 253–283, 1998.

[19] C. E. Kenig, G. Ponce, and L. Vega, "Global well-posedness for semi-linear wave equations," *Communications in Partial Differential Equations*, vol. 25, no. 9-10, pp. 1741–1752, 2000.

[20] G. Fonseca, F. Linares, and G. Ponce, "Global well-posedness for the modified Korteweg-de Vries equation," *Communications in Partial Differential Equations*, vol. 24, no. 3-4, pp. 683–705, 1999.

[21] J. Ginibre, "Le problem de Cauchy pour des EDP semi-lineaires preiodiques en variabkes d'espace (d'apres bourgain)," *Astérisque*, no. 237, pp. 163–187, 1996.

[22] L. Molinet and F. Ribaud, "The global Cauchy problem in Bourgain's-type spaces for a dispersive dissipative semilinear equation," *SIAM Journal on Mathematical Analysis*, vol. 33, no. 6, pp. 1269–1296, 2002.

[23] H. Pecher, "Global well-posedness below energy space for the 1-dimensional Zakharov system," *International Mathematics Research Notices*, vol. 2001, no. 19, pp. 1027–1056, 2001.

[24] C. X. Miao and B. Zhang, "H^S-global well-posedness for semilinear wave equations," *Journal of Mathematical Analysis and Applications*, vol. 283, no. 2, pp. 645–666, 2003.

[25] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation," *Mathematical Research Letters*, vol. 9, no. 5-6, pp. 659–682, 2002.

[26] J. Colliander, M. Grillakis, and N. Tzirakis, "Improved interaction Morawetz inequalities for the cubic nonlinear schrödinger equation on $\mathbb{R}^2$," *International Mathematics Research Notices*, vol. 2007, no. 23, pp. 90–119, 2007.

[27] C. X. Shen and B. L. Guo, "Almost conservation law and global rough solutions to a nonlinear Davey-Stewartson equation," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 365–379, 2006.

[28] B. Dodson, "Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d \geq 3$," *Journal of the American Mathematical Society*, vol. 25, no. 2, pp. 429–463, 2012.

[29] B. Dodson, "Global well-posedness and scattering for the defocusing, L^2-critical, nonlienar Schrödingerequation when $d = 2$," http://arxiv.org/abs/1006.1375.

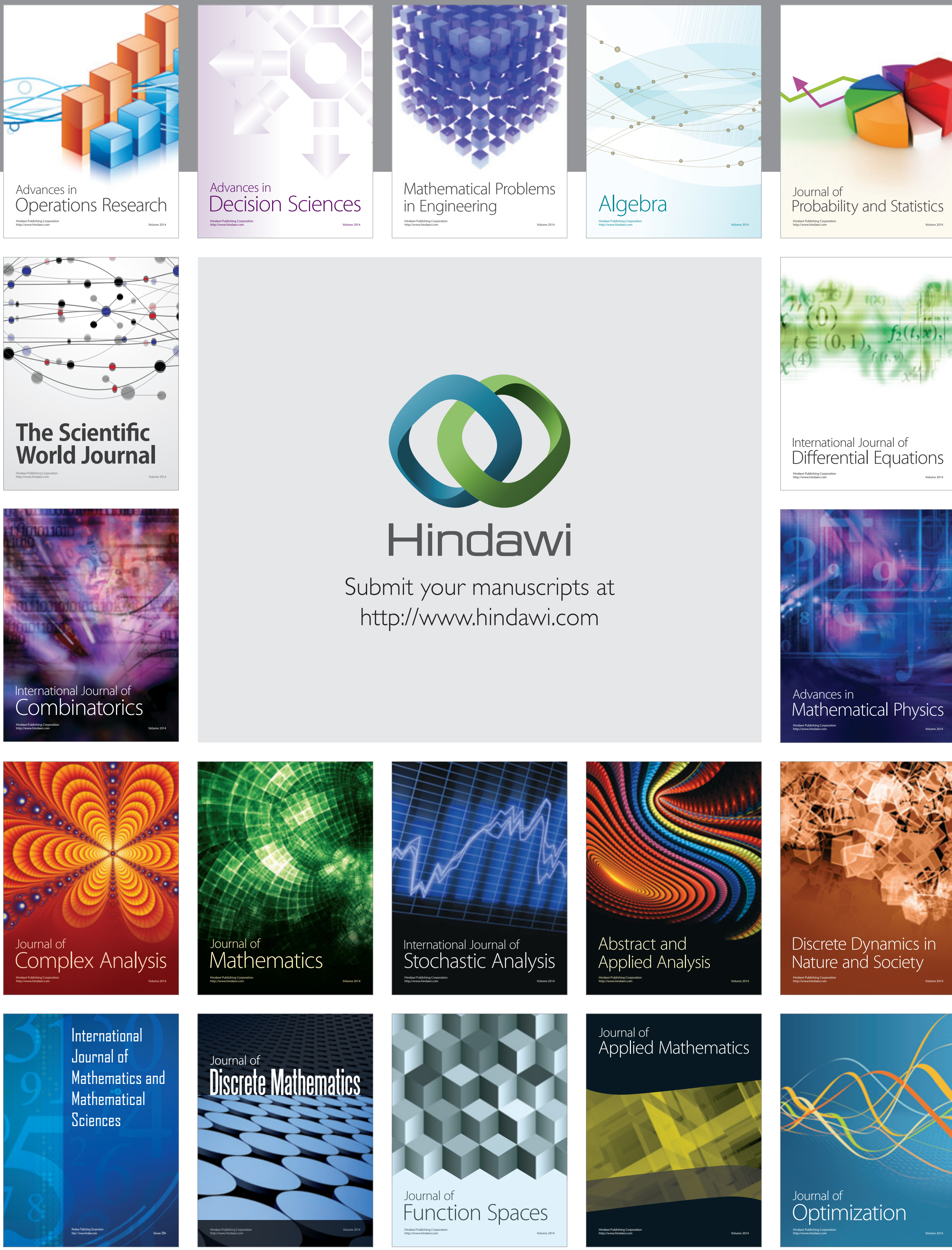
[30] B. Dodson, "Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 1$," http://arxiv.org/abs/1010.0040.

[31] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$," *Communications on Pure and Applied Mathematics*, vol. 57, no. 8, pp. 987–1014, 2004.

[32] J. Colliander, S. Raynor, C. Sulem, and J. D. Wright, "Ground state mass concentration in the L^2 critical nonlinear Schrödinger equation below H^1," *Mathematical Research Letters*, vol. 12, no. 3, pp. 357–375, 2005.

[33] J. E. Lin and W. A. Strauss, "Decay and scattering of solutions of a nonlinear Schrödinger equation," *Journal of Functional Analysis*, vol. 30, no. 2, pp. 245–263, 1978.

[34] Y. F. Fang and M. G. Grillakis, "On the global existence of rough solutions of the cubic defocusing Schrödinger equation in R^{2+1}," *Journal of Hyperbolic Differential Equations*, vol. 4, no. 2, pp. 233–257, 2007.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