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*Research Article*

# Multiplicity of Positive Solutions for Weighted Quasilinear Elliptic Equations Involving Critical Hardy-Sobolev Exponents and Concave-Convex Nonlinearities

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By variational methods and some analysis techniques, the multiplicity of positive solutions is obtained for a class of weighted quasilinear elliptic equations with critical Hardy-Sobolev exponents and concave-convex nonlinearities.

## 1. Introduction and Main Results

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$ and $0 \in \Omega$. We will study the multiplicity of positive solutions for the following quasilinear elliptic problem:

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \frac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}} + \lambda\frac{|u|^{q-2}u}{|x|^{dp^*(a,d)}} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda > 0$, $1 < p < N$, $0 \leq \mu < \overline{\mu}$, $\overline{\mu} \triangleq ((N-p)/p - a)^p$, $0 \leq a < (N-p)/p$, $a \leq b, d < a+1$, $1 \leq q < p$, $p^*(a,d) \triangleq Np/(N - p(a+1-d))$ is the critical Sobolev-Hardy exponent. Note that $p^*(0,0) = p^* \triangleq Np/(N-p)$ is the critical Sobolev exponent.

In this paper, $W \triangleq W_a^{1,p}(\Omega)$ denotes the space obtained as the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |x|^{-ap}|\nabla u|^p dx)^{1/p}$. The energy functional of (1.1) is defined on $W$ by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega\left(|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right)dx - \frac{1}{p^*(a,b)}\int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}dx - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}dx. \tag{1.2}$$

Then $J_\lambda \in C^1(W,\mathbb{R})$. $u \in W \setminus \{0\}$ is said to be a solution of (1.1) if $\langle J'_\lambda(u), v\rangle = 0$ for all $v \in W$ and a solution of (1.1) is a critical point of $J_\lambda$. By the standard elliptic regularity argument, we deduce that $u \in C^1(\Omega \setminus \{0\})$.

Problem (1.1) is related to the following Hardy inequality [1]:

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}dx\right)^{p/p^*(a,b)} \leq C\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx, \quad \forall u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.3}$$

which is also called the (general or weighted) Hardy-Sobolev inequality. For the sharp constants and extremal functions, see [2, 3]. If $b = a+1$, then $p^*(a,b) = p$ and the following (general or weighted) Hardy inequality holds [1, 4]:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|u|^{p(a+1)}}dx \leq \frac{1}{\overline{\mu}}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx, \quad \forall u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.4}$$

where $\overline{\mu} = ((N-p)/p - a)^p$ is the best Hardy constant.

In the space $W$, we employ the following norm if $\mu < \overline{\mu}$:

$$\|u\| = \|u\|_W \triangleq \left(\int_\Omega\left(|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right)dx\right)^{1/p}. \tag{1.5}$$

By (1.4) it is equivalent to the usual norm $(\int_\Omega |x|^{-ap}|\nabla u|^p dx)^{1/p}$ of the space $W$. According to (1.4), we can define the following best constant for $0 \leq a < (N-p)/p$, $a \leq b < a+1$ and $\mu < \overline{\mu}$:

$$S_{\mu,a,b}(\Omega) = \inf_{u\in W\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega |u|^{p^*(a,b)}/|x|^{bp^*(a,b)}dx\right)^{p/p^*(a,b)}}. \tag{1.6}$$

From Kang [5, Lemma 2.2], $S_{\mu,a,b}$ is independent of $\Omega \subset \mathbb{R}^N$. Thus, we will simply denote that $S_{\mu,a,b}(\Omega) = S_{\mu,a,b}(\mathbb{R}^N) = S_{\mu,a,b}$.

When $a = 0$, we set $s = dp^*(0,d)$ and $t = bp^*(0,b)$, then (1.1) is equivalent to the following quasilinear elliptic equations:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \mu\frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(t)-2}u}{|x|^t} + \lambda\frac{|u|^{q-2}u}{|x|^s} \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

where $\lambda > 0$, $1 < p < N$, $0 \leq \mu < \overline{\mu} = ((N-p)/p)^p$, $0 \leq s,t < p$, $1 \leq q < p$ and $p^*(t) = p(N-t)/(N-p)$.

Such kind of problem relative with (1.7) has been extensively studied by many authors. When $p = 2$, people have paid much attention to the existence of solutions for singular elliptic problems (see [6–16] and their references therein), besides, in the most of these papers, the operator $-\Delta - \mu/|x|^2$ with Sobolev-Hardy critical exponents (the case that $t = 0$) has been considered. Some authors also studied the singular problems with Sobolev-Hardy critical exponents (the case that $t \neq 0$) (see [17–22] and their references therein). In [23, 24], the authors deal with doubly-critical exponents.

When $p \neq 2$. The quasilinear problems related to Hardy inequality and Sobolev-Hardy inequality have been studied by some authors [25–32]. Here we recall the work in [25], where the extremal functions for the best Sobolev constant $S_{\mu,0,0}$ were studied. The results can be employed to study the problems with critical Sobolev exponents and Hard terms, see [25, 28]. In [26] it is investigated in $\mathbb{R}^N$ a quasilinear elliptic equation involving doubly critical exponents by the concentration compactness principle [33, 34].

We should note that the nonlinearities of problems studied in [11–14, 26, 28, 31] are all not sublinear or $p$-sublinear near the origin. To the best of our knowledge, there are few results of problem (1.7) with nonlinearities being $p$-sublinear near the origin for $1 < p < N$. We are only aware of the works [20, 30, 32] which studied the existence and multiplicity of positive solution of problem (1.7) with $1 \leq q < p < N$. In this paper, we study (1.1) and extend the results of [20, 30, 32] to the case $a \neq 0$ and $1 \leq q < p < N$.

For $0 \leq a < (N-p)/p$, $a \leq b < a+1$, and $0 \leq \mu < \overline{\mu}$, consider the following limiting problem:

$$-\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \frac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$
$$u > 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.8}$$
$$u \in W_a^{1,p}\left(\mathbb{R}^N\right),$$

where $W_a^{1,p}(\mathbb{R}^N)$ is the space obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx)^{1/p}$. From [5, Lemma 2.2], we know (1.8) has a unique ground state solution $U_{p,\mu}$ satisfying

$$U_{p,\mu}(1) = \left(\frac{p^*(a,b)(\overline{\mu}-\mu)}{p}\right)^{1/(p^*(a,b)-p)} \tag{1.9}$$

and all ground states must be of the form $\widetilde{U}_\varepsilon(x) = \varepsilon^{-(N-p)/p-a}U_{p,\mu}(x/\varepsilon)$ for some $\varepsilon > 0$, that is,

$$S_{\mu,a,b} = \inf_{u\in W_a^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^p - \mu|u|^p/|x|^{p(a+1)}\right)dx}{\left(\int_\Omega |u|^{p^*(a,b)}/|x|^{bp^*(a,b)}dx\right)^{p/p^*(a,b)}} \tag{1.10}$$

is achieved by $\widetilde{U}_\varepsilon$. Moreover, $U_{p,\mu}$ is radially symmetric and possesses the following properties:

$$\begin{aligned} &\lim_{r\to 0} r^{\alpha(\mu)}U_{p,\mu}(r) = c_1 > 0, \qquad &&\lim_{r\to 0} r^{\alpha(\mu)+1}\left|U'_{p,\mu}(r)\right| = c_1\alpha(\mu) \geq 0,\\ &\lim_{r\to+\infty} r^{\beta(\mu)}U_{p,\mu}(r) = c_2 > 0, \qquad &&\lim_{r\to+\infty} r^{\beta(\mu)+1}\left|U'_{p,\mu}(r)\right| = c_2\beta(\mu) > 0, \end{aligned} \tag{1.11}$$

where $c_i$ $(i = 1, 2)$ are positive constants and $\alpha(\mu)$, $\beta(\mu)$ are the zeros of the function

$$f(\tau) = (p-1)\tau^p - (N - p(a+1))\tau^{p-1} + \mu, \quad \tau \geq 0,\ 0 \leq \mu < \overline{\mu}, \tag{1.12}$$

which satisfy $0 \leq \alpha(\mu) < (N - p(a+1))/p < \beta(\mu) < (N - p(a+1))/(p-1)$. Furthermore, there exist the positive constants $c_3 = c_3(\mu, p, a, b)$ and $c_4 = c_4(\mu, p, a, b)$ such that

$$c_3 \leq U_{p,\mu}(x)\left(|x|^{\alpha(\mu)/\delta} + |x|^{\beta(\mu)/\delta}\right)^\delta \leq c_4, \quad \delta \triangleq \frac{N - p(a+1)}{p}. \tag{1.13}$$

Throughout this paper, let $R_0$ be the positive constant such that $\Omega \subset B(0; R_0)$, where $B(0; R_0) = \{x \in \mathbb{R}^N : |x| < R_0\}$. By Hölder and Sobolev-Hardy inequalities, for all $u \in W$, we obtain

$$\begin{aligned} \int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}} &\leq \left(\int_{B(0;R_0)} |x|^{-dp^*(a,d)}\right)^{(p^*(a,d)-q)/p^*(a,d)} \left(\int_\Omega \frac{|u|^{p^*(a,d)}}{|x|^{dp^*(a,d)}}\right)^{q/p^*(a,d)}\\ &\leq \left(N\omega_N\int_0^{R_0} r^{-dp^*(a,d)+N-1}dr\right)^{(p^*(a,d)-q)/p^*(a,d)} (S_{\mu,a,d})^{-(q/p)}\|u\|^q\\ &\leq \left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N - dp^*(a,d)}\right)^{(p^*(a,d)-q)/p^*(a,d)} (S_{\mu,a,d})^{-(q/p)}\|u\|^q. \end{aligned} \tag{1.14}$$

Set

$$\Lambda_0 = \left(\frac{p-q}{p^*(a,b)-q}\right)^{(p-q)/(p^*(a,b)-p)} \left(\frac{p^*(a,b)-p}{p^*(a,b)-q}\right) \left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{-((p^*(a,d)-q)/p^*(a,d))}$$
$$\times \left(S_{\mu,a,d}\right)^{q/p} \left(S_{\mu,a,b}\right)^{p^*(a,b)(p-q)/p(p^*(a,b)-p)}, \tag{1.15}$$

where $\omega_N = 2\pi^{N/2}/N\Gamma(N/2)$ is the volume of the unit ball in $\mathbb{R}^N$.

Furthermore, from $0 \le a < (N-p)/p$ and $a \le d < a+1$, we can deduce that

$$0 < p\left(\frac{N-p}{p} - a\right) + pd = N - p(a+1-d) < N, \tag{1.16}$$

which implies

$$\begin{aligned} p^*(a,d) &= \frac{pN}{N-p(a+1-d)} > p, \\ N - dp^*(a,d) &= \frac{Np}{N-p(a+1-d)}\left(\frac{N-p}{p} - a\right) > 0. \end{aligned} \tag{1.17}$$

Combining these with $1 \le q < p$, we get $\Lambda_0 > 0$.

We are now ready to state our main results.

**Theorem 1.1.** *Assume that $N \ge 3$, $0 \le \mu < \overline{\mu}$, $0 \le a < (N-p)/p$, $a \le b, d < a+1$, and $1 \le q < p < N$. Then one has the following results.*

(i) *If $\lambda \in (0, \Lambda_0)$, then* (1.1) *has at least one positive solution in $W$.*

(ii) *If $\lambda \in (0, (q/p)\Lambda_0)$, then* (1.1) *has at least two positive solutions in $W$.*

*Remark 1.2.* In [5], Kang considered (1.1) with $p$-sublinear perturbation of $p \le q < p^*(a,d)$. Via variational methods, he proved the existence of positive solutions of (1.1) when the parameters $a, b, d, p, q, \lambda, \mu$ satisfy suitable conditions. But the existence of positive solutions for (1.1) involving the $p$-sublinear of $1 \le q < p < N$ is not considered. In this paper, we will give a complement result.

This paper is organized as follows. In Sections 2 and 3, we give some preliminaries and some properties of Nehari manifold. In Section 4, we complete proofs of Theorem 1.1. At the end of this section, we explain some notations employed. In the following discussions, we will denote various positive constants as $C, C_i$ and omit $dx$ in the integral for convenience. We denote $B(0;R)$ as a ball centered at the origin with radius $R$, and $\omega_N = 2\pi^{N/2}/N\Gamma(N/2)$ is the volume of the unit ball $B(0;1)$ in $\mathbb{R}^N$. We denote the norm in $L^r(\Omega)$ by $|\cdot|_r$ for $1 \le r \le \infty$, and $L^r(\Omega, |x|^{-s})$, $1 \le r < \infty$ is the closure of $C_0^\infty(\Omega)$ with the norm $|\cdot|_{L^r(\Omega,|x|^{-s})} = (\int_\Omega |x|^{-s}|\cdot|^r)^{1/r}$. $W^{-1}$ denoting the dual space of $W$. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)/\varepsilon^t| \le C$, and $o(\varepsilon^t)$ means $|o(\varepsilon^t)/\varepsilon^t| \to 0$ as $\varepsilon \to 0$. By $o(1)$ we always mean it is a generic infinitesimal value.

## 2. Nehari Manifold

Since the functional $J_\lambda$ is not bounded from below on $W$, we will work on Nehari manifold. For $\lambda > 0$ we define

$$\mathcal{N}_\lambda = \{u \in W \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0\}. \tag{2.1}$$

We recall that any nonzero solutions of (1.1) belong to $\mathcal{N}_\lambda$. Moreover, by definition, we have that $u \in \mathcal{N}_\lambda$ if and only if

$$\|u\| \neq 0, \quad \|u\|^p - \frac{1}{p^*(a,b)} \int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} - \frac{\lambda}{q} \int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}} = 0. \tag{2.2}$$

The following result is concerned with the behavior of $J_\lambda$ on $\mathcal{N}_\lambda$.

**Lemma 2.1.** *$J_\lambda$ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

*Proof.* If $u \in \mathcal{N}_\lambda$, then by (1.14) and (2.2), we get

$$J_\lambda(u) = \frac{p^*(a,b) - p}{p^*(a,b)p} \|u\|^p - \lambda \left( \frac{p^*(a,b) - q}{p^*(a,b)q} \right) \int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}} \tag{2.3}$$

$$\geq \frac{p^*(a,b) - p}{p^*(a,b)p} \|u\|^p - \lambda \left( \frac{p^*(a,b) - q}{p^*(a,b)q} \right) \left( \frac{N\omega_N R_0^{N-dp^*(a,d)}}{N - dp^*(a,d)} \right)^{(p^*(a,d)-q)/p^*(a,d)} \times (S_{\mu,a,d})^{-q/p} \|u\|^q. \tag{2.4}$$

Since $0 \leq a < (N-p)/p$, $a \leq b, d < a+1$ and $1 \leq q < p < p^*(a,b)$, it follows that $J_\lambda$ is coercive and bounded from below on $\mathcal{N}_\lambda$. □

Define $\psi_\lambda : W \to \mathbb{R}$, by $\psi_\lambda(u) = \langle J'_\lambda(u), u \rangle$, that is,

$$\psi_\lambda(u) = \|u\|^p - \int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} - \lambda \int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}. \tag{2.5}$$

Note that $\psi_\lambda$ is of class $C^1$ with

$$\langle \psi'_\lambda(u), u \rangle = p\|u\|^p - p^*(a,b) \int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} - q\lambda \int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}. \tag{2.6}$$

Furthermore, if $u \in \mathcal{N}_\lambda$, then by (2.2), we have that

$$\langle \psi_\lambda'(u), u\rangle = (p-q)\|u\|^p - (p^*(a,b)-q)\int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \tag{2.7}$$

$$= (p-p^*(a,b))\|u\|^p - (q-p^*(a,b))\lambda\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}. \tag{2.8}$$

Now we split $\mathcal{N}_\lambda$ into three sets:

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u\in\mathcal{N}_\lambda : \langle \psi_\lambda'(u), u\rangle > 0\},\\
\mathcal{N}_\lambda^0 &= \{u\in\mathcal{N}_\lambda : \langle \psi_\lambda'(u), u\rangle = 0\},\\
\mathcal{N}_\lambda^- &= \{u\in\mathcal{N}_\lambda : \langle \psi_\lambda'(u), u\rangle < 0\}.
\end{aligned} \tag{2.9}$$

The following result shows that minimizers on $\mathcal{N}_\lambda$ are the "usual" critical points for $J_\lambda$.

**Lemma 2.2.** *Suppose $u_0$ is a local minimizer of $J_\lambda$ on $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$. Then, $J_\lambda'(u_0) = 0$ in $W^{-1}$.*

*Proof.* See [30, Lemma 2.2]. □

Motivated by the above result, we will get conditions for $\mathcal{N}_\lambda^0 = \emptyset$.

**Lemma 2.3.** *$\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \Lambda_0)$.*

*Proof.* We argue by contradiction. Suppose that there exists a $\lambda \in (0,\Lambda_0)$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. Let $u \in \mathcal{N}_\lambda^0$ be arbitrary, then by (2.2), (2.7), and (2.8), we have that

$$\begin{aligned}
0 < \|u\|^p &= \frac{p^*(a,b)-q}{p-q}\int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}},\\
0 < \|u\|^p &= \lambda\frac{p^*(a,b)-q}{p^*(a,b)-p}\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}.
\end{aligned} \tag{2.10}$$

By (1.14), (2.10), and Sobolev-Hardy inequality, we get

$$\begin{aligned}
\|u\| &\geq \left(\frac{p-q}{p^*(a,b)-q}\right)^{1/(p^*(a,b)-p)} (S_{\mu,a,b})^{p^*(a,b)/p(p^*(a,b)-p)},\\
\|u\| &\leq \left(\lambda\frac{p^*(a,b)-q}{p^*(a,b)-p}\right)^{1/(p-q)} \left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{(p^*(a,d)-q)/p^*(a,d)(p-q)} (S_{\mu,a,d})^{-(q/p(p-q))}.
\end{aligned} \tag{2.11}$$

Hence we must have

$$\begin{aligned}\lambda &\geq \left(\frac{p-q}{p^*(a,b)-q}\right)^{(p-q)/(p^*(a,b)-p)}\left(\frac{p^*(a,b)-p}{p^*(a,b)-q}\right)\left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{-((p^*(a,d)-q)/p^*(a,d))}\\ &\quad\times \left(S_{\mu,a,d}\right)^{q/p}\left(S_{\mu,a,b}\right)^{p^*(a,b)(p-q)/p(p^*(a,b)-p)}\\ &= \Lambda_0,\end{aligned} \tag{2.12}$$

which is a contradiction. □

For each $u \in W \setminus \{0\}$, let

$$\tau_{\max} = \left(\frac{(p-q)\|u\|^p}{(p^*(a,b)-q)\int_\Omega |u|^{p^*(a,b)}/|x|^{bp^*(a,b)}}\right)^{1/(p^*(a,b)-p)}. \tag{2.13}$$

**Lemma 2.4.** *If $\lambda \in (0,\Lambda_0)$, then for each $u \in W \setminus \{0\}$, the set $\{\tau u : \tau > 0\}$ intersects $\mathcal{N}_\lambda$ exactly twice. More specifically, there exist a unique $\tau^- = \tau^-(u) > 0$ such that $\tau^- u \in \mathcal{N}_\lambda^-$ and a unique $\tau^+ = \tau^+(u) > 0$ such that $\tau^+ u \in \mathcal{N}_\lambda^+$. Moreover, $\tau^+ < \tau_{\max} < \tau^-$ and*

$$J_\lambda(\tau^+ u) = \inf_{0\leq\tau\leq\tau_{\max}} J_\lambda(\tau u), \qquad J_\lambda(\tau^- u) = \sup_{\tau\geq\tau_{\max}} J_\lambda(\tau u). \tag{2.14}$$

*Proof.* The proof is similar to that of [29, Lemma 2.7] and is omitted. □

We remark that by Lemma 2.3 we have, $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all $\lambda \in (0,\Lambda_0)$. Furthermore, by Lemma 2.4 it follows that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are non-empty and by Lemma 2.1 we may define

$$\alpha_\lambda = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u), \qquad \alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u). \tag{2.15}$$

**Theorem 2.5.** (i) *If $\lambda \in (0,\Lambda_0)$, then one has $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.*

(ii) *If $\lambda \in (0,(q/p)\Lambda_0)$, then $\alpha_\lambda^- > d_0$ for some positive constant $d_0$. In particular, for each $\lambda \in (0,(q/p)\Lambda_0)$, one has $\alpha_\lambda = \alpha_\lambda^+ < 0 < \alpha_\lambda^-$.*

*Proof.* (i) Let $u \in \mathcal{N}_\lambda^+$. By (2.7),

$$\frac{p-q}{p^*(a,b)-q}\|u\|^p > \int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}, \tag{2.16}$$

and so also using (2.2),

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p}-\frac{1}{q}\right)\|u\|^p + \left(\frac{1}{q}-\frac{1}{p^*(a,b)}\right)\int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \\ &< \left[\left(\frac{1}{p}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{p^*(a,b)}\right)\left(\frac{p-q}{p^*(a,b)-q}\right)\right]\|u\|^p \\ &= -\frac{(p-q)(p^*(a,b)-p)}{pqp^*(a,b))}\|u\|^p < 0. \end{aligned} \tag{2.17}$$

Therefore, from the definition of $\alpha_\lambda$ and $\alpha_\lambda^+$, we can deduce that $\alpha_\lambda \le \alpha_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$. By (2.7),

$$\frac{p-q}{p^*(a,b)-q}\|u\|^p < \int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}. \tag{2.18}$$

Moreover, by Sobolev-Hardy inequality,

$$\int_\Omega \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \le \left(S_{\mu,a,b}\right)^{-(p^*(a,b)/p)}\|u\|^{p^*(a,b)}. \tag{2.19}$$

This implies

$$\|u\| > \left(\frac{p-q}{p^*(a,b)-q}\right)^{1/(p^*(a,b)-p)}\left(S_{\mu,a,b}\right)^{p^*(a,b)/p(p^*(a,b)-p)} \quad \forall u \in \mathcal{N}_\lambda^-. \tag{2.20}$$

By (2.4) and (2.20), we have

$$\begin{aligned} J_\lambda(u) &\ge \|u\|^q \left[\frac{p^*(a,b)-p}{p^*(a,b)p}\|u\|^{p-q} - \lambda\left(\frac{p^*(a,b)-q}{p^*(a,b)q}\right)\right. \\ &\qquad \left.\times\left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{(p^*(a,d)-q)/p^*(a,d)}\left(S_{\mu,a,d}\right)^{-(q/p)}\right] \\ &> \left(\frac{p-q}{p^*(a,b)-q}\right)^{q/(p^*(a,b)-p)}\left(S_{\mu,a,b}\right)^{qp^*(a,b)/p(p^*(a,b)-p)} \\ &\quad \times\left[\frac{p^*(a,b)-p}{p^*(a,b)p}\left(\frac{p-q}{p^*(a,b)-q}\right)^{(p-q)/(p^*(a,b)-p)}\left(S_{\mu,a,b}\right)^{p^*(a,b)(p-q)/p(p^*(a,b)-p)}\right. \\ &\qquad \left. -\lambda\left(\frac{p^*(a,b)-q}{p^*(a,b)q}\right)\left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{(p^*(a,d)-q)/p^*(a,d)}\left(S_{\mu,a,d}\right)^{-(q/p)}\right] \end{aligned}$$

$$= \left(\frac{q}{p}\Lambda_0 - \lambda\right)\left(\frac{p-q}{p^*(a,b)-q}\right)^{q/(p^*(a,b)-p)} (S_{\mu,a,b})^{qp^*(a,b)/p(p^*(a,b)-p)}\left(\frac{p^*(a,b)-q}{p^*(a,b)q}\right)$$
$$\times \left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{(p^*(a,d)-q)/p^*(a,d)} (S_{\mu,a,d})^{-(q/p)}. \tag{2.21}$$

Thus, if $\lambda \in (0, (q/p)\Lambda_0)$, then

$$J_\lambda(u) > d_0 \quad \forall u \in \mathcal{N}_\lambda^-, \tag{2.22}$$

for some positive constant $d_0$. □

*Remark 2.6.* If $\lambda \in (0, (q/p)\Lambda_0)$, then by Lemma 2.4 and Theorem 2.5, for each $u \in W \setminus \{0\}$, we can easily deduce that

$$\tau^- u \in \mathcal{N}_\lambda^-, \qquad J_\lambda(\tau^- u) = \sup_{\tau \geq 0} J_\lambda(\tau u) \geq \alpha_\lambda^- > 0. \tag{2.23}$$

## 3. Proof of the Main Results

First, we define the Palais-Smale (simply by (PS)) sequences, (PS)-values, and (PS)-conditions in $W$ for $J_\lambda$ as follows.

*Definition 3.1.* (i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence in $W$ for $J_\lambda$ if $J_\lambda(u_n) = c + o(1)$ and $J_\lambda'(u_n) = o(1)$ strongly in $W^{-1}$ as $n \to \infty$.

(ii) $c \in \mathbb{R}$ is a (PS)-value in $W$ for $J_\lambda$ if there exists a $(\mathrm{PS})_c$-sequence in $W$ for $J_\lambda$.

(iii) $J_\lambda$ satisfies the $(\mathrm{PS})_c$-condition in $W$ if any $(\mathrm{PS})_c$-sequence $\{u_n\}$ in $W$ for $J_\lambda$ contains a convergent subsequence.

Now, we use the Ekeland variational principle [35] to get the following results.

**Proposition 3.2.** (i) *If $\lambda \in (0, \Lambda_0)$, then $J_\lambda$ has a $(PS)_{\alpha_\lambda}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda$.*

(ii) *If $\lambda \in (0, (q/p)\Lambda_0)$, then $J_\lambda$ has a $(PS)_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$.*

*Proof.* The proof is similar to [29, Proposition 3.3] and the details are omitted. □

Now, we establish the existence of a local minimum for $J_\lambda$ on $\mathcal{N}_\lambda$.

**Theorem 3.3.** *Assume that $N \geq 3$, $0 \leq \mu < \overline{\mu}$, $0 \leq a < (N-p)/p$, $a \leq b, d < a+1$, and $1 \leq q < p < N$. If $\lambda \in (0, \Lambda_0)$, then there exists $u_\lambda \in \mathcal{N}_\lambda^+$ such that*

(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$,

(ii) $u_\lambda$ *is a positive solution of* (1.1),

(iii) $\|u_\lambda\| \to 0$ *as* $\lambda \to 0^+$.

*Proof.* By Proposition 3.2(i), there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o(1), \quad J'_\lambda(u_n) = o(1) \quad \text{in } W^{-1}. \tag{3.1}$$

Since $J_\lambda$ is coercive on $\mathcal{N}_\lambda$ (see Lemma 2.1), we get that $\{u_n\}$ is bounded in $W$. From [5, Lemma 2.1], we deduce that the embedding $W \hookrightarrow L^r(\Omega, |x|^{-dp^*(a,d)})$ is compact for $1 \le r < p^*(a,d)$. Thus, there exists $u_\lambda \in W$, passing to a subsequence if necessary, using similar arguments found in [27, 36], then one can get that as $n \to \infty$

$$\begin{aligned}
&u_n \rightharpoonup u_\lambda \text{ weakly in } W,\\
&u_n \longrightarrow u_\lambda \text{ strongly in } L^q\left(\Omega, |x|^{-dp^*(a,d)}\right) \quad \text{for } 1 \le q < p,\\
&u_n \longrightarrow u_\lambda \text{ a.e. in } \Omega,\\
&\nabla u_n \longrightarrow \nabla u_\lambda \text{ a.e. in } \Omega,\\
&\frac{u_n}{|x|^{a+1}} \rightharpoonup \frac{u_\lambda}{|x|^{a+1}} \text{ weakly in } L^p(\Omega),\\
&\int_\Omega \frac{|u_n|^{p^*(a,b)-2}u_n}{|x|^{bp^*(a,b)}}v \longrightarrow \int_\Omega \frac{|u_\lambda|^{p^*(a,b)-2}u_\lambda}{|x|^{bp^*(a,b)}}v, \quad \forall v \in W.
\end{aligned} \tag{3.2}$$

Consequently, passing to the limit in $\langle J'_\lambda(u_n), v\rangle$, by (3.1) and (3.2), as $n \to \infty$, we have

$$\int_\Omega \left( \frac{|\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla v}{|x|^{ap}} - \mu \frac{|u_\lambda|^{p-2}u_\lambda v}{|x|^{p(a+1)}} \right) - \int_\Omega \frac{|u_\lambda|^{p^*(a,b)-2}u_\lambda v}{|x|^{bp^*(a,b)}} - \lambda \int_\Omega \frac{|u_\lambda|^{q-2}u_\lambda v}{|x|^{dp^*(a,d)}} = 0, \tag{3.3}$$

for all $v \in W$. That is, $\langle J'_\lambda(u_\lambda), v\rangle = 0$. Thus $u_\lambda$ is a weak solution of (1.1). Furthermore, from $u_n \in \mathcal{N}_\lambda$ and (2.3), we deduce that

$$\lambda \int_\Omega \frac{|u_n|^q}{|x|^{dp^*(a,d)}} = \frac{q(p^*(a,b)-p)}{p(p^*(a,b)-q)}\|u_n\|^p - \frac{p^*(a,b)q}{p^*(a,b)-q}J_\lambda(u_n). \tag{3.4}$$

Let $n \to \infty$ in (3.4), by (3.1) and (3.2), and since $\alpha_\lambda < 0$ by (i) of Theorem 2.5, we get

$$\lambda \int_\Omega \frac{|u_\lambda|^q}{|x|^{dp^*(a,d)}} \ge -\frac{p^*(a,b)q}{p^*(a,b)-q}\alpha_\lambda > 0. \tag{3.5}$$

Thus $u_\lambda \not\equiv 0$, and since $J'_\lambda(u_\lambda) = 0$, it follows that $u_\lambda \in \mathcal{N}_\lambda$ and, in particular, $J_\lambda(u_\lambda) \ge \alpha_\lambda$.

Next, we will show, up to a subsequence, that $u_n \to u_\lambda$ strongly in $W$ and $J_\lambda(u_\lambda) = \alpha_\lambda$. From the fact $u_n,\ u \in \mathcal{N}_\lambda$, (2.3) and the Fatou's lemma, it follows that

$$\begin{aligned}
\alpha_\lambda \le J_\lambda(u_\lambda) &= \frac{p^*(a,b)-p}{pp^*(a,b)}\|u_\lambda\|^p - \lambda\left(\frac{p^*(a,b)-q}{p^*(a,b)q}\right)\int_\Omega \frac{|u_\lambda|^q}{|x|^{dp^*(a,d)}} \\
&\le \liminf_{n\to\infty}\left[\frac{p^*(a,b)-p}{p^*(a,b)p}\|u_n\|^p - \lambda\left(\frac{p^*(a,b)-q}{p^*(a,b)q}\right)\int_\Omega \frac{|u_n|^q}{|x|^{dp^*(a,d)}}\right] \\
&= \liminf_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda,
\end{aligned} \tag{3.6}$$

which implies that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n\to\infty}\|u_n\|^p = \|u_\lambda\|^p$. Standard argument shows that $u_n \to u_\lambda$ strongly in $W$. Moreover, $u_\lambda \in \mathcal{N}_\lambda^+$. Otherwise, if $u_\lambda \in \mathcal{N}_\lambda^-$, by Lemma 2.4, there exist unique $\tau_\lambda^+$ and $\tau_\lambda^-$ such that $\tau_\lambda^+ u_\lambda \in \mathcal{N}_\lambda^+$, $\tau_\lambda^- u_\lambda \in \mathcal{N}_\lambda^-$ and $\tau_\lambda^+ < \tau_\lambda^- = 1$. Since

$$\frac{d}{d\tau}J_\lambda(\tau_\lambda^+ u_\lambda) = 0, \qquad \frac{d^2}{d\tau^2}J_\lambda(\tau_\lambda^+ u_\lambda) > 0, \tag{3.7}$$

there exists $\overline{\tau} \in (\tau_\lambda^+, \tau_\lambda^-)$ such that $J_\lambda(\tau_\lambda^+ u_\lambda) < J_\lambda(\overline{\tau} u_\lambda)$. By Lemma 2.4 we get that

$$J_\lambda(\tau_\lambda^+ u_\lambda) < J_\lambda(\overline{\tau} u_\lambda) \le J_\lambda(\tau_\lambda^- u_\lambda) = J_\lambda(u_\lambda), \tag{3.8}$$

which is a contradiction. Since $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{N}_\lambda^+$, by Lemma 2.2, we may assume that $u_\lambda$ is a nontrivial nonnegative solution of (1.1). By [5, Lemma 2.3], it follows that $u_\lambda > 0$ in $\Omega$. Finally, by (1.14) and (2.8), we obtain

$$\|u_\lambda\|^{p-q} < \lambda\left(\frac{p^*(a,b)-q}{p^*(a,b)-p}\right)\left(\frac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{(p^*(a,d)-q)/p^*(a,d)} \left(S_{\mu,a,d}\right)^{-(q/p)}, \tag{3.9}$$

which implies that $\|u_\lambda\| \to 0$ as $\lambda \to 0^+$. □

Next, we will establish the existence of the second positive solution of (1.1) by proving that $J_\lambda$ satisfies the $(\mathrm{PS})_{\alpha_\lambda^-}$-condition.

**Lemma 3.4.** *Let $\{u_n\}$ be a bounded sequence in $W$. If $\{u_n\}$ is a $(PS)_c$-sequence for $J_\lambda$ with*

$$c \in \left(0, \frac{p^*(a,b)-p}{p^*(a,b)p}\left(S_{\mu,a,b}\right)^{p^*(a,b)/(p^*(a,b)-p)}\right), \tag{3.10}$$

*then there exists a subsequence of $\{u_n\}$ converging weakly to a nonzero solution of* (1.1).

*Proof.* Let $\{u_n\} \subset W$ be a $(\mathrm{PS})_c$-sequence for $J_\lambda$ with $c \in (0, ((p^*(a,b) - p)/p^*(a,b)p)\ (S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)})$. Since $\{u_n\}$ is bounded in $W$ and the embedding

$W \hookrightarrow L^r(\Omega, |x|^{-dp^*(a,d)})$ is compact for $1 \le r < p^*(a,d)$ (see [5, Lemma 2.1]), thus passing to a subsequence if necessary, we may assume that as $n \to \infty$

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ weakly in } W, \\ &u_n \rightharpoonup u_0 \text{ weakly in } L^{p^*(a,b)}\left(\Omega, |x|^{-bp^*(a,b)}\right) \\ &u_n \longrightarrow u_0 \text{ strongly in } L^q\left(\Omega, |x|^{-dp^*(a,d)}\right) \quad \text{for } 1 \le q < p, \\ &u_n \longrightarrow u_0 \text{ a.e. in } \Omega. \end{aligned} \tag{3.11}$$

Using the same argument in Theorem 3.3, we deduce that $J'_\lambda(u_0) = 0$ and

$$\lambda \int_\Omega \frac{|u_n|^q}{|x|^{dp^*(a,d)}} = \lambda \int_\Omega \frac{|u_0|^q}{|x|^{dp^*(a,d)}} + o(1). \tag{3.12}$$

Next we verify that $u_0 \not\equiv 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. Set

$$l = \lim_{n\to\infty} \int_\Omega \frac{|u_n|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}. \tag{3.13}$$

Since $J'_\lambda(u_0) = 0$ and $\{u_n\}$ is bounded in $W$, then by (3.12), we can deduce that

$$0 = \left\langle \lim_{n\to\infty} J'_\lambda(u_n), u_n \right\rangle = \lim_{n\to\infty} \left( \|u_n\|^p - \int_\Omega \frac{|u_n|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \right) = \lim_{n\to\infty} \|u_n\|^p - l, \tag{3.14}$$

that is,

$$\lim_{n\to\infty} \|u_n\|^p = l. \tag{3.15}$$

If $l = 0$, then by (3.12)–(3.15), we get

$$c = \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty} \left( \frac{1}{p}\|u_n\|^p - \int_\Omega \frac{|u_n|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} - \lambda \int_\Omega \frac{|u_n|^q}{|x|^{dp^*(a,d)}} \right) = 0, \tag{3.16}$$

which contradicts $c > 0$. Thus we conclude that $l > 0$. Furthermore, the Sobolev-Hardy inequality implies that

$$\|u_n\|^p \ge S_{\mu,a,b} \left( \int_\Omega \frac{|u_n|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \right)^{p/p^*(a,b)}. \tag{3.17}$$

Then as $n \to \infty$, we have $l = \lim_{n\to\infty} \|u_n\|^p \ge S_{\mu,a,b} l^{p/p^*(a,b)}$, which implies that

$$l \ge (S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)}. \tag{3.18}$$

Hence, from (3.12)–(3.18) we get

$$\begin{aligned}
c &= \lim_{n\to\infty} J_\lambda(u_n) \\
&= \frac{1}{p}\lim_{n\to\infty}\|u_n\|^p - \frac{1}{p^*(a,b)}\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} - \frac{\lambda}{q}\lim_{n\to\infty}\int_\Omega \frac{|u_n|^q}{|x|^{dp^*(a,d)}} \\
&= \left(\frac{1}{p} - \frac{1}{p^*(a,b)}\right) l \\
&\geq \frac{p^*(a,b)-p}{p^*(a,b)p}\left(S_{\mu,a,b}\right)^{p^*(a,b)/(p^*(a,b)-p)}.
\end{aligned} \tag{3.19}$$

This contradicts the definition of $c$. Therefore, $u_0$ is a nontrivial solution of (1.1). □

**Lemma 3.5.** *If $N \geq 3$, $0 \leq \mu < \overline{\mu}$, $0 \leq a < (N-p)/p$, $a \leq b, d < a+1$, and $1 \leq q < p < N$, then for any $\lambda > 0$, there exists $v_\lambda \in W$ such that*

$$\sup_{\tau\geq 0} J_\lambda(\tau v_\lambda) < \frac{p^*(a,b)-p}{p^*(a,b)p}\left(S_{\mu,a,b}\right)^{p^*(a,b)/(p^*(a,b)-p)}. \tag{3.20}$$

*In particular, $\alpha_\lambda^- < ((p^*(a,b)-p)/p^*(a,b)p)(S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)}$ for all $\lambda \in (0, \Lambda_0)$.*

*Proof.* Let $U_{p,\mu}$ be a ground state solution of (1.8), $\rho > 0$ small enough such that $B(0;\rho) \subset \Omega$, $\eta \in C_0^\infty(\Omega)$, $0 \leq \eta(x) \leq 1$, $\eta(x) = 1$ for $|x| < \rho/2$, $\eta(x) = 0$ for $|x| \geq \rho$. Set $\widetilde{U}_\varepsilon(x) = \varepsilon^{-((N-p)/p)-a} U_{p,\mu}(x/\varepsilon)$ and $u_\varepsilon(x) = \eta(x)\widetilde{U}_\varepsilon(x)$, $\varepsilon > 0$. Then, following the same lines as in [5], we get the following estimates as $\varepsilon \to 0$:

$$\|u_\varepsilon\|^p = \left(S_{\mu,a,b}\right)^{p^*(a,b)/(p^*(a,b)-p)} + O\left(\varepsilon^{\beta(\mu)p+p(a+1)-N}\right), \tag{3.21}$$

$$\int_\Omega \frac{|u_\varepsilon|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} = \left(S_{\mu,a,b}\right)^{p^*(a,b)/(p^*(a,b)-p)} + O\left(\varepsilon^{(\beta(\mu)+b)p^*(a,b)-N}\right), \tag{3.22}$$

$$\int_\Omega \frac{|u_\varepsilon|^q}{|x|^{dp^*(a,d)}} \geq \begin{cases} C\varepsilon^{N-dp^*(a,d)-q\delta}, & \dfrac{N-dp^*(a,d)}{\beta(\mu)} < q < p^*(a,d), \\ C\varepsilon^{q(\beta(\mu)-\delta)}|\ln\varepsilon|, & q = \dfrac{N-dp^*(a,d)}{\beta(\mu)}, \\ C\varepsilon^{q(\beta(\mu)-\delta)}, & 1 \leq q < \dfrac{N-dp^*(a,d)}{\beta(\mu)}, \end{cases} \tag{3.23}$$

where $b(\mu)$ is given in the introduction satisfying $\delta = (N-p(a+1))/p < b(\mu) < (N-p(a+1))/(p-1)$.

Now we consider the following functions:

$$g(\tau) = J_\lambda(\tau u_\varepsilon) = \frac{\tau^p}{p}\|u_\varepsilon\|^p - \frac{\tau^{p^*(a,b)}}{p^*(a,b)}\int_\Omega \frac{|u_\varepsilon|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} - \lambda\frac{\tau^q}{q}\int_\Omega \frac{|u_\varepsilon|^q}{|x|^{dp^*(a,d)}},$$
$$\overline{g}(\tau) = \frac{\tau^p}{p}\|u_\varepsilon\|^p - \frac{\tau^{p^*(a,b)}}{p^*(a,b)}\int_\Omega \frac{|u_\varepsilon|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}. \tag{3.24}$$

Using the definitions of $g$ and $u_\varepsilon$, we get

$$g(\tau) = J_\lambda(\tau u_\varepsilon) \le \frac{\tau^p}{p}\|u_\varepsilon\|^p, \quad \forall\tau \ge 0,\ \lambda > 0. \tag{3.25}$$

Combining this with (3.21), let $\varepsilon \in (0,1)$, then there exists $\tau_0 \in (0,1)$ independent of $\varepsilon$ such that

$$\sup_{0\le\tau\le\tau_0} g(\tau) < \frac{p^*(a,b)-p}{p^*(a,b)p}(S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)}, \quad \forall\lambda > 0,\ \forall\varepsilon \in (0,1). \tag{3.26}$$

On the other hand, by the fact for $B_1, B_2 > 0$

$$\max_{\tau\ge0}\left(\frac{\tau^p}{p}B_1 - \frac{\tau^{p^*(a,b)}}{p^*(a,b)}B_2\right) = \frac{p^*(a,b)-p}{p^*(a,b)p}(B_1)^{p^*(a,b)/(p^*(a,b)-p)}(B_2)^{-(p/(p^*(a,b)-p))}, \tag{3.27}$$

and by (3.21) and (3.22), we can get that

$$\begin{aligned}
\max_{\tau\ge0}\overline{g}(\tau) &= \frac{p^*(a,b)-p}{p^*(a,b)p}\|u_\varepsilon\|^{p^*(a,b)/(p^*(a,b)-p)}\left(\int_\Omega \frac{|u_\varepsilon|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\right)^{-(p/(p^*(a,b)-p))} \\
&= \frac{p^*(a,b)-p}{p^*(a,b)p}\left((S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)} + O\left(\varepsilon^{\beta(\mu)p+p(a+1)-N}\right)^{p^*(a,b)/(p^*(a,b)-p)}\right. \\
&\quad \times\left.\left((S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)} + O\left(\varepsilon^{(\beta(\mu)+b)p^*(a,b)-N}\right)\right)^{-(p/(p^*(a,b)-p))}\right. \\
&= \frac{p^*(a,b)-p}{p^*(a,b)p}(S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)} + O\left(\varepsilon^{\beta(\mu)p+p(a+1)-N}\right).
\end{aligned} \tag{3.28}$$

Hence as $\lambda > 0$, $1 \le q < p$, by (3.28) we have that

$$\begin{aligned}
\sup_{\tau\ge\tau_0} g(\tau) &= \sup_{\tau\ge\tau_0}\left(\overline{g}(\tau) - \lambda\frac{\tau^q}{q}\int_\Omega \frac{|u_\varepsilon|^q}{|x|^{dp^*(a,d)}}\right) \\
&\le \frac{p^*(a,b)-p}{p^*(a,b)p}(S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)} + O\left(\varepsilon^{\beta(\mu)p+p(a+1)-N}\right) - \lambda\frac{\tau_0^q}{q}\int_\Omega \frac{|u_\varepsilon|^q}{|x|^{dp^*(a,d)}}.
\end{aligned} \tag{3.29}$$

(i) If $1 \le q < (N - dp^*(a,d))/\beta(\mu)$, then by (3.23) we have that

$$\int_\Omega \frac{|u_\varepsilon|^q}{|x|^{dp^*(a,d)}} \ge C\varepsilon^{q(\beta(\mu)-\delta)} \tag{3.30}$$

and since $b(\mu) > \delta = (N - p(a+1))/p$, then

$$\beta(\mu)p + p(a+1) - N = p(\beta(\mu) - \delta) > q(\beta(\mu) - \delta). \tag{3.31}$$

Combining this with (3.26) and (3.29), for any $\lambda > 0$, we can choose $\varepsilon_\lambda$ small enough such that

$$\sup_{\tau \ge 0} J_\lambda(\tau u_{\varepsilon_\lambda}) < \frac{p^*(a,b) - p}{p^*(a,b)p} (S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)}. \tag{3.32}$$

(ii) If $(N - dp^*(a,d))/\beta(\mu) \le q < p$, then by (3.23) and $b(\mu) > \delta = (N - p(a+1))/p$ we have that

$$\int_\Omega \frac{|u_\varepsilon|^q}{|x|^{dp^*(a,d)}} \ge \begin{cases} C\varepsilon^{N-dp^*(a,d)-q\delta}, & \dfrac{N - dp^*(a,d)}{\beta(\mu)} < q < p^*(a,d), \\ C\varepsilon^{q(\beta(\mu)-\delta)}|\ln \varepsilon|, & q = \dfrac{N - dp^*(a,d)}{\beta(\mu)}, \end{cases} \tag{3.33}$$

$$N - dp^*(a,d) - q\delta \le q\beta(\mu) - q\delta < p(\beta(\mu) - \delta) = \beta(\mu)p + p(a+1) - N.$$

Combining this with (3.26) and (3.29), for any $\lambda > 0$, we can choose $\varepsilon_\lambda$ small enough such that

$$\sup_{\tau \ge 0} J_\lambda(\tau u_{\varepsilon_\lambda}) < \frac{p^*(a,b) - p}{p^*(a,b)p} (S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)}. \tag{3.34}$$

From (i) and (ii), (3.20) holds by taking $v_\lambda = u_{\varepsilon_\lambda}$.

From Lemma 2.4, the definition of $\alpha_\lambda^-$, and (3.20), for any $\lambda \in (0, \Lambda_0)$, we obtain that there exists $\tau_\lambda^- > 0$ such that $\tau_\lambda^- v_\lambda \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda^- \le J_\lambda(\tau_\lambda^- v_\lambda) \le \sup_{t \ge 0} J_\lambda(\tau v_\lambda) < \frac{p^*(a,b) - p}{p^*(a,b)p} (S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)}. \tag{3.35}$$

The proof is thus complete. □

Now, we establish the existence of a local minimum of $J_\lambda$ on $\mathcal{N}_\lambda^-$.

**Theorem 3.6.** *Assume that $N \ge 3$, $0 \le \mu < \overline{\mu}$, $0 \le a < (N - p)/p$, $a \le b, d < a + 1$ and $1 \le q < p < N$. If $\lambda \in (0, (q/p)\Lambda_0)$, then there exists $U_\lambda \in \mathcal{N}_\lambda^-$ such that*

(i) $J_\lambda(U_\lambda) = \alpha_\lambda^-$,

(ii) $U_\lambda$ *is a positive solution of* (1.1).

*Proof.* If $\lambda \in (0,(q/p)\Lambda_0)$, then by Theorem 2.5 (ii), Proposition 3.2 (ii), and Lemma 3.5, there exists a $(\mathrm{PS})_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in $W$ for $J_\lambda$ with $\alpha_\lambda^- \in (0,((p^*(a,b)-p)/p^*(a,b)p)(S_{\mu,a,b})^{p^*(a,b)/(p^*(a,b)-p)})$. Since $J_\lambda$ is coercive on $\mathcal{N}_\lambda$ (see Lemma 2.1), we get that $\{u_n\}$ is bounded in $W$. From Lemma 3.4, there exists a subsequence still denoted by $\{u_n\}$ and a nontrivial solution $U_\lambda \in W$ of (1.1) such that $u_n \rightharpoonup U_\lambda$ weakly in $W$.

First, we prove that $U_\lambda \in \mathcal{N}_\lambda^-$. Arguing by contradiction, we assume $U_\lambda \in \mathcal{N}_\lambda^+$. Since $\mathcal{N}_\lambda^-$ is closed in $W$, we have $\|U_\lambda\| < \liminf_{n\to\infty}\|u_n\|$. Thus, by Lemma 2.4, there exists a unique $\tau_\lambda^-$ such that $\tau_\lambda^- U_\lambda \in \mathcal{N}_\lambda^-$. If $u \in \mathcal{N}_\lambda$, then it is easy to see that

$$J_\lambda(u) = \frac{p^*(a,b)-p}{p^*(a,b)p}\|u\|^p - \lambda\left(\frac{p^*(a,b)-q}{p^*(a,b)q}\right)\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}. \tag{3.36}$$

From Remark 2.6, $u_n \in \mathcal{N}_\lambda^-$, $\|U_\lambda\| < \liminf_{n\to\infty}\|u_n\|$, and (3.36), we can deduce that

$$\alpha_\lambda^- \le J_\lambda(\tau_\lambda^- U_\lambda) < \lim_{n\to\infty} J_\lambda(\tau_\lambda^- u_n) \le \lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^-. \tag{3.37}$$

This is a contradiction. Thus, $U_\lambda \in \mathcal{N}_\lambda^-$.

Next, by the same argument as that in Theorem 3.3, we get that $u_n \to U_\lambda$ strongly in $W$ and $J_\lambda(U_\lambda) = \alpha_\lambda^- > 0$ for all $\lambda \in (0,(q/p)\Lambda_0)$. Since $J_\lambda(U_\lambda) = J_\lambda(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{N}_\lambda^-$, by Lemma 2.2, we may assume that $U_\lambda$ is a nontrivial nonnegative solution of (1.1). Finally, by [5, Lemma 2.3], we obtain that $U_\lambda$ is a positive solution of (1.1). □

Now, we complete the proof of Theorem 1.1. The part (i) of Theorem 1.1 immediately follows from Theorem 3.3. When $0 < \lambda < (q/p)\Lambda_0 < \Lambda_0$, by Theorems 3.3 and 3.6, we obtain (1.1) has two positive solutions $u_\lambda$ and $U_\lambda$ such that $u_\lambda \in \mathcal{N}_\lambda^+$, $U_\lambda \in \mathcal{N}_\lambda^-$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that $u_\lambda$ and $U_\lambda$ are distinct. This completes the proof of Theorem 1.1.

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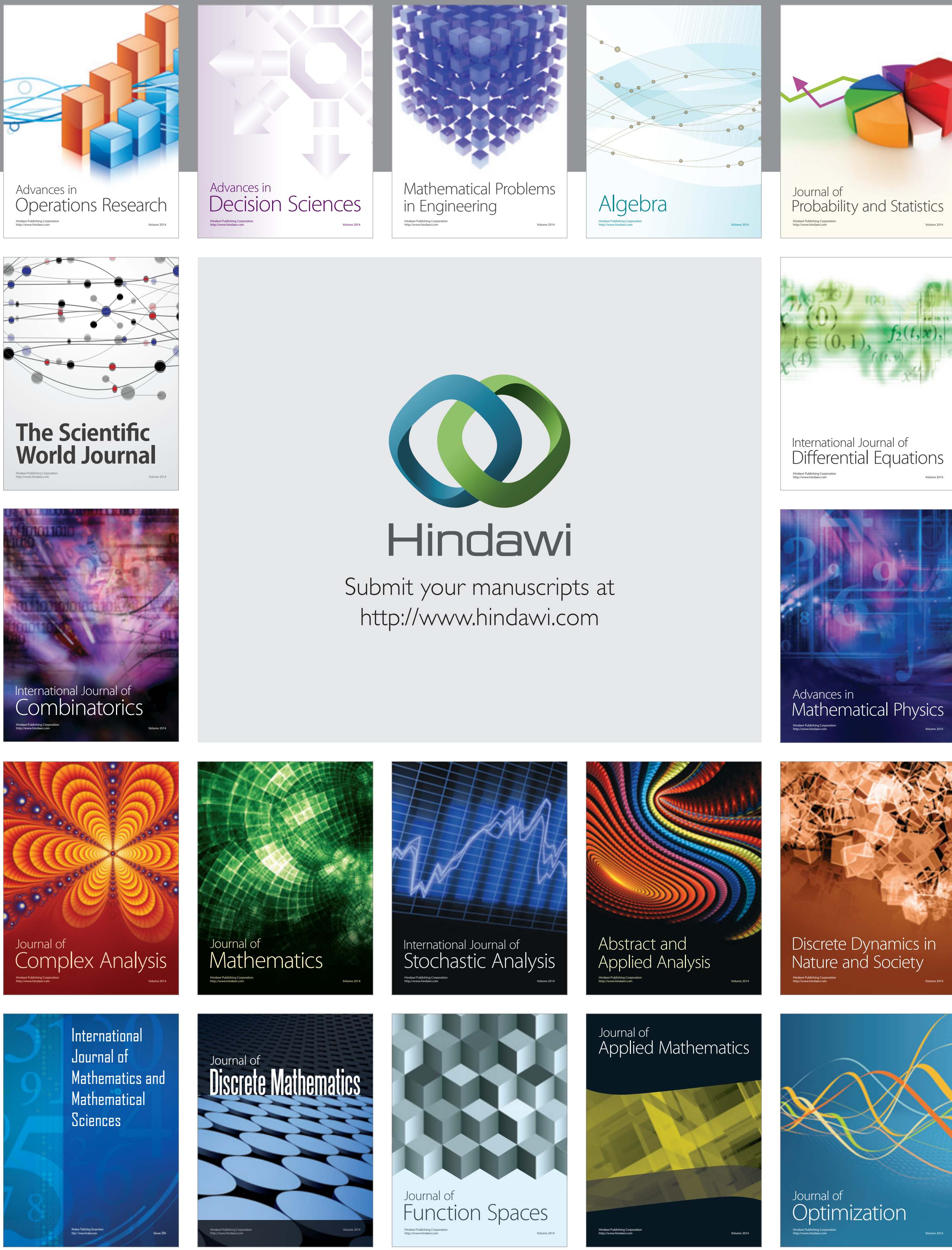
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